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# Uniform convexity of Musielak-Orlicz spaces with Luxemburg's norm

**Abstract.** We give some sufficient conditions in order that the Cartesian product $L_M^\mu(T, X) \times L_N^\nu(S, Y)$ of Musielak–Orlicz spaces $L_M^\mu(T, X)$ and $L_N^\nu(S, Y)$ with Luxemburg's norm be uniformly convex and we prove necessity of some of them. Next, we give some corollaries and some examples of $\varphi$-functions with parameter, which generate uniformly convex Musielak–Orlicz spaces. These results are generalizations of the respective results of Luxemburg [9] and Nakano [11].

**0. Introduction.** $(T, \Sigma, \mu)$ and $(S, \Sigma_0, \nu)$ are spaces of non-negative, $\sigma$-finite, atomless and complete measures, $R = (-\infty, \infty)$, $R_+ = [0, \infty)$, $R^n$ denotes the $n$-dimensional Euclidean space, $C$ is the space of all complex numbers, $X$ and $Y$ are complex or real Banach spaces with norms $\|\cdot\|$ and $|||\cdot|||$, respectively. We say that a map $M\colon T \times R_+ \to R_+$ is a *$\varphi$-function with parameter* if:

(i) $M(t, 0) = 0$ and $M(t, u) \to \infty$ as $u \to \infty$ for $\mu$-almost every $t \in T$,

(ii) there exists a set $T_0$ of measure zero such that $M(t, au + \beta v) \leqslant aM(t, u) + \beta M(t, v)$ for every $u, v, a, \beta \geqslant 0$, $a + \beta = 1$, $t \in T \setminus T_0$,

(iii) $M(t, u)$ is a $\mu$-measurable function of $t$ for every fixed $u \geqslant 0$.

We define the Musielak–Orlicz space $L_M = L_M^\mu(T, X)$ as the set of all strongly $\mu$-measurable functions $x(\cdot)$ defined on $T$ with values in $X$ such that $I_M\big(\lambda x(\cdot)\big) < \infty$ for some $\lambda > 0$ depending on $x(\cdot)$, where $I_M\big(x(\cdot)\big) = \int_T M\big(t, \|x(t)\|\big)\, d\mu$. Analogously we define the space $L_N^\nu(S, Y)$.

We say that a $\varphi$-function $M$ with parameter is *strictly convex* if there exists a set $T_1$ with $\mu(T_1) = 0$ such that

$$M\big(t, au + (1-a)v\big) < aM(t, u) + (1-a)M(t, v)$$

for every $t \in T \setminus T_1$ and $0 \leqslant u < v < \infty$, $0 < a < 1$.

Recall that a $\varphi$-function $M$ with parameter is *uniformly convex* if there exist a set $T_2$ with $\mu(T_2) = 0$ and a function $\delta(\cdot)\colon (0, 1) \to (0, 1)$ such that

$$M\left(t, \frac{1+b}{2} u\right) \leqslant \big(1 - \delta(a)\big) \frac{M(t, u) + M(t, bu)}{2}$$

for every $t \in T \setminus T_2$, $0 < a < 1$, $0 \leqslant b \leqslant a$, $u \geqslant 0$ (see [9] and [1]).

We say that $M$ satisfies the condition $\Delta_2$ if there exist a set $T_3$ with $\mu(T_3) = 0$, a constant $K > 0$ and a non-negative function $h(\cdot) \in L_1^\mu(T, R)$ such that

$$M(t, 2u) \leqslant KM(t, u) + h(t) \quad \text{for every } t \in T \setminus T_3,\ u \geqslant 0.$$

We say that a Banach space $X = (X, \|\cdot\|)$ is *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $x, y \in X$, $\|x\| = \|y\| = 1$, $\|x-y\| \geqslant \varepsilon$ imply $\|x+y\| < 2(1-\delta(\varepsilon))$ (see e.g. [8]).

A modular $\varrho$ is called *uniformly convex* if for every $\varepsilon > 0$ there exists $q(\varepsilon) > 0$ such that the conditions $\varrho(x) = \varrho(y) = 1$, $\varrho(x-y) \geqslant \varepsilon$ imply $\varrho\big((x+y)/2\big) \leqslant 1 - q(\varepsilon)$ (see [9]).

We shall consider the Cartesian product $L \doteq L_M^\mu(T, X) \times L_N^\nu(S, Y)$ with the norm

$$\|x(\cdot)\|_L = \inf\{u > 0 : I\big(x(\cdot)/u\big) \leqslant 1\},$$

where $x(\cdot) = \big(x_1(\cdot), x_2(\cdot)\big)$ and $I\big(x(\cdot)\big) = I_M\big(x_1(\cdot)\big) + I_N\big(x_2(\cdot)\big)$.

## Results

**1. Uniform convexity of $L$.** First, we shall prove some lemmas.

1.1. Lemma. *A Banach space $X$ is uniformly convex if and only if for every $\varepsilon > 0$ there exists $\delta_1(\varepsilon) > 0$ such that for every $x, y \in X$, $x \neq 0$, $y \neq 0$ with $\|x\| \leqslant 1$ and $\|y\| \leqslant 1$, the condition $\|x+y\| \geqslant 2 - \delta_1$ implies $\|x-y\| \leqslant \varepsilon$.*

Proof. It is obvious that if the condition from the lemma is satisfied, then $X$ is uniformly convex. Conversely, let $X$ be uniformly convex and let $x \neq 0$, $y \neq 0$, $x, y \in X$, $\|x\| \leqslant 1$, $\|y\| \leqslant 1$, $\|x+y\| \geqslant 2 - \delta_1(\varepsilon)$, where $\delta_1(\varepsilon) = \frac{1}{3}\min\big(\delta(\varepsilon/2), \varepsilon/2\big)$. It follows from the assumptions that $\|x\| \geqslant 1 - \delta_1$ and $\|y\| \geqslant 1 - \delta_1$. Moreover, writing $a = 1/\|x\|$, $b = 1/\|y\|$, we have

$$\begin{aligned}\big|\|ax+by\| - \|x+y\|\big| &\leqslant \|(a-1)x + (b-1)y\| \leqslant (a-1)\|x\| + (b-1)\|y\| \\ &= 2 - (\|x\| + \|y\|) \leqslant 2 - 2(1-\delta_1) = 2\delta_1.\end{aligned}$$

Hence

$$\|ax+by\| \geqslant \|x+y\| - 2\delta_1 \geqslant 2 - \delta_1 - 2\delta_1 = 2 - 3\delta_1 \geqslant 2 - \delta(\varepsilon/2).$$

Since $\|ax\| = \|by\| = 1$, by uniform convexity of $X$ we obtain $\|ax - by\| \leqslant \varepsilon/2$. Thus, we have

$$\begin{aligned}\big|\|x-y\| - \|ax-by\|\big| &\leqslant \|(1-a)x + (1-b)y\| \leqslant (a-1)\|x\| + (b-1)\|y\| \\ &= 2 - (\|x\| + \|y\|) \leqslant 2 - 2(1-\delta_1) = 2\delta_1 < \varepsilon/2.\end{aligned}$$

Hence

$$\|x-y\| \leqslant \|ax-by\| + \varepsilon/2 \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

1.2. LEMMA. *If $M$ is a uniformly convex $\varphi$-function with parameter and $X$ is a uniformly convex Banach space, then for every $\varepsilon > 0$ there exists $p(\varepsilon) > 0$ such that the inequality*

$$M(t, \|x+y\|/2) \leqslant (1-p(\varepsilon))[M(t, \|x\|)+M(t, \|y\|)]/2$$

*holds for every $x, y \in X$ satisfying the inequality $\|x-y\| \geqslant \varepsilon \max(\|x\|, \|y\|)$, and for every $t \in T \setminus T_2$.*

Proof. Let $\delta_0$ and $\delta^0$ denote moduli of convexity of $M$ and $X$, respectively. Let $b(\varepsilon) = \sqrt{1-\delta^0(\varepsilon)}$. Without loss of generality it can be assumed that $0 < \varepsilon \leqslant 1$ and $\|y\| \leqslant \|x\|$. Thus the inequality $\|x-y\| \geqslant \varepsilon \max(\|x\|, \|y\|)$ is equivalent to $\|x-y\| \geqslant \varepsilon \|x\|$. We consider two cases.

1. $\|y\| \leqslant b(\varepsilon)\|x\|$. Then, by uniform convexity of $M$, we have for $t \in T \setminus T_2$

$$M(t, \|x+y\|/2) \leqslant M(t, (\|x\|+\|y\|)/2) = M\left(t, \left(\|x\| + \frac{\|y\|}{b\|x\|} b\|x\|\right)/2\right)$$
$$\leqslant \left(1-\delta_0(b(\varepsilon))\right)[M(t, \|x\|)+M(t, \|y\|)]/2.$$

2. Let be $b(\varepsilon)\|x\| < \|y\| \leqslant \|x\|$. Then, we have

$$\big\|(x-y)/\|x\|\,\big\| \geqslant \varepsilon \quad \text{and} \quad \big\|x/\|x\|\,\big\| = 1, \quad \big\|y/\|x\|\big\| \leqslant 1.$$

Thus, by uniform convexity of $X$ and by Lemma 1.1, we get

$$\big\|(x+y)/2\|x\|\big\| \leqslant 1-\delta^0(\varepsilon).$$

Hence,

$$\|x+y\| \leqslant 2(1-\delta^0(\varepsilon))\|x\| = (1-\delta^0(\varepsilon))(\|x\|+\|x\|) \leqslant (1-\delta^0(\varepsilon))(\|x\|+\|y\|/b(\varepsilon))$$
$$\leqslant [(1-\delta^0(\varepsilon))/b(\varepsilon)](\|x\|+\|y\|) = b(\varepsilon)(\|x\|+\|y\|).$$

Therefore, we have for $t \in T \setminus T_2$

$$M(t, \|x+y\|/2) \leqslant b(\varepsilon)M(t, (\|x\|+\|y\|)/2) \leqslant b(\varepsilon)[M(t, \|x\|)+M(t, \|y\|)]/2.$$

Taking $p(\varepsilon) = \min(\delta_0(b(\varepsilon)), 1-b(\varepsilon))$, we obtain our lemma.

1.3. LEMMA. *If $M$ and $N$ are $\varphi$-functions with parameter satisfying the condition $\Delta_2$, then $I(x(\cdot)) = 1$ if and only if $\|x(\cdot)\|_I = 1$.*

The proof of this fact is analogous to the proof of Lemma 1 from [3], so it is omitted here.

1.4. LEMMA. *If $M$ and $N$ are $\varphi$-functions with parameter satisfying the condition $\Delta_2$, then for every $\varepsilon > 0$ there exists $\varepsilon_1(\varepsilon) > 0$ such that $\|x(\cdot)\|_I > \varepsilon$ implies $I(x(\cdot)) > \varepsilon_1$.*

Proof. It suffices to show that for every $\varepsilon > 0$ there exists $\varepsilon_1(\varepsilon) > 0$ such that $I(x(\cdot)) \leqslant \varepsilon_1$ implies $\|x(\cdot)\|_I \leqslant \varepsilon$. This follows from the equivalence: $I(x_n(\cdot)) \to 0$ as $n \to \infty$ if and only if $\|x_n(\cdot)\|_I \to 0$ as $n \to \infty$ (by the condition $\Delta_2$, see [4] and [6]).

1.5. LEMMA. *If $M$ and $N$ are $\varphi$-functions with parameter satisfying the condition $\Delta_2$, then for every $\varepsilon > 0$ there exists a number $\eta(\varepsilon) > 0$ such that $I(x(\cdot)) \leqslant 1-\varepsilon$ implies $\|x(\cdot)\|_I \leqslant 1-\eta(\varepsilon)$.*

Proof. If the statement is false there exist a number $\varepsilon > 0$ and a sequence $\{x_n(\cdot)\}_{n=1}^{\infty}$ such that $I(x_n(\cdot)) \leqslant 1-\varepsilon$ and $\|x_n(\cdot)\|_I \uparrow 1$. Then, for $a_n = \|x_n(\cdot)\|_I^{-1}$ we have $\|a_n x_n(\cdot)\|_I = 1$. Hence $I(a_n x_n(\cdot)) = 1$, by Lemma 1.3. So, we have

$$1 = I(a_n x_n(\cdot)) = I((a_n-1)2x_n + (2-a_n)x_n) \leqslant (a_n-1)I(2x_n(\cdot)) + \\ + (2-a_n)I(x_n(\cdot)) \leqslant (a_n-1)\big(KI(x_n(\cdot))+b\big) + (2-a_n)I(x_n(\cdot))$$

(by the condition $\Delta_2$ for $M$ and $N$), which contradicts $\varepsilon > 0$ for sufficiently large $n$.

1.6. LEMMA. *If $M$ is a $\varphi$-function with parameter satisfying the condition $\Delta_2$ and $M(t, u) = 0$ iff $u = 0$ for $\mu$-a.e. $t \in T$, then there exists a set $A \in \Sigma$ of measure zero such that for every $\varepsilon > 0$ there exist a non-negative function $h_\varepsilon(\cdot)$ with $\int_T h_\varepsilon(t)d\mu \leqslant \varepsilon$ and a constant $K_\varepsilon > 0$ such that for every $u \geqslant 0$ and for every $t \in T \setminus A$ there holds*

$$M(t, 2u) \leqslant K_\varepsilon M(t, u) + h_\varepsilon(t).$$

Proof. Let $T\setminus A = \{t \in T\setminus T_0 \colon M(t, u) = 0$ iff $u = 0$ and $M(t, \cdot)$ satisfies the condition $\Delta_2\}$.(1) We have $\mu(A) = 0$. Denote

$$h_n(t) = \sup_{u\geqslant 0}[M(t, 2u) - 2^n M(t, u)].$$

By continuity of $M(t, \cdot)$ we have for every $t \in T\setminus A$

$$h_n(t) = \sup_{i\in N}[M(t, 2u_i) - 2^n M(t, u_i)],$$

where $\{u_i\}_{i=1}^{\infty}$ denotes the sequence of all non-negative rational numbers Thus, $h_n(\cdot)$ are $\mu$-measurable functions for every $n \in N$. It is obvious that $0 \leqslant h_{n+1}(t) \leqslant h_n(t)$ for every $t \in T\setminus A$, $n \in N$. So, for every $t \in T\setminus A$ there exists the limit $\lim_{n\to\infty} h_n(t)$. Now, we shall show that $h_n(t)\downarrow 0$ for every $t \in T\setminus A$. Let us assume, to the contrary, that this does not hold. Then, there exists a set $B \subset T\setminus A$ (nonempty and measurable) such that for every $t \in B$, $h_n(t) \not\to 0$. Hence, there exists a function $a(\cdot)\colon B \to (0, \infty)$ such that $h_n(t) \to a(t)$ for every $t \in B$. Thus

$$h_n(t) = \sup_{u\geqslant 0}[M(t, 2u) - 2^n M(t, u)] \geqslant a(t) \quad \text{for every } t \in B,\ n \in N.$$

Hence, we have

$$(1) \quad \forall n \in N\ \forall t \in B\ \exists \varepsilon_t\ \exists u_{n,t}\colon M(t, 2u_{n,t}) - 2^n M(t, u_{n,t}) \geqslant a(t) - \varepsilon_t > 0.$$

(1) $T_0$ is the null-set given in the definition of $M$ being a $\varphi$-function with parameter.

Let $a(t)-\varepsilon_t = b_t$. Applying the condition $\Delta_2$ for $t \in B\setminus A$, we obtain

$$KM(t, u_{n,t})+h(t)-2^n M(t, u_{n,t}) \geqslant b_t.$$

Hence, we get

$$M(t, u_{n,t})(1-K2^{-n}) \leqslant 2^{-n}\big(h(t)-b_t\big)$$

and

$$0 \leqslant \limsup_{n\to\infty} M(t, u_{n,t}) \leqslant \lim_{n\to\infty} 2^{-n}\big(h(t)-b_t\big) = 0.$$

Moreover, we have

$$0 \leqslant \liminf_{n\to\infty} M(t, u_{n,t}) \leqslant \limsup_{n\to\infty} M(t, u_{n,t}) = 0.$$

Thus

$$\lim_{n\to\infty} M(t, u_{n,t}) = M(t, \lim_{n\to\infty} u_{n,t}) = 0, \quad \forall t \in B. \tag{2}$$

Hence it follows that $\lim\limits_{n\to\infty} u_{n,t} = 0$ and $\lim\limits_{n\to\infty} M(t, 2u_{n,t}) = 0$, which contradicts the condition (1). Thus, $\lim\limits_{n\to\infty} h_n(t) = 0$ for every $t \in T\setminus A$. It follows by the condition $\Delta_2$ that there exists a natural number $n_0$ such that $\int_T h_{n_0}(t)\,d\mu < \infty$. Applying the Lebesgue theorem on bounded convergence, we obtain

$$\lim_{n\to\infty} \int_T h_n(t)\,d\mu = 0.$$

Thus, for every $\varepsilon > 0$ there exists $n_1 \in N$ such that $\int_T h_{n_1}(t)\,d\mu \leqslant \varepsilon$. It suffices to put $K_\varepsilon = 2^{n_1}$, $h_\varepsilon = h_{n_1}$, and the proof is completed.

1.7. Lemma. *Let $f$ be a convex function on $R$ with values in $R_+$ and let there be numbers $-\infty < a < b < \infty$, $0 < \lambda_0 < 1$, such that*

$$f\big(\lambda_0 a+(1-\lambda_0)b\big) = \lambda_0 f(a)+(1-\lambda_0)f(b). \tag{3}$$

*Then for every $\lambda \in [0, 1]$ there holds*

$$f\big(\lambda a+(1-\lambda)b\big) = \lambda f(a)+(1-\lambda)f(b).$$

Proof. Write $x_0 = \lambda_0 a+(1-\lambda_0)b$ and assume, to the contrary, that there exists $\lambda_1 \in (0, 1)$ such that $\lambda_1 \neq \lambda_0$ and

$$f\big(\lambda_1 a+(1-\lambda_1)b\big) < \lambda_1 f(a)+(1-\lambda_1)f(b). \tag{4}$$

Writing $x_1 = \lambda_1 a+(1-\lambda_1)b$, we shall consider two cases:

(i) $a < x_1 < x_0 < b$. There exists a number $\alpha \in (0, 1)$ such that

$$x_0 = \alpha x_1+(1-\alpha)b = \alpha\lambda_1 a+(1-\alpha\lambda_1)b.$$

Thus, we have $a\lambda_1 = \lambda_0$, and hence

$$f(x_0) \leqslant af(x_1)+(1-a)f(b) < a\big(\lambda_1 f(a)+(1-\lambda_1)f(b)\big)+(1-a)f(b)$$
$$= \lambda_0 f(a)+(1-\lambda_0)f(b) = f(x_0),$$

a contradiction.

(ii) $a < x_0 < x_1 < b$. There exists a number $\beta \in (0, 1)$ such that

$$x_0 = \beta a+(1-\beta)x_1.$$

In the same way as in the first case we obtain again a contradiction. Thus our lemma is proved.

1.8. LEMMA. *If the assumptions of Lemma* 1.7 *are satisfied, then*

$$f\big(\lambda c+(1-\lambda)d\big) = \lambda f(c)+(1-\lambda)f(d)$$

*for every* $0 \leqslant \lambda \leqslant 1$, $a \leqslant c \leqslant d \leqslant b$.

Proof. We may assume that $a < c < d < b$. There exist numbers $\lambda_1 \in (0, 1)$ and $\lambda_2 \in (0, 1)$ such that

$$c = \lambda_1 a+(1-\lambda_1)b, \quad d = \lambda_2 a+(1-\lambda_2)b.$$

Hence, we get for $\lambda \in [0, 1]$

$$\lambda c+(1-\lambda)d = (\lambda\lambda_1+\lambda_2-\lambda\lambda_2)a+[1-(\lambda\lambda_1+\lambda_2-\lambda\lambda_2)]b.$$

We have, by Lemma 1.7,

$$f\big(\lambda c+(1-\lambda)d\big) = (\lambda\lambda_1+\lambda_2-\lambda\lambda_2)f(a)+[1-(\lambda\lambda_1+\lambda_2-\lambda\lambda_2)]f(b). \tag{5}$$

Moreover, we have also

$$\lambda f(c) = \lambda\lambda_1 f(a)+\lambda f(b)-\lambda\lambda_1 f(b),\ (1-\lambda)f(d) = (1-\lambda)[\lambda_2 f(a)+(1-\lambda_2)f(b)].$$

Combining both last equalities and equality (5), we obtain the desired result.

1.9. COROLLARY. *If* $f\colon R\to R_+$ *is convex and* $f$ *is strictly convex on* $[a, b]$, *then for every* $u \in [a, b)$ *and* $v > a$, $u \neq v$ *or* $u \in (a, b]$ *and* $v < b$, $u \neq v$, *and for every* $0 < \lambda < 1$ *there holds*

$$f\big(\lambda u+(1-\lambda)v\big) < \lambda f(u)+(1-\lambda)f(v). \tag{6}$$

Proof. We may assume, by assumptions, that $v \notin [a, b]$. Let $u \in [a, b)$, $v > b$, and let us assume for a contrary that

$$f\big(\lambda_0 u+(1-\lambda_0)v\big) = \lambda_0 f(u)+(1-\lambda_0)f(v)$$

for some $0 < \lambda_0 < 1$. Then, by Lemma 1.8, we have

$$f\big(\lambda c+(1-\lambda)d\big) = \lambda f(c)+(1-\lambda)f(d)$$

for every $\lambda \in [0, 1]$, $u \leqslant c \leqslant d \leqslant v$, a contradiction with strict convexity of $f$ on $[u, b]$. In the same way we can prove that (6) holds for every $\lambda \in (0, 1)$, $u \in (a, b]$, $v < a$.

1.10. THEOREM. *If $X$ and $Y$ are uniformly convex, $M$ and $N$ are $\varphi$-functions with parameter from $T$ and from $S$, respectively, satisfying the condition $\Delta_2$ and uniformly convex, then $I$ is uniformly convex.*

Proof. Let $I(x(\cdot)) = I(y(\cdot)) = 1$, $I(x(\cdot)-y(\cdot)) \geqslant \varepsilon$. Without loss of generality we may assume that $T_2 \cup T_3 = \emptyset$ and $S_2 \cup S_3 = \emptyset$, where the sets $T_2$, $T_3$ and $S_2$, $S_3$ are from definition of strict convexity and from the condition $\Delta_2$ for the functions $M$ and $N$, respectively. We may assume also that $\varepsilon \leqslant 1$. Let $a = \varepsilon/4$ and

$$E = \{t \in T\colon \|x_1(t)-y_1(t)\| \geqslant a\max(\|x_1(t)\|, \|y_1(t)\|)\},$$
$$F = \{s \in S\colon |||x_2(s)-y_2(s)||| \geqslant a\max(|||x_2(s)|||, |||y_2(s)|||)\}.$$

If $t \in T\setminus E$, $s \in S\setminus F$, then, by Lemma 1.2, we have

$$M\big(t, 2^{-1}(\|x_1(t)\|+\|y_1(t)\|)\big) \leqslant (1-p_1(a))2^{-1}\big[M(t, \|x_1(t)\|)+M(t, \|y_1(t)\|)\big],$$
$$N\big(s, 2^{-1}(|||x_2(s)|||+|||y_2(s)|||)\big) \leqslant (1-p_2(a))2^{-1}\big[N(s, |||x_2(s)|||)+N(s, |||y_2(s)|||)\big].$$

Hence

$$\begin{aligned} I-I\big((x(\cdot)+y(\cdot))/2\big) &\geqslant \{I_M(x_1(\cdot)\chi_E(\cdot))+I_M(y_1(\cdot)\chi_E(\cdot))\}/2 - \\ &\quad - I_M\big[\big((x_1(\cdot)+y_1(\cdot))/2\big)\chi_E(\cdot)\big]+\{I_N(x_2(\cdot)\chi_F(\cdot))+I_N(y_2(\cdot)\chi_F(\cdot))\}/2 - \\ &\quad - I_N\big[\big((x_2(\cdot)+y_2(\cdot))/2\big)\chi_F(\cdot)\big] \\ &\geqslant p_1(a)\{I_M(x_1(\cdot)\chi_E(\cdot))+I_M(y_1(\cdot)\chi_E(\cdot))\}/2 + \\ &\quad + p_2(a)\{I_N(x_2(\cdot)\chi_F(\cdot))+I_N(y_2(\cdot)\chi_F(\cdot))\}/2. \end{aligned}$$

If $t \in T\setminus E$ and $s \in S\setminus F$, then $\|x_1(t)-y_1(t)\| \leqslant a(\|x_1(t)\|+\|y_1(t)\|)$ and $|||x_2(s)-y_2(s)||| \leqslant a(|||x_2(s)|||+|||y_2(s)|||)$, and hence

$$\begin{aligned} M(t, \|x_1(t)-y_1(t)\|) &\leqslant M\big(t, 2a(\|x_1(t)\|+\|y_1(t)\|)/2\big) \\ &\leqslant a\big[M(t, \|x_1(t)\|)+M(t, \|y_1(t)\|)\big], \end{aligned}$$
$$N(s, |||x_2(s)-y_2(s)|||) \leqslant a\big[N(s, |||x_2(s)|||)+N(s, |||y_2(s)|||)\big].$$

Thus

$$(7)\qquad I_M\big[(x_1(\cdot)-y_1(\cdot))\chi_{T\setminus E}(\cdot)\big]+I_N\big[(x_2(\cdot)-y_2(\cdot))\chi_{S\setminus F}(\cdot)\big] \leqslant 2a = \varepsilon/2.$$

But $I(x(\cdot)-y(\cdot)) \geqslant \varepsilon$ by hypothesis, so

$$(8)\qquad I_M\big[(x_1(\cdot)-y_1(\cdot))\chi_E(\cdot)\big]+I_N\big[(x_2(\cdot)-y_2(\cdot))\chi_F(\cdot)\big] \geqslant \varepsilon/2.$$

We have

$$\begin{aligned} (9)\qquad & I_M\big[(x_1(\cdot)-y_1(\cdot))\chi_E(\cdot)\big]+I_N\big[(x_2(\cdot)-y_2(\cdot))\chi_F(\cdot)\big] \\ &\leqslant 2^{-1}\big[I_M(2x_1(\cdot)\chi_E(\cdot))+I_M(2y_1(\cdot)\chi_E(\cdot))+I_N(2x_2(\cdot)\chi_F(\cdot))+I_N(2y_2(\cdot)\chi_F(\cdot))\big]. \end{aligned}$$

Applying Lemma 1.6 with $\varepsilon/8$ instead of $\varepsilon$, we obtain for $t \in E$, $s \in F$

$$M\big(t, 2\|x_1(t)\|\big) + M\big(t, 2\|y_1(t)\|\big) \leqslant K_1\big[M\big(t, \|x_1(t)\|\big) + M\big(t, \|y_1(t)\|\big)\big] + 2h_1(t),$$

$$N\big(s, 2|||x_2(s)|||\big) + N\big(s, 2|||y_2(s)|||\big) \leqslant K_2\big[N\big(s, |||x_2(s)|||\big) + N\big(s, |||y_2(s)|||\big)\big] + 2h_2(s),$$

and further

$$I_M\big(2x_1(\cdot)\chi_E(\cdot)\big) + I_M\big(2y_1(\cdot)\chi_E(\cdot)\big) + I_N\big(2x_2(\cdot)\chi_F(\cdot)\big) + I_N\big(2y_2(\cdot)\chi_F(\cdot)\big)$$
$$\leqslant K\big[I_M\big(x_1(\cdot)\chi_E(\cdot)\big) + I_M\big(y_1(\cdot)\chi_E(\cdot)\big) + I_N\big(x_2(\cdot)\chi_F(\cdot)\big) + I_N\big(y_2(\cdot)\chi_F(\cdot)\big)\big] + \varepsilon/2.$$

Hence and from (9), we obtain

$$(10)\quad I_M\big[\big(x_1(\cdot) - y_1(\cdot)\big)\chi_E(\cdot)\big] + I_N\big[\big(x_2(\cdot) - y_2(\cdot)\big)\chi_F(\cdot)\big]$$
$$\leqslant 2^{-1}K\big[I_M\big(x_1(\cdot)\chi_E(\cdot)\big) + I_M\big(y_1(\cdot)\chi_E(\cdot)\big) + I_N\big(x_2(\cdot)\chi_F(\cdot)\big) + I_N\big(y_2(\cdot)\chi_F(\cdot)\big)\big] + \varepsilon/4.$$

Hence and by (8), we get

$$(11)\quad I_M\big(x_1(\cdot)\chi_E(\cdot)\big) + I_M\big(y_1(\cdot)\chi_E(\cdot)\big) + I_N\big(x_2(\cdot)\chi_F(\cdot)\big) + I_N\big(y_2(\cdot)\chi_F(\cdot)\big)$$
$$\geqslant \frac{2}{K}\Big(I_M\big(x_1(\cdot) - y_1(\cdot)\big)\chi_E(\cdot)\big) + I_N\Big(\big(x_2(\cdot) - y_2(\cdot)\big)\chi_F(\cdot)\Big) - \varepsilon/2K \geqslant \varepsilon/2K.$$

So, taking into account (6) an (11), we obtain

$$1 - I\Big(\big(x(\cdot) + y(\cdot)\big)/2\Big) \geqslant \varepsilon \min\big(p_1(a), p_2(a)\big)/4K = q(\varepsilon),$$

which is the desired result.

1.11. Theorem. *If $X$ and $Y$ are uniformly convex, $M$ and $N$ are uniformly convex $\varphi$-functions with parameter from $T$ and from $S$, respectively, and both satisfy the condition $\Delta_2$, then the space $(L, \|\cdot\|_I)$ is uniformly convex.*

Proof. Let $\varepsilon > 0$, $\|x(\cdot)\|_I = \|y(\cdot)\|_I = 1$ and $\|x(\cdot) - y(\cdot)\|_I \geqslant \varepsilon$. Then, by Lemma 1.3, $I\big(x(\cdot)\big) = I\big(y(\cdot)\big) = 1$ and, by Lemma 1.4, $I\big(x(\cdot) - y(\cdot)\big) \geqslant \varepsilon_1(\varepsilon)$. Hence, by Theorem 1.10, $I\Big(\big(x(\cdot) + y(\cdot)\big)/2\Big) \leqslant 1 - q(\varepsilon_1)$ and so $\big\|\big(x(\cdot) + y(\cdot)\big)/2\big\|_I \leqslant 1 - \eta(q)$, by Lemma 1.5. Putting $\delta(\varepsilon) = \eta\Big(q\big(\varepsilon_1(\varepsilon)\big)\Big)$, we obtain the desired result.

1.12. Theorem. *If $(L, \|\cdot\|_I)$ is uniformly convex, then $M$ and $N$ satisfy the condition $\Delta_2$ and the spaces $X = (X, \|\cdot\|)$, $Y = (Y, |||\cdot|||)$ are uniformly convex* (2).

Proof. If $M$ or $N$ do not satisfy the condition $\Delta_2$, then $(L, \|\cdot\|_I)$ is not strictly convex, see [3]. Let $M$ and $N$ satisfy the condition $\Delta_2$ and let $X$ be not uniformly convex. First, we shall show that there exists a function $0 \neq x(\cdot) \in L_M^\mu$. Let $A = \{t \in T\colon M(t, 1) < \infty\}$. We have $\mu(A) = \mu(T)$.

(2) These theorems are also true for finite Cartesian product of Orlicz's spaces.

Let $B \subset A$, $B \in \Sigma$, be such that $0 < \mu(B) < \infty$. We define

$$B_n = \{t \in B: M(t, 1) \leqslant n\}, \quad n = 1, 2, \ldots$$

We have $B_n \subset B_{n+1}$ for each $n \in N$. Thus, $0 < \lim_{n\to\infty} \mu(B_n) = \mu(B)$. It suffices to put $x(t) = \chi_{B_{n_0}}(t)$ for sufficiently large $n_0$. Further, taking into account the condition $\Delta_2$ for $M$ and choosing a sequence $\{a_k\}_{k=1}^{\infty}$ with $a_k \geqslant 1$, $a_k \to \infty$ as $k \to \infty$, we get (³)

$$\infty > \int_{B_{n_0}} M(t, a_k)\, d\mu \geqslant a_k \int_{B_{n_0}} M(t, 1)\, d\mu \to \infty \quad \text{as} \quad k \to \infty.$$

Thus, there exists a number $a_{k_0}$ such that $I_M\big(a_{k_0}\chi_{B_{n_0}}(\cdot)\big) \geqslant 1$. Next, there exists a set $C \subset B_{n_0}$, $C \in \Sigma$ such that $I_M\big(a_{k_0}\chi_C(\cdot)\big) = 1$.

There exists $\varepsilon > 0$ such that for every $\delta > 0$ there are $x, y \in X$ such that $\|x\| = \|y\| = 1$, $\|x-y\| \geqslant \varepsilon$ and $\|(x+y)/2\| > 1-\delta$. Putting $x_1(t) = xa_{k_0}\chi_C(t)$, $y_1(t) = ya_{k_0}\chi_C(t)$, $x_2(s) = y_2(s) \equiv 0$, $x(\cdot) = \big(x_1(\cdot), x_2(\cdot)\big)$, $y(\cdot) = \big(y_1(\cdot), y_2(\cdot)\big)$, we have $x(\cdot) \neq y(\cdot)$ and

$$\big\|\big(x_1(t) - y_1(t)\big)/\varepsilon\big\| \geqslant a_{k_0}\chi_C(t), \qquad \big\|\big(x_1(t) + y_1(t)\big)/2(1-\delta)\big\| > a_{k_0}\chi_C(t).$$

Hence, we have $I\Big(\big(x(\cdot) + y(\cdot)\big)/\varepsilon\Big) \geqslant 1$, $I\Big(\big(x(\cdot) + y(\cdot)\big)/2(1-\delta)\Big) > 1$. So $\big\|\big(x(\cdot) - y(\cdot)\big)/\varepsilon\big\|_I \geqslant 1$ and $\big\|\big(x(\cdot) + y(\cdot)\big)/2(1-\delta)\big\|_I > 1$, by Lemma 1.3. The proof is completed.

**2. Examples and corollaries.** First, we shall prove the following lemma:

2.1. Lemma. *Let $M_1$ and $M_2$ be $\varphi$-functions with parameter and let at least one of them be uniformly (strictly) convex. Then the functions $f = M_1 \circ M_2$ and $g = M_1 \cdot M_2$ are uniformly (strictly) convex $\varphi$-functions with parameter.*

Proof. It suffices to prove the uniform (strict) convexity of $f$ and $g$. We prove only the uniform convexity of $f$ and $g$.

(i) *Uniform convexity of $f$.* First, let $M_1$ be uniformly convex with modulus of convexity $\delta_1(a)$. Let $0 < a < 1$, $0 \leqslant b \leqslant a$, $u > 0$. We have for $\mu$-a.e. $t \in T$, by $M_2(t, bu)/M_2(t, u) \leqslant b$,

$$\begin{aligned} f\big(t, (u+bu)/2\big) &= M_1\big[t, M_2\big(t, (u+bu)/2\big)\big] \leqslant M_1\big[t, \big(M_2(t, u) + M_2(t, bu)\big)/2\big] \\ &= M_1\left[t, \frac{M_2(t,u) + \big(M_2(t,bu)/M_2(t,u)\big)M_2(t,u)}{2}\right] \\ &\leqslant \big(1-\delta_1(a)\big)\frac{M_1[t, M_2(t,u)] + M_1[t, M_2(t, bu)]}{2} \\ &= \big(1-\delta_1(a)\big)\frac{f(t,u) + f(t, bu)}{2}. \end{aligned}$$

(³) Since, by [3] Theorem 1.3, $M(t, u) = 0$ iff $u = 0$ for $\mu$-a.e. $t \in T$.

Now, let $M_2$ be uniformly convex with modulus of convexity $\delta_2(a)$; then, by the inequality $M_1(t, au) \leqslant aM_1(t, u)$ for $0 \leqslant a \leqslant 1$, $u \geqslant 0$, we have for $\mu$-a.e. $t \in T$

$$f\big(t, (u+bu)/2\big) = M_1\big[t, M_2\big(t, (u+bu)/2\big)\big]$$
$$\leqslant M_1\left[t, \big(1-\delta_2(a)\big) \frac{M_2(t, u)+M_2(t, bu)}{2}\right] \leqslant \big(1-\delta_2(a)\big)\frac{f(t, u)+f(t, bu)}{2}.$$

(ii) *Uniform convexity of $g$.* Let $M_1$ be uniformly convex. $M_1(t, u)$ and $M_2(t, u)$ are increasing functions of $u$ for every $t \in T \setminus T_0$, where $\mu(T_0) = 0$, so we have for $a$, $b$, $u$ and $t$ as in the case (i)

$$[M_1(t, u)-M_1(t, bu)]\cdot[M_2(t, bu)-M_2(t, u)] \leqslant 0.$$

This inequality is equivalent to

$$M_1(t, bu)M_2(t, u)+M_1(t, u)M_2(t, bu)$$
$$\leqslant M_1(t, u)M_2(t, u)+M_1(t, bu)M_2(t, bu).$$

Thus, we have

$$g\big(t, (u+bu)/2\big) = M_1\big(t, (u+bu)/2\big)M_2\big(t, (u+bu)/2\big)$$
$$\leqslant \tfrac{1}{4}\big(1-\delta_1(a)\big)\{M_1(t, u)M_2(t, u)+M_1(t, bu)M_2(t, bu)+M_1(t, u)M_2(t, u)+$$
$$+M_1(t, bu)M_2(t, bu)\}$$
$$= \big(1-\delta_1(a)\big)\{g(t, u)+g(t, bu)\}/2.$$

2.2. COROLLARY. *Each uniformly convex $\varphi$-function $M$ with parameter is strictly convex.*

This follows immediately from definitions.

2.3. EXAMPLE. There exists a strictly convex $\varphi$-function $M$, which is not uniformly convex. Let $M(u) = u\ln(1+u)$. We have $M'(u) = \ln(1+u)+u/(1+u)$. Since $M'(u)$ is strictly increasing function on $[0, \infty)$, $M$ is strictly convex. Now, we shall prove that $M$ is not uniformly convex. Uniform convexity of $M$ is equivalent to the condition:

$$(2.1) \qquad 0 < \sup_{0\leqslant b\leqslant a}\big[\sup_{u>0} 2M\big((u+bu)/2\big)/\big(M(u)+M(bu)\big)\big] = \eta(a) < 1$$

for every $0 < a < 1$. Let us write

$$f(b, u) = 2M\big((1+b)u/2\big)/\big(M(u)+M(bu)\big).$$

Applying the L'Hospital formula, we obtain for $0 \leqslant b \leqslant a$, $\lim_{u\to\infty} f(b, u) = 1$. So, condition (2.1) holds for no number $0 < \eta(a) < 1$, and hence $M$ is not uniformly convex.

2.4. COROLLARY. *If $X_i$, $i = 1, 2, \ldots, n$, are uniformly (strictly) convex Banach spaces and $M_i$, $i = 1, 2, \ldots, n$, are uniformly (strictly) convex*

*$\varphi$-functions without parameter, satisfying the condition $\Delta_2$ for large $u \geqslant 0$, then the space $X = X_1 \times \ldots \times X_n$ with norm*

$$\|x\|_{\varrho_0} = \inf\{u > 0 \colon \varrho_0(x/u) \leqslant 1\},$$

*for $x = \{x_i\}_{i=1}^n$, $x_i \in X_i$, where*

$$\varrho_0(x) = \sum_{i=1}^{n} M_i(\|x_i\|_i),$$

*($x_i$ are components of $x$ and $\|\cdot\|_i$ are norms in $X_i$), is uniformly (strictly) convex.*

Proof. Let $\mu_1 = \mu_2 = \ldots = \mu_n =$ Lebesgue measure in $[0, 1]$, $T_1 = T_2 = \ldots = T_n = [0, 1]$. Let $F$ denote the space of all functions from $[0, 1]$ into $X$ of the form $x(\cdot) = \big(x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot)\big)$, where $x_i(\cdot) = x_i\chi_{[0,1]}(\cdot)$, $x_i \in X_i$, for $i = 1, 2, \ldots, n$. $F$ is a subspace of $L = L_{M_1}^{\mu_1}(T_1, X_1) \times \ldots \times L_{M_n}^{\mu_n}(T_n, X_n)$. We define the operator $A \colon X \to F$ by the formula

$$(Ax)(t) = \big(x_1\chi_{[0,1]}(\cdot), \ldots, x_n\chi_{[0,1]}(\cdot)\big),$$

where $x_i$ are components of $x$. We have $I\big((Ax)(\cdot)\big) = \varrho_0(x)$ and thus $\|(Ax)(\cdot)\|_I = \|x\|_{\varrho_0}$ for $x \in X$. Since $F$ is uniformly (strictly) convex (see Theorem 1.10 of this paper and [3]), so $X$ is uniformly (strictly) convex.

2.5. *If $M$ is a uniformly convex $\varphi$-function with parameter satisfying the condition $\Delta_2$, then for every non-negative integer $k$, the Orlicz–Sobolev space* (for definition see [1]) *$W_M^k(\Omega)$ with norm*

$$\|x(\cdot)\|_{W_M^k} = \inf\big\{u > 0 \colon \varrho_1\big(x(\cdot)/u\big) \leqslant 1\big\},$$

*where*

$$\varrho_1\big(x(\cdot)\big) = \sum_{|\alpha| \leqslant k} \int_{\Omega} M\big(t, |D^\alpha x(t)|\big)\,dt,$$

*is uniformly convex. Here $\Omega$ is an open set in $R^n$ and $D^\alpha x(\cdot)$ denotes distributional derivatives of $x(\cdot)$.*

This may be deduced in the same way as strict convexity of $W_M^k(\Omega)$ in [3].

2.6. Examples. Let us consider the following functions:

1. $M_1(u) = u^p$, $1 < p < \infty$,
2. $M_2(u) = u^p \ln^p(1+u)$, $1 < p < \infty$,
3. $M_3(u) = u^p \ln(1+u^p)$, $1 < p < \infty$,
4. $M_4(u) = u^{p+1} \ln(1+u)$, $1 < p < \infty$,
5. $M_5(u) = \begin{cases} u^p & \text{for } 0 \leqslant u \leqslant 1, \\ u^q & \text{for } u > 1, \end{cases}$ $\quad 1 < p \leqslant q < \infty$,

6. $M_6(t, u) = [M(u)]^{p(t)}$,

7. $M_7(t, u) = M(p(t)u)$.

Functions $M_i$, $i = 1, 2, \ldots, 5$, are uniformly convex. This follows from Lemma 2.1. In the case of the function $M_1$ we have $\delta(a) = 2^{1-p}(1+a)^p/(1+a^p)$. All functions $M_i, i = 1, \ldots, 5$, satisfy the condition $\Delta_2$ for all $u \geqslant 0$. If $M$ is a uniformly convex $\varphi$-function without parameter and $1 \leqslant p(t) < \infty$ is a $\mu$-measurable function on $T$, then $M_6$ and $M_7$(4) are uniformly convex $\varphi$-functions with parameter. Moreover, if additionally $M$ satisfies the condition $\Delta_2$ for all $u \geqslant 0$ if $\mu(T) = \infty$ and for large $u \geqslant 0$ if $\mu(T) < \infty$, and $1 \leqslant p(t) \leqslant K_1 < \infty$, then $M_6$ and $M_7$ satisfy the condition $\Delta_2$(5). If $1 < K_2 \leqslant p(t) < \infty$, then $M_6$ is a uniformly convex $\varphi$-function with parameter for each $\varphi$-function $M$ without parameter. H. Nakano [11] considered Orlicz spaces generated by $\varphi$-function $M_6$ with $M(u) = u$ and proved that then $L^{\mu}_{M_6}(T, C)$ is uniformly convex if $1 < K_2 \leqslant p(t) \leqslant K_1 < \infty$. W. A. Luxemburg proved uniform convexity of $L^{\mu}_{M}(T, C)$ for uniformly convex $\varphi$-function $M$ without parameter satisfying the condition $\Delta_2$ for every $u \geqslant 0$.

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(4) $M_7$ is uniformly convex under weaker assumption $0 < p(t) < \infty$.

(5) $M_7$ satisfies the condition $\Delta_2$ under weaker assumption $0 < p(t) < \infty$.

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