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# Linear operators in modular spaces

**Abstract.** The problem of modular convergence in a modular space and of modular continuity and boundedness of linear operators between such spaces is studied. Adjoint linear operators are also investigated.

**§ 1. Modular convergence.** Let $X$ be a real vector space. A *pseudomodular* in $X$ is a functional $\varrho: X \to [0, +\infty]$ such that $\varrho(0) = 0$, $\varrho(-x) = \varrho(x)$ and $\varrho(ax+\beta y) \leqslant \varrho(x)+\varrho(y)$ for $x, y \in X$, $a, \beta \geqslant 0$, $a+\beta = 1$. If, moreover, $\varrho(ax+\beta y) \leqslant a\varrho(x)+\beta\varrho(y)$ for such $x, y, a, \beta$, then $\varrho$ is called a *convex pseudomodular*. If $\varrho(x) = 0$ implies $x = 0$, then $\varrho$ is called a *modular*. If $\varrho(ax) = 0$ for all $a > 0$ implies $x = 0$, we shall say that $\varrho$ is a *semimodular*. If a pseudomodular $\varrho$ satisfies the condition $\lim_{a\to 1-}\varrho(ax) = \varrho(x)$ for all $x \in X$, it is called *left-continuous* or *normal* (for this terminology, see [2], pp. 5 and 9, or [3], pp. 661–663, [5], pp. 439–440). The vector subspace $X_\varrho = \{x \in X: \varrho(ax) \to 0 \text{ as } a \to 0+\}$ of $X$ is called a *modular space*; if $\varrho$ is a pseudomodular (a semimodular), then $|x|_\varrho = \inf\{u > 0: \ \varrho(x/u) \leqslant u\}$ is an *F-seminorm* (an *F-norm*) in $X_\varrho$, and if it is a convex pseudomodular (a convex semimodular), then $\|x\|_\varrho = \inf\{u > 0: \varrho(x/u) \leqslant 1\}$ is a *seminorm* (a *norm*) in $X_\varrho$, equivalent to $|\ |_\varrho$. Moreover, if $|x|_\varrho < 1$ then $\varrho(x) \leqslant |x|_\varrho$ and if $\|x\|_\varrho < 1$, then $\varrho(x) \leqslant \|x\|_\varrho$. If $\varrho$ is a left-continuous convex pseudomodular, then the conditions $\varrho(x) \leqslant 1$ and $\|x\|_\varrho \leqslant 1$ are equivalent for all $x \in X_\varrho$ (see [2], pp. 6 and 10, or [3], p. 52, [4], p. 662, and [1], p. 235).

Let $(x_n)$ be a sequence of elements of the modular space $X_\varrho$; $(x_n)$ is convergent in the norm $|\ |_\varrho$ (or $\|\ \|_\varrho$) to an element $x \in X_\varrho$ if and only if $\varrho\big(a(x_n-x)\big) \to 0$ as $n \to \infty$ for every $a > 0$ (see [2], p. 7, or [4], p. 662). There is also a notion of modular convergence (briefly $\varrho$-convergence): $(x_n)$ is called *modular convergent* (*$\varrho$-convergent*) to $x$ if there exists an $a > 0$ such that $\varrho\big(a(x_n-x)\big) \to 0$ as $n \to \infty$; we write it $x_n \xrightarrow{\varrho} x$ (see [3], pp. 50–53). Obviously, convergence in norm in $X_\varrho$ implies $\varrho$-convergence to the same limit. The converse implication does not hold in general, as may be shown by easy examples of Orlicz spaces. Namely, let $\varphi$ be a $\varphi$-function (i.e. $\varphi(0) = 0, \varphi(u) > 0$ for $u > 0$, $\varphi(u)$ is non-decreasing and continuous,

$\varphi(u)\to\infty$ as $u\to\infty$), and let $L^{\varphi}(0,1)$ be the respective Orlicz space, that is, the modular space $X_\varrho$ with the modular $\varrho(x)=\int_0^1\varphi(|x(t)|)\,dt$. Then $\varrho(x)<\infty$ implies $\varrho(2x)<\infty$ for all $x\in X_\varrho$ if and only if $\varphi$ satisfies the following condition $(\Delta_2)$ for large $u$: *there exist positive constants $k$ and $u_0$ such that $\varphi(2u)\leqslant k\varphi(u)$ for all $u>u_0$* (see e.g. [2], pp. 13–14). If $(\Delta_2)$ does not hold, then we may choose a function $x(t)\geqslant 0$ such that $\varrho(x)<\infty$ and $\varrho(2x)=\infty$. Taking $x_n(t)=x(t)$ if $x(t)\leqslant n$ and $x_n(t)=0$ if $x(t)>n$, one may easily check that $\varrho(x_n-x)\to 0$, but $\varrho(2(x_n-x))=\infty$ for all $n$, and so $x_n\xrightarrow{\varrho}x$, but $(x_n)$ does not converge to $x$ in norm. One may show more, namely

PROPOSITION 1.1. *Let $X_\varrho=L^{\varphi}(0,1)$ be an Orlicz space with $\varphi$ not satisfying $(\Delta_2)$ for large $u$. Then there exists a sequence $(x_n)$ of elements of $X_\varrho$ which is $\varrho$-convergent, but contains no subsequence convergent in the norm of $X_\varrho$.*

This follows immediately since in the converse case, taking $x_n\xrightarrow{\varrho}x$ in $L^{\varphi}(0,1)$ and an arbitrary subsequence $(x_{n_k})$ of $(x_n)$, one could extract a norm-convergent sequence from $(x_{n_k})$, and this would imply $x_n\to x$ in the norm of $L^{\varphi}(0,1)$.

Now, let $\varrho$ be any pseudomodular.

DEFINITION 1.2. A set $A\subset X_\varrho$ is called *$\varrho$-bounded* if for any sequence of elements $x_n\in A$ and any sequence of numbers $\varepsilon_n\to 0$, there holds $\varepsilon_n x_n\xrightarrow{\varrho}0$.

We shall examine the connections between the following conditions for a set $A\subset X_\varrho$:

(a) *$A$ is $\varrho$-bounded,*

(b) *there exist positive constants $M$ and $k$ such that $\varrho(kx)\leqslant M$ for all $x\in A$,*

(c) *there exists a positive constant $k$ such that $\varrho(kx)\leqslant 1$ for all $x\in A$,*

(d) *there is a positive number $K$ such that $\|x\|_\varrho\leqslant K$ for all $x\in A$,*

(e) *for every sequence of elements $x_n\in A$ and any sequence of numbers $\varepsilon_n\to 0$, there holds $\varrho(\varepsilon_n x_n)\to 0$.*

PROPOSITION 1.3. *If $\varrho$ is convex, then all conditions* (a)–(e) *are pairwise equivalent. In the case of general pseudomodular $\varrho$, there hold the implications* (e) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b). Suppose that (a) is satisfied, but (b) does not hold; then there exists a sequence of elements $x_n\in A$ such that $\varrho(x_n n^{-2})>1$ for $n=1,2,\ldots$ Taking $a>0$ in such a manner that $\varrho(ax_n n^{-1})\to 0$ as $n\to\infty$ and $an>1$, we get

$$1<\varrho(x_n n^{-2})=\varrho\left(\frac{1}{an}\,ax_n n^{-1}\right)\leqslant\varrho(ax_n n^{-1})\to 0,$$

a contradiction. The implication (e) $\Rightarrow$ (a) and the equivalence (b) $\Leftrightarrow$ (c) are obvious. Now, let $\varrho$ be convex. Let us suppose (c) and let $\varepsilon_n \to 0$, $x_n \in A$; then $\varrho(\varepsilon_n x_n) \leqslant \frac{\varepsilon_n}{k}\varrho(kx_n) \leqslant \frac{\varepsilon_n}{k} \to 0$ as $n\to\infty$, because $\varepsilon_n/k < 1$ for $n$ sufficiently large. Hence (c) $\Rightarrow$ (e). The equivalence (b) $\Leftrightarrow$ (d) follows immediately from the definition of the pseudonorm $\|\ \|_\varrho$.

DEFINITION 1.4. A set $A \subset X_\varrho$ will be called *$\varrho$-closed* if $x_n \in A$, $x_n \xrightarrow{\varrho} x$, imply $x \in A$. The smallest $\varrho$-closed set containing the set $A \subset X_\varrho$ will be called the *$\varrho$-closure* of $A$ and denoted by $\overline{A}^\varrho$.

Evidently, the empty set $\emptyset$, the whole space $X_\varrho$ and finite sets $A \subset X_\varrho$ are $\varrho$-closed.

PROPOSITION 1.5. (a) *A set $A \subset X_\varrho$ is $\varrho$-closed if and only if $A = \overline{A}^\varrho$.*

(b) *If $A$ is $\varrho$-closed, then it is closed with respect to the $F$-norm (or norm) in $X$.*

(c) *$A \subset \overline{A} \subset \overline{A}^\varrho$, where $\overline{A}$ means the closure of $A$ with respect to the $F$-norm (or norm) in $X_\varrho$.*

Let us observe that a set $A$ closed in norm does not need to be $\varrho$-closed. Indeed, let $X_\varrho = L^{\varphi}(0,1)$ with $\varphi$ not satisfying $(\Delta_2)$ for large $u$. By 1.1, there exists a sequence $(x_n)$ of elements of $X_\varrho$ such that $x_n \xrightarrow{\varrho} x \in X_\varrho$, but no subsequence $(x_{n_k})$ of $(x_n)$ is convergent in the $F$-norm (or norm) in $X_\varrho$. Taking as $A$ the set of all elements of the sequence $(x_n)$, we easily observe that $A$ is closed in the $F$-norm (or norm) but is not $\varrho$-closed. Consequently, we see also that none of the inclusions $A \subset \overline{A} \subset \overline{A}^\varrho$ needs to be an identity.

PROPOSITION 1.6. *If $\varrho$ is a pseudomodular and a set $A \subset X_\varrho$ is $\varrho$-bounded, then its $\varrho$-closure $\overline{A}^\varrho$ is a sum of countable family of $\varrho$-bounded sets.*

Proof. Let, for every fixed $a > 0$, $\overline{A}^\varrho_a$ denote the set of all $x \in X_\varrho$ for which there exists a sequence $x_n \in A$, $n = 1, 2, \ldots$, such that $\varrho\big(a(x-x_n)\big) \to 0$ as $n\to\infty$. We have $\overline{A}^\varrho = \bigcup_{a>0} \overline{A}^\varrho_a = \bigcup_{n\in N} \overline{A}^\varrho_{1/n}$ and thus it suffices to prove that the sets $\overline{A}^\varrho_a$ are $\varrho$-bounded for every $a > 0$. Let $x_n \in \overline{A}^\varrho_a$ for $n = 1, 2, \ldots$ and let $0 \leqslant \varepsilon_n \to 0$. There exists a sequence $y_n \in A$, $n = 1, 2, \ldots$, such that $\varrho\big(a(x_n - y_n)\big) < 1/n$ for $n = 1, 2, \ldots$ Hence and from the properties of $\varrho$ we get

$$\varrho(a\varepsilon_n x_n) \leqslant \varrho\big(2a\varepsilon_n(x_n - y_n)\big) + \varrho(2a\varepsilon_n y_n) \to 0 \quad \text{as} \quad n\to\infty.$$

DEFINITION 1.7. A set $A \subset X_\varrho$ will be called *relatively $\varrho$-compact* if every sequence of elements $x_n \in A$ contains a subsequence $\varrho$-convergent to an element $x \in X_\varrho$; $A$ will be called *$\varrho$-compact* if every such sequence contains a subsequence $\varrho$-convergent to an $x \in A$.

It is easily observed that a set $A \subset X_\varrho$ compact (relatively compact) with respect to the norm in $X_\varrho$ is also $\varrho$-compact (relatively $\varrho$-compact).

Moreover, a set $A \subset X_\varrho$ is $\varrho$-compact if and only if $A$ is both relatively $\varrho$-compact and $\varrho$-closed.

PROPOSITION 1.8. *A relatively $\varrho$-compact set $A \subset X_\varrho$ is $\varrho$-bounded.*

Proof. Let $x_n \in A$ and $\varepsilon_n \to 0$ and let us write $a_n = \varrho(\varepsilon_n x_n)$. Let $(n_k)$ be any increasing sequence of indices. There exist a number $a > 0$, an element $x \in X_\varrho$ and a subsequence $(n_{k_l})$ of the sequence $(n_k)$ such that $\varrho\big(a(x_{n_{k_l}} - x)\big) \to 0$ as $l \to \infty$. Taking $l$ so large that $2\varepsilon_{n_{k_l}} < a$, we obtain

$$a_{n_{k_l}} = \varrho(\varepsilon_{n_{k_l}} x_{n_{k_l}}) \leqslant \varrho\big(a(x_{n_{k_l}} - x)\big) + \varrho(2\varepsilon_{n_{k_l}} x) \to 0 \quad \text{as} \quad l \to \infty.$$

Thus $a_n \to 0$, and so $A$ is $\varrho$-bounded.

**§ 2. Conjugate spaces to modular spaces.** We give now some remarks on linear continuous functionals over a modular space. If $f$ is a linear functional over a modular space $X_\varrho$, then two notions of continuity of $f$ can be considered: this of continuity in norm and that of modular continuity.

DEFINITION 2.1. A functional $f$ over $X_\varrho$ is called *modular continuous* (or briefly, *$\varrho$-continuous*) if $x_n \xrightarrow{\varrho} x$ implies $f(x_n) \to f(x)$ for any $x \in X_\varrho$.

If $f$ is linear, then obviously $\varrho$-continuity of $f$ is equivalent to $\varrho$-continuity at 0, i.e. to the condition $x_n \xrightarrow{\varrho} 0$ implies $f(x_n) \to 0$. Let $X_\varrho^*$ be the conjugate space to $X_\varrho$ with respect to the norm in $X_\varrho$, and let $X_\varrho^{*\varrho}$ be $\varrho$-conjugate to $X_\varrho$, i.e. $X_\varrho^{*\varrho}$ is the space of $\varrho$-continuous linear functionals over $X_\varrho$. It is evident that $X_\varrho^{*\varrho} \subset X_\varrho^*$. This inclusion may be proper, as shows the example of an Orlicz space $L^\varphi(0, 1)$, where $\varphi$ is an $N$-function (that is, $\varphi$ is convex $\varphi$-function satisfying the conditions $\varphi(u)/u \to 0$ as $u \to 0$, $\varphi(u)/u \to \infty$ as $u \to \infty$) not satisfying the condition $(\Delta_2)$ for large $u$ (see [4], p. 664). In the following, the elements of $X_\varrho^*$ will be denoted by $x^*$, $x_1^*$, etc.

Let us remark that a linear functional $x^*$ over $X_\varrho$ belongs to $X_\varrho^*$ if and only if there exists a constant $K > 0$ such that $|x^*(x)| \leqslant K\big(\varrho(x) + 1\big)$ for every $x \in X_\varrho$, in the case of convex $\varrho$ (this is also true for $s$-convex modulars $\varrho$; see [6], p. 159). The right-hand side of this inequality cannot be changed in general to $K\varrho(x)$ since taking e.g. $X = \boldsymbol{R}$ and $\varrho(x) = x^2$, $x^*$ defined by $x^*(x) = xy$ with arbitrary $y \in \boldsymbol{R}$ belongs to $X_\varrho^*$ but does not satisfy the inequality $|x^*(x)| \leqslant Kx^2$.

H. Nąkano defined in the space $X_\varrho^*$ a conjugate semimodular $\varrho^*$ to $\varrho$ ([5], p. 442) by means of the formula

$$\varrho^*(x^*) = \sup_{x \in X_\varrho} \big(|x^*(x)| - \varrho(x)\big)$$

and has shown that if $\varrho$ is a pseudomodular in $X_\varrho$, then $\varrho^*$ is a convex left-continuous semimodular in $X_\varrho^*$. Hence, supposing $\varrho$ to be convex semimodular, one may define two norms in the conjugate space $X_\varrho^*$. The first one is the norm of a linear continuous functional $x^*$ over a normed space $\langle X_\varrho, \|\ \|_\varrho\rangle$:

$$\|x^*\|_\varrho^* = \sup_{\|x\|_\varrho \leqslant 1} |x^*(x)|,$$

and the second one is the norm defined in $X_\varrho^*$ by means of the modular $\varrho^*$:

$$\|x^*\|_{\varrho^*} = \inf\{u > 0: \varrho^*(x^*/u) \leqslant 1\}.$$

The following inequalities hold.

PROPOSITION 2.2. *Let $\varrho$ be a convex left continuous semimodular. Then*

$$\|x^*\|_{\varrho^*} \leqslant \|x^*\|_\varrho^* \leqslant 2\|x^*\|_{\varrho^*} \quad \textit{for every } x^* \in X_\varrho^*.$$

Proof. We have, by left-continuity of $\varrho^*$, that $\varrho^*(x^*/\|x^*\|_{\varrho^*}) \leqslant 1$ for every $x^* \in X_\varrho^*$. Moreover, by our assumptions the inequalities $\|x\|_\varrho \leqslant 1$ and $\varrho(x) \leqslant 1$ are equivalent. Hence

$$\big\|x^*/\|x^*\|_{\varrho^*}\big\|_\varrho^* = \sup_{\varrho(x)\leqslant 1} |x^*(x)| \leqslant \sup_{\varrho(x)\leqslant 1} \big(\varrho(x) + \varrho^*(x^*/\|x^*\|_{\varrho^*})\big) \leqslant 2$$

and the right-hand side inequality is proved. In order to prove the left-hand side inequality, we shall prove first that for every $x^* \in X_\varrho^*$ with $\|x^*\|_\varrho^* \leqslant 1$, we have

$$(+) \qquad \varrho^*(x^*) = \sup_{\varrho(x)\leqslant 1} \big(|x^*(x)| - \varrho(x)\big).$$

We have

$$\varrho^*(x^*) = \max\Big(\sup_{\varrho(x)\leqslant 1} \big(|x^*(x)| - \varrho(x)\big), \sup_{\varrho(x)>1} \big(|x^*(x)| - \varrho(x)\big)\Big),$$

thus it suffices to prove that $\sup\limits_{\varrho(x)>1} \big(|x^*(x)| - \varrho(x)\big) \leqslant 0$. If $\varrho(x) > 1$, then $\|x\|_\varrho \leqslant \varrho(x)$ and thus

$$\sup_{\varrho(x)>1} \big(|x^*(x)| - \varrho(x)\big) \leqslant \sup_{\varrho(x)>1} \big(\|x^*\|_\varrho^* \|x\|_\varrho - \varrho(x)\big) \leqslant \sup_{\varrho(x)>1} \big(\|x\|_\varrho - \varrho(x)\big) \leqslant 0.$$

Applying equality $(+)$ and the inequality $\varrho(x) \leqslant \|x\|_\varrho$ for $\varrho(x) \leqslant 1$, we get

$$\varrho^*(x^*/\|x^*\|_\varrho^*) = \sup_{\varrho(x)\leqslant 1} \big(|x^*(x)|/\|x^*\|_\varrho^* - \varrho(x)\big) \leqslant \sup_{\varrho(x)\leqslant 1} \big(\|x\|_\varrho - \varrho(x)\big) \leqslant 1,$$

and thus the inequality $\|x^*\|_{\varrho^*} \leqslant \|x^*\|_\varrho^*$ holds for every $x^* \in X_\varrho^*$.

EXAMPLE I. Taking as $X$ the space of all Lebesgue measurable, almost everywhere finite functions in the interval $[0, 1]$ and putting $\varrho(x) = \int_0^1 \varphi(|x(t)|)\,dt$ with an $N$-function $\varphi$ satisfying the condition $(\Delta_2)$ for large $u$, we have $X_\varrho = L^\varphi(0, 1)$ and $x^* \in X_\varrho^*$ are exactly of the form $x^*(x) = \int_0^1 x(t)y(t)\,dt$ with $y \in L^{\varphi^*}(0, 1)$, where $\varphi^*(u) = \sup\limits_{v\geqslant 0} \big(uv - \varphi(v)\big)$ for $u \geqslant 0$

is the function complementary to $\varphi$ in the sense of Young. Moreover, we have then $\varrho^*(x^*) = \int_0^1 \varphi^*(|y(t)|)dt$ and $\|x^*\|_{\varrho^*} = \|x^*\|_\varrho^*$.

**§ 3. Modularly continuous and bounded linear operators.** Let $X$, $Y$ be real vector spaces, $\varrho$ a pseudomodular in $X$, $\sigma$ a pseudomodular in $Y$, and let $X_\varrho$, $Y_\sigma$ be the respective modular spaces. Let $T\colon X_\varrho \to Y_\sigma$ be a linear map of $X_\varrho$ in $Y_\sigma$. Then

DEFINITION 3.1. $T$ will be called $(\varrho, \sigma)$-*continuous* if $x_n \in X_\varrho$, $x_n \xrightarrow{\varrho} x$, imply $Tx_n \xrightarrow{\varrho} Tx$, and $T$ will be called $(\varrho, \sigma)$-*bounded* if it maps $\varrho$-bounded sets in $X_\varrho$ on $\sigma$-bounded sets in $Y_\sigma$. $T$ will be called a $(\varrho, \sigma)$-*contraction* if $\sigma(Tx) \leqslant \varrho(x)$ for all $x \in X_\varrho$, and a *restricted* $(\varrho, \sigma)$-*contraction* if $\sigma(Tx) \leqslant \varrho(x)$ for all $x \in X_\varrho$ satisfying the inequality $\varrho(x) \leqslant 1$.

Obviously, there is in general no inclusion between the set of all $(\varrho, \sigma)$-continuous linear maps of $X_\varrho$ in $Y_\sigma$ and the set of all continuous linear maps of the normed space $\langle X_\varrho, \|\ \|_\varrho\rangle$ in the normed space $\langle Y_\sigma, \|\ \|_\sigma\rangle$. As regards bounded operators, there holds

PROPOSITION 3.2. *If $\varrho$ and $\sigma$ are convex pseudomodulars, then a linear operator $T\colon X_\varrho \to Y_\sigma$ is $(\varrho, \sigma)$-bounded if and only if there holds the following condition:*

(B) *There exist positive constants $k$ and $M$ such that $\sigma(kTx) \leqslant M\|x\|_\varrho$ for all $x \in X_\varrho$ satisfying the inequality $\varrho(x) \leqslant 1$.*

Proof. Let $A \subset X_\varrho$ be $\varrho$-bounded and let (B) hold. Let $x_n \in A$, $\varepsilon_n \to 0$, $0 < \varepsilon_n < 1$; then $\varrho(\sqrt{\varepsilon_n} x_n) \to 0$, by 1.3. Hence $\|\sqrt{\varepsilon_n} x_n\|_\varrho \leqslant 1$ for sufficiently large $n$, and so

$$\sigma(k\varepsilon_n Tx) \leqslant \sqrt{\varepsilon_n}\,\sigma(kT\sqrt{\varepsilon_n} x_n) \leqslant \sqrt{\varepsilon_n} M \|\sqrt{\varepsilon_n} x_n\|_\varrho \leqslant \sqrt{\varepsilon_n} M \to 0$$

as $n \to \infty$. Consequently, $\varepsilon_n T x_n \xrightarrow{\sigma} 0$, and so $T$ is $(\varrho, \sigma)$-bounded. Conversely, let us suppose that $T$ is $(\varrho, \sigma)$-bounded. Since the set $A = \{x/\|x\|_\varrho : x \in X_\varrho\}$ is $\varrho$-bounded, so $T(A)$ is $\sigma$-bounded. By 1.3, there are $k, M > 0$ such that $\sigma(kTx) \leqslant M$ for all $x \in A$. Thus

$$\sigma(kTx) \leqslant \|x\|_\varrho \sigma\big(kT(x/\|x\|_\varrho)\big) \leqslant M\|x\|_\varrho$$

for all $x \in X_\varrho$ such that $\|x\|_\varrho \leqslant 1$. Thus we proved (B).

PROPOSITION 3.3. *If $\varrho$ is a convex left-continuous pseudomodular and $\sigma$ is an arbitrary pseudomodular, and if there holds the condition*

($B_1$) *there exist positive constants $k$ and $M$ such that $\sigma(kTx) \leqslant M\varrho(x)$ for all $x \in X_\varrho$ satisfying the inequality $\varrho(x) \leqslant 1$,*

*then condition* (B) *is also satisfied.*

Let us observe that in general conditions (B) and ($B_1$) are not equivalent. To show this we take as $X_\varrho$ an Orlicz space $L^\varphi(0, 1)$ generated by an $N$-function $\varphi$ not satisfying the condition $(\Delta_2)$ for large $u$, and as $Y_\sigma$ the space of real numbers, $\varrho(x) = \int_0^1 \varphi(|x(t)|)\,dt$, $\sigma(y) = |y|$. Then there exists a linear functional $T$ over $X_\varrho$ which is continuous with respect to the norm $\| \|_\varrho$ (and thus satisfies (B)), but is not $\varrho$-continuous (see § 2), whence it cannot satisfy $(B_1)$.

We shall also deal later with a more restrictive condition than $(B_1)$:

$(B_2)$ *there exist positive constants $k$ and $M$ such that $\sigma(kTx) \leqslant M\varrho(x)$ for all $x \in X_\varrho$.*

Linear operators satisfying $(B_2)$ will be called *strongly $(\varrho, \sigma)$-bounded.*

If $\sigma$ is convex, then the constant $M$ in (B), $(B_1)$ and $(B_2)$ may be taken equal 1.

Let us still remark that if $\langle X, \| \|\rangle$ and $\langle Y, ||| \; |||\rangle$ are normed spaces and $\varrho(x) = \|x\|$, $\sigma(y) = |||y|||$, then every one of the conditions: $(\varrho, \sigma)$-boundedness of $T$, (B), $(B_1)$, $(B_2)$ is equivalent to continuity of $T$ with respect to the norms in $X$ and $Y$. Moreover, from 3.1 and 3.2 it follows in general

PROPOSITION 3.5. *If $\varrho$ and $\sigma$ are left-continuous, convex pseudomodulars, then every linear restricted $(\varrho, \sigma)$-contraction $T$ is a contraction with respect to $\| \|_\varrho$, $\| \|_\sigma$.*

Proof. Taking $x \in X_\varrho$, $\|x\|_\varrho \leqslant 1$, we get $\varrho(x) \leqslant 1$ and so $\sigma(Tx) \leqslant \varrho(x) \leqslant 1$. Consequently, $\|Tx\|_\sigma \leqslant 1$. Hence $\|Tx\|_\sigma \leqslant \|x\|_\varrho$ for all $x \in X_\varrho$.

The converse statement to Proposition 3.4 is not true. As an example it is sufficient to take $X = Y =$ the space of real numbers, $\sigma(x) = \frac{1}{2}(e^x - 1)$, $\varrho(x) = \frac{1}{2}(e^{x^2} - 1)$ for $x \geqslant 0$, and $Tx = x$. We have $\|x\|_\sigma = |x|/\ln 3$ and $\|x\|_\varrho = |x|/\sqrt{\ln 3}$, and so $\|Tx\|_\sigma = \|x\|_\sigma \leqslant \|x\|_\varrho$ for all $x \in X$, but $\sigma(Tx) = \sigma(x) > \varrho(x)$ and $\varrho(x) \leqslant 1$ for all $0 < x < 1$. However, let us still remark that if we take two Orlicz spaces $L^\varphi(S)$ and $L^\psi(S)$ over a set $S$ with respect to an atomless and infinite measure $\mu$, and we assume that $L^\varphi(S) \subset L^\psi(S)$, then supposing the identity map $T$ of $L^\varphi(S)$ in $L^\psi(S)$ to be a contraction with respect to the norm, it is also a $(\varrho, \sigma)$-contraction. It is sufficient to show that $\psi(u) \leqslant \varphi(u)$ for all $u \geqslant 0$. However, in the other case we would have $\psi(u_0) > \varphi(u_0)$ for $u_0 > 0$. Choosing a set $A \subset S$ such that $\mu(A) = 1/\psi(u_0)$ and $x_0(s) = u_0$ for $s \in A$, $x_0(s) = 0$ for $s \in S \setminus A$, we then obtain $\sigma(x_0) = 1$, $\varrho(x_0) < 1$ and so $\|x_0\|_\sigma = 1$, $\|x_0\|_\varrho < 1$, a contradiction.

**§ 4. Adjoint operators.** Let $X_\varrho$ and $Y_\sigma$ be two modular spaces with convex pseudomodulars $\varrho$ and $\sigma$, and let $X_\varrho^*$ and $Y_\sigma^*$ be the conjugate spaces with respect to $\langle X_\varrho, \| \|_\varrho\rangle$ and $\langle Y_\sigma, \| \|_\sigma\rangle$, respectively. Then the formulae $\varrho^*(x^*) = \sup_{x \in X_\varrho}(|x^*(x)| - \varrho(x))$ and $\sigma^*(y^*) = \sup_{y \in Y_\sigma}(|y^*(y)| - \sigma(y))$ define convex left-continuous semimodulars in $X_\varrho^*$, respectively $Y_\sigma^*$ (see § 2). Now, let $T$ be a linear map of $X_\varrho$ in $Y_\sigma$, then the formula $(T^*y^*)(x) = (y^*T)(x)$ for $x \in X_\varrho$ defines a linear operator from $Y_\sigma^*$ to $X_\varrho^*$, called the *adjoint operator* to $T$. We shall prove the following

THEOREM 4.1. (a) *If* $T\colon X_\varrho \to Y_\sigma$ *is strongly* $(\varrho, \sigma)$*-bounded, then* $T^*\colon Y_\sigma^* \to X_\varrho^*$ *is strongly* $(\sigma^*, \varrho^*)$*-bounded.*

(b) *If* $T\colon X_\varrho \to Y_\sigma$ *is a* $(\varrho, \sigma)$*-contraction* (see § 3, Def. 3.1 and condition $(B_2)$), *then* $T^*\colon Y_\sigma^* \to X_\varrho^*$ *is a* $(\sigma^*, \varrho^*)$*-contraction.*

Proof. If $T\colon X_\varrho \to Y_\sigma$ is strongly $(\varrho, \sigma)$-bounded with constants $k > 0$ and $M > 0$, i.e. $\sigma(kTx) \leqslant M\varrho(x)$ for all $x \in X_\varrho$, then

$$\varrho^*\left(\frac{k}{M} T^* y^*\right) = \sup_{x \in X_\varrho}\left(\frac{k}{M} y^*(Tx) - \varrho(x)\right) \leqslant \sup_{x \in X_\varrho}\left(\frac{k}{M} y^*(Tx) - \frac{1}{M}\sigma(kTx)\right)$$
$$= \frac{1}{M}\sup_{y \in TX_\varrho}\big(y^*(y) - \sigma(y)\big) \leqslant \frac{1}{M}\sigma^*(y^*),$$

which proves that $T^*$ is strongly $(\sigma^*, \varrho^*)$-bounded with constants $k/M$ and $1/M$. This proves both (a) and (b).

Remark 4.2. The result above makes it possible to define some categories of modular spaces in analogy to the known categories of normed spaces, where as morphism one takes continuous linear operators or contractions. Namely, taking as objects all real modular spaces $X_\varrho$ with convex modulars $\varrho$, we obtain a category $\mathrm{Md}_b$ taking as morphism the strongly $(\varrho, \sigma)$-bounded linear operators, and a category $\mathrm{Md}_1$ taking as morphism the $(\varrho, \sigma)$-contractions. Associating with every $X_\varrho$ the conjugate space (with respect to the norm $\|\ \|_\varrho$) $X_\varrho^* = f^*(X_\varrho)$ and with linear operators $T$, the adjoint linear operators $T^* = f^*(T)$, $f^*$ is a contravariant functor in each of the categories $\mathrm{Md}_b$ and $\mathrm{Md}_1$.

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