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Multivalent Harmonic Functions defined by m-tuple Integral operators

Abstract. In this paper a multivalent harmonic function is defined by m-tuple integral operators and some classes of these multivalent harmonic functions are studied in terms of inequalities involving Wright generalized hypergeometric functions. Some special cases of our results are also mentioned.

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1. Introduction and Preliminaries. Duren, Hengartner and Laugesen [15] has given the concept of multivalent harmonic functions by proving argument principle for harmonic complex valued functions. Using this concept, Ahuja and Jahagiri [8], [9] introduced the family $H(p)$, $p \in N = \{1, 2, 3, ...\}$ of all p-valent, harmonic and orientation preserving functions in the open disc $\triangle = \{z : |z| < 1\}$. A function f in $H(p)$ can be expressed as:

$$f = h + \overline{g} \tag{1.1}$$

where h and g are p-valent analytic functions in the unit disk $\triangle$ and of the form:

$$h(z) = \sum_{k=p}^{\infty} h_k z^k;\ h_p = 1 \text{ and } g(z) = \sum_{k=p}^{\infty} g_k z^k;\ |g_p| < 1.$$

Let $S_H^*(p, \alpha)$, $K_H(p, \alpha)$ and $Q_H(p, \alpha)$ be the classes of functions $f = h + \overline{g} \in H(p)$ satisfying the conditions

$$\frac{\partial}{\partial\theta}(arg(f(re^{i\theta}))) \geqslant p\alpha,\ \frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right)\right) \geqslant p\alpha \text{ and } Re\left(\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{\frac{\partial}{\partial\theta}z^p}\right) \geqslant \alpha$$

respectively for each $z = re^{i\theta}$, $0 \leqslant \theta < 2\pi$ and $0 \leqslant r < 1$, $\alpha(0 \leqslant \alpha < 1)$.

Whereas $TH(p)$ denote the subclass of functions $f = h + \overline{g} \in H(p)$ such that

$$h(z) = z^p - \sum_{k=p+1}^{\infty} |h_k| z^k \text{ and } g(z) = \sum_{k=p}^{\infty} |g_k| z^k;\ |g_p| < 1. \tag{1.2}$$

Hence, we denote $TS_H^*(p,\alpha) \approx S_H^*(p,\alpha) \cap TH(p)$, $TK_H(p,\alpha) \approx K_H(p,\alpha) \cap TH(p)$, and $TQ_H(p,\alpha) \approx Q_H(p,\alpha) \cap TH(p)$.

We formulate following lemmas from the work of Ahuja and Jahangiri [7, 8, 9, 10]:

Lemma 1.1 Let $f = h + \overline{g}$ given by (1.1) satisfies

$$\sum_{k=p+1}^{\infty} \frac{k - p\alpha}{p(1-\alpha)} |h_k| + \sum_{k=p}^{\infty} \frac{k + p\alpha}{p(1-\alpha)} |g_k| \leqslant 1 \tag{1.3}$$

for $p \geqslant 1$, $0 \leqslant \alpha < 1$, then f is sense preserving, p-valent and $f \in S_H^*(p,\alpha)$. Furthermore, $f = h + \overline{g} \in TS_H^*(p,\alpha)$ if and only if (1.3) holds.

Lemma 1.2 Let $f = h + \overline{g}$ given by (1.1) satisfies

$$\sum_{k=p+1}^{\infty} \frac{k(k - p\alpha)}{p^2(1-\alpha)} |h_k| + \sum_{k=p}^{\infty} \frac{k(k + p\alpha)}{p^2(1-\alpha)} |g_k| \leqslant 1 \tag{1.4}$$

for $p \geqslant 1$, then $f \in K_H(p,\alpha)$. Also $f = h + \overline{g} \in TK_H(p,\alpha)$ if and only if (1.4) holds.

Lemma 1.3 Let $f = h + \overline{g}$ given by (1.1) satisfies

$$\sum_{k=p+1}^{\infty} \frac{k}{p(1-\alpha)} |h_k| + \sum_{k=p}^{\infty} \frac{k}{p(1-\alpha)} |g_k| \leqslant 1 \tag{1.5}$$

for $p \geqslant 1$, $0 \leqslant \alpha < 1$, then $f \in Q_H(p,\alpha)$. Also $f = h + \overline{g} \in TQ_H(p,\alpha)$ if and only if (1.5) holds.

Recently several fractional calculus operators have found their applications in geometric function theory. Many research papers [11],[12],[13],[14] on harmonic functions defined by certain operators such as Dziok and Srivastava operator [5], Hohlov operator [19], Carlson and shaffer operator [1] have been published.

Motivated with their works, we intend to apply m-tuple integral operator [17, 18] which is a generalized form of previously introduced operators in the space of analytic functions and is defined as follows:

Definition 1.4 Let h be an analytic function in Δ for $m \in N = \{1, 2, 3, ...\}$, $\beta_i \in R_+, \delta_i \in R_+ \cup \{0\}, \nu_i \in R \; \forall \, i = 1, 2, ..., m$, an m-tuple integral operator, by means of m-repeated Erdélyi-Kober integral operators is defined as:

$$I_{\beta_i,m}^{(\nu_i),(\delta_i)} h(z) = \left[\prod_{i=1}^{m} I_{\beta_i,m}^{\nu_i,\delta_i} \right] h(z), \; \sum_{i=1}^{m} \delta_i > 0; \; I_{\beta_i,m}^{(\nu_i),(0)} h(z) = h(z); \; z \in \triangle \tag{1.6}$$

where $I_{\beta}^{\nu,\delta}$ is the Erdélyi-Kober integral operator [16], defined for $\beta \in R_+, \nu \in R$ as:

$$I_{\beta}^{\nu,\delta} h(z) = \frac{1}{\Gamma(\delta)} \int_0^1 (1-t)^{\delta-1} t^{\nu} h(z t^{\frac{1}{\beta}}) dt, \; \delta \in R_+ \text{ and } I_{\beta,m}^{\nu,0} h(z) = h(z).$$

The image of power function z^k [17, 18] under the operator defined in (1.6) is given as follows:

$$I_{\beta_i,m}^{(\nu_i),(\delta_i)} z^k = \lambda_k z^k \tag{1.7}$$

where

$$\lambda_k = \prod_{i=1}^{m} \frac{\Gamma\left(\nu_i + 1 + \frac{k}{\beta_i}\right)}{\Gamma\left(\nu_i + \delta_i + 1 + \frac{k}{\beta_i}\right)} \tag{1.8}$$

for each $k > \max_{1 \leqslant i \leqslant m} [-\beta_i(\nu_i + 1)]$.

In the subsequent work, we use Wright generalized hypergeometric function [3, 16] which is defined as follows:

Definition 1.5 Let $a_i \; (i = 1, 2, ...q)$, $b_i \; (i = 1, 2, ...s)$ be positive real numbers and $A_i \; (i = 1, 2, ...q)$, $B_i \; (i = 1, 2, ...s)$ be positive integers such that $1 + \sum_{i=1}^{s} B_i - \sum_{i=1}^{q} A_i \geqslant 0$, a Wright generalized hypergeometric (Wgh) function:

$${}_q\Psi_s[(a_1, A_1), ..., (a_q, A_q); (b_1, B_1), ..., (b_s, B_s); z] \approx \; {}_q\Psi_s[(a_i, A_i)_{1,q}; (b_i, B_i)_{1,s}; z]$$

is defined as:

$${}_q\Psi_s[(a_i, A_i)_{1,q}; (b_i, B_i)_{1,s}; z] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{q} \Gamma(a_i + A_i k) z^k}{\prod_{i=1}^{s} \Gamma(b_i + B_i k) k!}, \; z \in \triangle \tag{1.9}$$

which is an analytic function in the unit disk $\triangle$ if $q = s + 1$. Also for positive real a and for positive integer A [[6], 240, Eq. (1.26)]:

$$\Gamma(a+kA)=\Gamma(a)\left(\frac{a}{A}\right)_k\left(\frac{a+1}{A}\right)_k...\left(\frac{a+A-1}{A}\right)_k(A)^{kA},k=0,1,2,..$$

when used together with the result [[4], p.57]:

$$\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(d+k)}=k^{a+b-c-d}\left[1+O\left(\frac{1}{k}\right)\right],k=1,2,3,..$$

we observe that at $z=1$ the series (1.9) converges absolutely for $\sum\limits_{i=1}^{s}b_i-\sum\limits_{i=1}^{q}a_i>0$.

In particular if $A_1=...=A_q=B_1=...=B_s=1$, we have

$$_q\Psi_s[(a_i,1)_{1,q};(b_i,1)_{1,s};z]=\sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{q}\Gamma(a_i)}{\prod\limits_{i=1}^{s}\Gamma(b_i)}{}_qF_s((a_i)_{1,q};(b_i)_{1,s};z) \tag{1.10}$$

where $_qF_s((a_i)_{1,q};(b_i)_{1,s};z)\approx\ _qF_s(a_1,...a_q;b_1,....b_s;z)$ is the generalized hypergeometric function defined as:

$$_qF_s((a_i)_{1,q};(b_i)_{1,s};z)=\sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{q}\Gamma(a_i)_kz^k}{\prod\limits_{i=1}^{s}\Gamma(b_i)_kk!},\ z\in\triangle.$$

The symbol $(\lambda)_n$ is called Pochhammer symbol defined as:

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\lambda(\lambda+1)...(\lambda+n-1)\,\text{and}\,(\lambda)_0=1.$$

We shall also make use of the Hadmard product (convolution) "*óf two power series converging in $\triangle$ and defined as:

$$\sum_{k=0}^{\infty}h_kz^k*\sum_{k=0}^{\infty}g_kz^k=\sum_{k=0}^{\infty}h_kg_kz^k.$$

2. A Multivalent Harmonic Function. In this section, a multivalent harmonic function belonging to the class $H(p)$ involving m-tuple integral operators defined as in (1.6) with the use of (1.7) is defined first and then some of its special forms are also mentioned. Some notations and identities which we use throughout the work are also given.

Definition 2.1 Let $f(z)$ be given by (1.1) and with the parameters $m\in N=\{1,2,3,...\},\beta_i,\beta_i'\in R_+,\delta_i,\delta_i'\in R_+\cup\{0\},\nu_i,\nu_i'\geqslant-1,\forall i=1,2,...,m,\ |\sigma|<1$, a multivalent harmonic function in the class $H(p)$ as an operator $Wf(z)$ is defined as:

$$Wf(z) \approx W\left[\begin{array}{c}(\nu_i),(\delta_i);(\nu'_i),(\delta'_i)\\ \beta_i;\beta'_i;m\end{array}\right]f(z) \tag{2.1}$$

$$\begin{aligned}
&= \frac{1}{\lambda_p} I^{(\nu_i),(\delta_i)}_{\beta_i,m} h(z) + \overline{\frac{\sigma}{\lambda'_p} I^{(\nu'_i),(\delta'_i)}_{\beta'_i,m} g(z)} \\
&= \sum_{k=p}^{\infty} \frac{\lambda_k}{\lambda_p} h_k z^k + \overline{\sigma \sum_{k=p}^{\infty} \frac{\lambda'_k}{\lambda'_p} g_k z^k} \\
&= \sum_{k=p}^{\infty} \theta_k z^k * \sum_{k=p}^{\infty} h_k z^k + \overline{\sigma \sum_{k=p}^{\infty} \theta'_k z^k * \sum_{k=p}^{\infty} g_k z^k} \\
&= \left\{\frac{z^p}{\lambda_p}\Psi_1(z)\right\} * h(z) + \overline{\left\{\frac{\sigma z^p}{\lambda'_p}\Psi'_1(z)\right\} * g(z)}
\end{aligned}$$

where

$$\lambda_p = \prod_{i=1}^{m} \frac{\Gamma\left(\nu_i + 1 + \frac{p}{\beta_i}\right)}{\Gamma\left(\nu_i + \delta_i + 1 + \frac{p}{\beta_i}\right)}, \ \lambda'_p = \prod_{i=1}^{m} \frac{\Gamma\left(\nu'_i + 1 + \frac{p}{\beta'_i}\right)}{\Gamma\left(\nu'_i + \delta_i + 1 + \frac{p}{\beta'_i}\right)}; \tag{2.2}$$

$$\Psi_1(z) \approx \ {}_{m+1}\Psi_m((1,1),(\nu_i + 1 + \frac{p}{\beta_i}, \frac{1}{\beta_i})_{1,m}; (\nu_i + \delta_i + 1 + \frac{p}{\beta_i}, \frac{1}{\beta_i})_{1,m}; z),$$

$$\Psi'_1(z) \approx \ {}_{m+1}\Psi_m((1,1),(\nu'_i + 1 + \frac{p}{\beta'_i}, \frac{1}{\beta'_i})_{1,m}; (\nu'_i + \delta'_i + 1 + \frac{p}{\beta'_i}, \frac{1}{\beta'_i})_{1,m}; z); \tag{2.3}$$

and

$$\theta_k = \prod_{i=1}^{m} \frac{\Gamma\left(\nu_i + 1 + \frac{k}{\beta_i}\right)\Gamma\left(\nu_i + \delta_i + 1 + \frac{p}{\beta_i}\right)}{\Gamma\left(\nu_i + \delta_i + 1 + \frac{k}{\beta_i}\right)\Gamma\left(\nu_i + 1 + \frac{p}{\beta_i}\right)},$$

$$\theta'_k = \prod_{i=1}^{m} \frac{\Gamma\left(\nu'_i + 1 + \frac{k}{\beta'_i}\right)\Gamma\left(\nu'_i + \delta'_i + 1 + \frac{p}{\beta'_i}\right)}{\Gamma\left(\nu'_i + \delta'_i + 1 + \frac{k}{\beta'_i}\right)\Gamma\left(\nu'_i + 1 + \frac{p}{\beta'_i}\right)} \tag{2.4}$$

are non-increasing functions of $k(\geqslant p)$ such that $0 < \theta_k \leqslant \theta_{p+1} < \theta_p = 1$ and $0 < \theta'_k \leqslant \theta'_{p+1} < \theta'_p = 1$.

In particular, taking $\beta_i = 1 = \beta'_i, \nu_i = a_i - 1 - p$, $\nu'_i = c_i - 1 - p$, $\delta_i = b_i - a_i$, $\delta'_i = d_i - c_i$ for $i = 1, 2, ..., m$, the operator $Wf(z)$ reduces to $\Omega f(z)$ which is Dziok and Srivastava type operator involving generalized hypergeometric functions:

$$\Omega f(z) = \prod_{i=1}^{m} \frac{\Gamma(b_i)}{\Gamma(a_i)} I^{(\nu_i),(\delta_i)}_{1,m} h(z) + \overline{\sigma \prod_{i=1}^{m} \frac{\Gamma(d_i)}{\Gamma(c_i)} I^{(\nu'_i),(\delta'_i)}_{1,m} g(z)} \tag{2.5}$$

$$= z^p F_1(z) * h(z) + \overline{\sigma z^p F_1'(z) * g(z)}$$

where

$$F_1(z) \approx {}_{m+1}F_m((1,(a_i)_{1,m};(b_i)_{1,m};z),$$

$$F_1'(z) \approx {}_{m+1}F_m'((1,(c_i)_{1,m};(d_i)_{1,m};z). \tag{2.6}$$

If we take $m = 2, \nu_1 = a_1 - 1 - p, \nu_2 = b_1 - 1 - p$, $\delta_1 = 1 - a_1$, $\delta_2 = c_1 - b_1$; $\nu_1' = a_2 - 1 - p$, $\nu_2' = b_2 - 1 - p$, $\delta_1' = 1 - a_2$, $\delta_2' = c_2 - b_2$ and $\beta_i = 1 = \beta_i'(i = 1, 2)$, the operator $Wf(z)$ reduces to $H^p f(z)$ which is Hohlov type operator involving Gauss hypergeometric functions:

$$H^p f(z) = \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(b_1)} I_{1,2}^{(\nu_i),(\delta_i)} h(z) + \overline{\sigma \frac{\Gamma(c_2)}{\Gamma(a_2)\Gamma(b_2)} I_{1,2}^{(\nu_i'),(\delta_i')} g(z)} \tag{2.7}$$

$$= z^p \ {}_2F_1(a_1, b_1; c_1; z) * h(z) + \overline{\sigma z^p \ {}_2F_1'(a_2, b_2; c_2; z) * g(z)}.$$

Taking $m = 1$, $\nu = a_1 - 1 - p$, $\delta = c_1 - a_1$, $\nu' = a_2 - 1 - p$, $\delta' = c_2 - a_2$ and $\beta_i = 1 = \beta_i'$, the operator $Wf(z)$ reduces to $L^p f(z)$ which is Carlson Shaffer type operator involving incomplete beta functions:

$$L^p f(z) = \frac{\Gamma(c_1)}{\Gamma(a_1)} I_{1,1}^{a_1-1-p, c_1-a_1} h(z) + \overline{\sigma \frac{\Gamma(c_2)}{\Gamma(a_2)} I_{1,1}^{a_2-1-p, c_2-a_2} g(z)} \tag{2.8}$$

$$= z^p \ {}_2F_1(1, a_1; c_1; z) * h(z) + \overline{\sigma z^p \ {}_2F_1'(1, a_2; c_2; z) * g(z)}.$$

For convenience throughout the work we use following notations:

$$\Psi_n \approx {}_{m+1}\Psi_m((n,1),(\nu_i + 1 + \frac{(p+n-1)}{\beta_i}, \frac{1}{\beta_i})_{1,m}; (\nu_i + \delta_i + 1 + \frac{(p+n-1)}{\beta_i}, \frac{1}{\beta_i})_{1,m}; 1),$$

$$\Psi_n' \approx {}_{m+1}\Psi_m((n,1),(\nu_i' + 1 + \frac{(p+n-1)}{\beta_i'}, \frac{1}{\beta_i'})_{1,m}; (\nu_i' + \delta_i' + 1 + \frac{(p+n-1)}{\beta_i'}, \frac{1}{\beta_i'})_{1,m}; 1) \tag{2.9}$$

for $n = 1, 2, 3, ...$ which are the representations of absolute convergent series of type (1.9) at $z = 1$ provided $\sum_{i=1}^{m} \delta_i > n$ and $\sum_{i=1}^{m} \delta_i' > n$ respectively.

In the proof of our theorems, we use some identities which we prove in the form of following Lemma:

Lemma 2.2 Let Ψ_n ,Ψ_n' be given in (2.9) with the conditions $\sum_{i=1}^{m} \delta_i > n$, $\sum_{i=1}^{m} \delta_i' > n$ and λ_p, λ_p' in (2.2), then for $n = 1, 2, 3, ...$

$$\frac{1}{\lambda_p}\Psi_n = \sum_{k=p+n-1}^{\infty}(k-p-n+2)_{n-1}\theta_k \tag{2.10}$$

and

$$\frac{1}{\lambda'_p}\Psi'_n = \sum_{k=p+n-1}^{\infty}(k-p-n+2)_{n-1}\theta'_k. \tag{2.11}$$

PROOF Using the series expansion (1.9) for Ψ_n given in (2.9), λ_p given in (2.2) and θ_k in (2.4), we get

$$\begin{aligned}\frac{1}{\lambda_p}\Psi_n &= \sum_{k=0}^{\infty}\Gamma(n+k)\prod_{i=1}^{m}\frac{\Gamma(\nu_i+1+(p+n-1+k)/\beta_i)\Gamma(\nu_i+\delta_i+1+p/\beta_i)}{\Gamma(\nu_i+\delta_i+1+(p+n-1+k)/\beta_i)\Gamma(\nu_i+1+p/\beta_i)\Gamma(k+1)}\\ &= \sum_{k=0}^{\infty}(k+1)_{n-1}\prod_{i=1}^{m}\frac{\Gamma(\nu_i+1+(p+n-1+k)/\beta_i)\Gamma(\nu_i+\delta_i+1+p/\beta_i)}{\Gamma(\nu_i+\delta_i+1+(p+n-1+k)/\beta_i)\Gamma(\nu_i+1+p/\beta_i)}\\ &= \sum_{k=p+n-1}^{\infty}(k-p-n+2)_{n-1}\theta_k\end{aligned}$$

which proves identity (2.10). Similarly, identity (2.11) can be proved. ■

In particular, taking $\beta_i = 1 = \beta'_i, \nu_i = a_i-1-p$, $\nu'_i = c_i-1-p$, $\delta_i = b_i-a_i$, $\delta'_i = d_i-c_i$ for $(i = 1,2,...,m)$, and using relation (1.10), for $n = 1,2,3,...$, we get

$$\frac{1}{\lambda_p}\Psi_n = (1)_{n-1}\prod_{i=1}^{m}\frac{(a_i)_{n-1}}{(b_i)_{n-1}}F_n \text{ for } \sum_{i=1}^{m}(b_i-a_i) > n \tag{2.12}$$

and

$$\frac{1}{\lambda'_p}\Psi'_n = (1)_{n-1}\prod_{i=1}^{m}\frac{(c_i)_{n-1}}{(d_i)_{n-1}}F'_n \text{ for } \sum_{i=1}^{m}(d_i-c_i) > n \tag{2.13}$$

where

$$F_n \approx {}_{m+1}F_m((n,(a_i+n-1)_{1,m};(b_i+n-1)_{1,m};1)$$

and

$$F'_n \approx {}_{m+1}F_m((n,(c_i+n-1)_{1,m};(d_i+n-1)_{1,m};1).$$

We also have the following well known result:

$${}_2F_1(a+k,b+k;c+k;1) = \frac{(c)_k}{(c-a-b-k)_k}{}_2F_1(a,b;c;1),\ (c-a-b-k) > 0 \tag{2.14}$$

for $k = 0, 1, 2, ...$ and

$$_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \ (c-a-b) > 0.$$

3. Main results. In this section, some inequalities for $Wf(z)$ to be in the classes $S^*_H(p, \alpha)$, $K_H(p, \alpha)$ and $Q_H(p, \alpha)$ as sufficient conditions in terms of Wgh functions are examined and proved . Some consequences of these results are also derived as their corollaries. The results are given in the form of following theorems:

THEOREM 3.1 Let $f \in H(p)$ and $Wf(z)$ be given by (2.1) under the same parameter conditions along with $\sum_{i=1}^{m} \delta_i > 2$, $\sum_{i=1}^{m} \delta'_i > 2$ and if inequality

$$\frac{1}{\lambda_p}\Psi_2 + p(1-\alpha)\frac{1}{\lambda_p}\Psi_1 + |\sigma|\frac{1}{\lambda'_p}\Psi'_2 + |\sigma|p(1+\alpha)\frac{1}{\lambda'_p}\Psi'_1 \leqslant 2p(1-\alpha) \tag{3.1}$$

holds, then $Wf(z) \in S^*_H(p, \alpha)$.

PROOF To prove the theorem it is required by Lemma 1.1, to show

$$S_1 := \sum_{k=p+1}^{\infty} (k - p\alpha)|h_k|\theta_k + |\sigma|\sum_{k=p}^{\infty}(k + p\alpha)|g_k|\theta'_k \leqslant p(1-\alpha). \tag{3.2}$$

Since $f \in H(p)$, we have $|h_k| \leqslant 1$ and $|g_k| \leqslant 1$, $k \geqslant p$. Hence, by simple arrangement of terms in (3.2), we get

$$S_1 \leqslant \sum_{k=p+1}^{\infty} (k-p)\theta_k + p(1-\alpha) \sum_{k=p+1}^{\infty} \theta_k + |\sigma|\sum_{k=p}^{\infty}(k-p)\theta'_k + |\sigma|p(1+\alpha)\sum_{k=p}^{\infty}\theta'_k \tag{3.3}$$

Now applying identities (2.10) and (2.11) for $n = 1$ and 2, to the right hand side of (3.3), we see that

$$S_1 \leqslant \frac{1}{\lambda_p}\Psi_2 + p(1-\alpha)\frac{1}{\lambda_p}\Psi_1 - p(1-\alpha) + |\sigma|\frac{1}{\lambda'_p}\Psi'_2 + |\sigma|p(1+\alpha)\frac{1}{\lambda'_p}\Psi'_1 \leqslant p(1-\alpha) \text{ if (3.1)}$$

holds. This proves the result. ∎

On using (2.12) and (2.13), Theorem 3.1 yeilds following result:

COROLLARY 3.2 ([11]) Let $f \in H(p)$ and $\Omega f(z)$ be given by (2.5) along with $\sum_{i=1}^{m}(b_i - a_i) > 2$, $\sum_{i=1}^{m}(d_i - c_i) > 2$ and if inequality

$$\prod_{i=1}^{m}\frac{a_i}{b_i}F_2 + p(1-\alpha)F_1 + |\sigma|\prod_{i=1}^{m}\frac{c_i}{d_i}F'_2 + |\sigma|p(1+\alpha)F'_1 \leqslant 2p(1-\alpha)$$

holds, then $\Omega(f) \in S^*_H(p, \alpha)$.

Again, Corollary 3.2 for $m = 2$ together with the result (2.14) yeilds:

Corollary 3.3 Let $f \in H(p)$ and $H^p(f)$ be given by (2.7) along with

$$(c_1 - a_1 - b_1) > 1,\ (c_2 - a_2 - b_2) > 1$$

and if inequality

$$\left[\frac{a_1 b_1}{(c_1 - a_1 - b_1 - 1)} + p(1-\alpha)\right]\ {}_2F_1(a_1, b_1; c_1; 1)$$

$$+|\sigma|\left[\frac{a_2 b_2}{(c_2 - a_2 - b_2 - 1)} + p(1+\alpha)\right]\ {}_2F_1(a_2, b_2; c_2; 1) \leqslant 2p(1-\alpha)$$

holds, then $H^p(f) \in S_H^*(p, \alpha)$.

Further, on using summation formula: ${}_2F_1(a, 1; c; 1) = \frac{(c-1)}{(c-a-1)}, (c-a) > 1$, Corollary 3.3 yields the following result:

Corollary 3.4 Let $f \in H(p)$ and $L^p(f)$ be given by (2.8) along with $(c_1 - a_1) > 2,\ (c_2 - a_2) > 2$ and if inequality

$$\left[\frac{a_1 + p(1-\alpha)(c_1 - a_1 - 2)}{(c_1 - a_1 - 1)(c_1 - a_1 - 2)}\right](c_1 - 1) + |\sigma|\left[\frac{a_2 + p(1+\alpha)(c_2 - a_2 - 2)}{(c_2 - a_2 - 1)(c_2 - a_2 - 2)}\right](c_2 - 1) \leqslant 2p(1-\alpha)$$

holds, then $L^p(f) \in S_H^*(p, \alpha)$.

Theorem 3.5 Let $f \in H(p)$ and $Wf(z)$ be given by (2.1) under the same parameter conditions along with $\sum\limits_{i=1}^{m} \delta_i > 3$, $\sum\limits_{i=1}^{m} \delta'_i > 3$ if inequality

$$\frac{1}{\lambda_p}\Psi_3 + \{(2-\alpha)p+1\}\frac{1}{\lambda_p}\Psi_2 + p^2(1-\alpha)\frac{1}{\lambda_p}\Psi_1 + |\sigma|\frac{1}{\lambda'_p}\Psi'_3 + |\sigma|\{(2+\alpha)p+1\}\frac{1}{\lambda'_p}\Psi'_2$$

$$+|\sigma| p^2(1+\alpha)\frac{1}{\lambda'_p}\Psi'_1 \leqslant 2p^2(1-\alpha) \tag{3.4}$$

holds, then $Wf(z) \in K_H(p, \alpha)$.

Proof To prove the theorem it is required by Lemma 1.2, to show

$$S_2 := \sum_{k=p+1}^{\infty} k(k-p\alpha)|h_k|\theta_k + |\sigma|\sum_{k=p}^{\infty} k(k+p\alpha)|g_k|\theta'_k \leqslant p^2(1-\alpha). \tag{3.5}$$

Since $f \in H(p)$, we have $|h_k| \leqslant 1$ and $|g_k| \leqslant 1,\ k \geqslant p$. Hence, by simple arrangement of terms in (3.5) and applying identities (2.10) and (2.11)for $n = 1, 2$ and 3, we get

$$S_2 \leqslant \sum_{k=p+1}^{\infty} k^2\theta_k - p\alpha \sum_{k=p+1}^{\infty} k\theta_k + |\sigma| p^2(1+\alpha) + |\sigma| \sum_{k=p+1}^{\infty} k^2\theta'_k + |\sigma| p\alpha \sum_{k=p+1}^{\infty} k\theta'_k$$

$$= \sum_{k=p+1}^{\infty} [(k-p)(k-p-1) + (2p+1)(k-p) + p^2]\theta_k - p\alpha \sum_{k=p+1}^{\infty} (k-p)\theta_k$$

$$-p^2\alpha \sum_{k=p+1}^{\infty} \theta_k + |\sigma| p^2(1+\alpha)$$

$$+|\sigma| \sum_{k=p+1}^{\infty} [(k-p)(k-p-1)+(2p+1)(k-p)+p^2]\theta'_k + |\sigma| p\alpha \sum_{k=p+1}^{\infty} (k-p)\theta'_k + |\sigma| p^2\alpha \sum_{k=p+1}^{\infty} \theta'_k$$

$$= \sum_{k=p+2}^{\infty} (k-p)(k-p-1)\theta_k + \{(2-\alpha)p+1\} \sum_{k=p+1}^{\infty} (k-p)\theta_k + p^2(1-\alpha) \sum_{k=p+1}^{\infty} \theta_k$$

$$+|\sigma| p^2(1+\alpha) + |\sigma| \left[\sum_{k=p+2}^{\infty} (k-p)(k-p-1)\theta'_k + \{(2+\alpha)p+1\} \sum_{k=p+1}^{\infty} (k-p)\theta'_k \right.$$

$$\left. + p^2(1+\alpha) \sum_{k=p+1}^{\infty} \theta'_k \right]$$

$$= \frac{1}{\lambda_p}\Psi_3 + \{(2-\alpha)p+1\}\frac{1}{\lambda_p}\Psi_2 + p^2(1-\alpha)\frac{1}{\lambda_p}\Psi_1 - p^2(1-\alpha) + |\sigma|\frac{1}{\lambda'_p}\Psi'_3$$

$$+|\sigma|\{(2+\alpha)p+1\}\frac{1}{\lambda'_p}\Psi'_2 + p^2(1+\alpha)\frac{1}{\lambda'_p}\Psi'_1 \leqslant p^2(1-\alpha) \text{ if (3.4) holds.}$$

This proves the result. ■

On using (2.12) and (2.13), Theorem 3.5 yields following result:

COROLLARY 3.6 Let $f \in H(p)$ and $\Omega f(z)$ be given by (2.5) along with $\sum_{i=1}^{m}(b_i - a_i) > 3, \sum_{i=1}^{m}(d_i - c_i) > 3$ and if inequality

$$2\prod_{i=1}^{m}\frac{a_i(a_i+1)}{b_i(b_i+1)}F_3+\{p(2-\alpha)+1\}\prod_{i=1}^{m}\frac{a_i}{b_i}F_2+p^2(1-\alpha)F_1+|\sigma|\prod_{i=1}^{m}\frac{c_i(c_i+1)}{d_i(d_i+1)}F_3'+$$

$$|\sigma|\{p(2+\alpha)+1\}\prod_{i=1}^{m}\frac{c_i}{d_i}F_2'+|\sigma|p^2(1+\alpha)F_1'\leqslant 2p^2(1-\alpha)$$

holds, then $\Omega(f)\in K_H(p,\alpha)$.

Again, Corollary 3.6 for $m=2$ together with the result (2.14) yeilds:

Corollary 3.7 Let $f\in H(p)$ and $H^p(f)$ be given by (2.7) along with $(c_1-a_1-b_1)>2$, $(c_2-a_2-b_2)>2$ and if inequality

$$\left[\frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}+\{p(2-\alpha)+1\}\frac{a_1b_1}{(c_1-a_1-b_1-1)}+p^2(1-\alpha)\right]{}_2F_1(a_1,b_1;c_1;1)+$$

$$|\sigma|\left[\frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}+\{p(2+\alpha)+1\}\frac{a_2b_2}{(c_2-a_2-b_2-1)}+p^2(1+\alpha)\right]{}_2F_1(a_2,b_2;c_2;1)$$

$$\leqslant 2p^2(1-\alpha)$$

holds, then $H^p(f)\in K_H(p,\alpha)$.

Further, on using summation formula: ${}_2F_1(a,1;c;1)=\frac{(c-1)}{(c-a-1)}$, $(c-a)>1$, Corollary 3.7 yields the following result:

Corollary 3.8 Let $f\in H(p)$ and $L^p(f)$ be given by (2.8) along with $(c_1-a_1)>3$, $(c_2-a_2)>3$ and if inequality

$$\left[\frac{2(a_1)_2}{(c_1-a_1-3)_2}+\{p(2-\alpha)+1\}\frac{a_1}{(c_1-a_1-2)}+p^2(1-\alpha)\right]\frac{(c_1-1)}{(c_1-a_1-1)}+$$

$$|\sigma|\left[\frac{2(a_2)_2}{(c_2-a_2-3)_2}+\{p(2+\alpha)+1\}\frac{a_2}{(c_2-a_2-2)}+(1+\alpha)\right]\frac{(c_2-1)}{(c_2-a_2-1)}$$

$$\leqslant 2p^2(1-\alpha)$$

holds, then $L^p(f)\in K_H(p,\alpha)$.

Theorem 3.9 Let $f\in H(p)$ and $Wf(z)$ be given by (2.1) under the same parameter conditions along with $\sum\limits_{i=1}^{m}\delta_i>2$, $\sum\limits_{i=1}^{m}\delta_i'>2$ and if inequality

$$\frac{1}{\lambda_p}\Psi_2+p\frac{1}{\lambda_p}\Psi_1+|\sigma|\frac{1}{\lambda_p'}\Psi_2'+|\sigma|p\frac{1}{\lambda_p'}\Psi_1'\leqslant p(2-\alpha) \tag{3.6}$$

holds, then $Wf(z)\in Q_H(p,\alpha)$.

Proof To prove the theorem it is required by Lemma 1.3, to show

$$S_3 := \sum_{k=p+1}^{\infty} k|h_k|\theta_k + |\sigma|\sum_{k=p}^{\infty} k|g_k|\theta'_k \leqslant p(1-\alpha). \tag{3.7}$$

Since $f \in H(p)$, we have $|h_k| \leqslant 1$ and $|g_k| \leqslant 1$, $k \geqslant p$. Hence, by simple arrangement of terms in (3.7) and applying identities (2.10) and (2.11) for $n = 1$ and 2, we get

$$\begin{aligned} S_3 &\leqslant \sum_{k=p+1}^{\infty} (k-p)\theta_k + p\sum_{k=p+1}^{\infty} \theta_k + |\sigma|\sum_{k=p}^{\infty}(k-p)\theta'_k + |\sigma|p\sum_{k=p}^{\infty}\theta'_k \\ &= \frac{1}{\lambda_p}\Psi_2 + p\frac{1}{\lambda_p}\Psi_1 - p + |\sigma|\frac{1}{\lambda'_p}\Psi'_2 + |\sigma|p\frac{1}{\lambda'_p}\Psi'_1 \\ &\leqslant p(2-\alpha) \text{ if (3.6) holds.} \end{aligned}$$

This proves the result. ■

On using relations (2.12) and (2.13), Theorem 3.9 yeilds following result:

Corollary 3.10 Let $f \in H(p)$ and $\Omega f(z)$ be given by (2.5) along with $\sum_{i=1}^{m}(b_i - a_i) > 2, \sum_{i=1}^{m}(d_i - c_i) > 2$ and if inequality

$$\prod_{i=1}^{m}\frac{a_i}{b_i}F_2 + pF_1 + |\sigma|\prod_{i=1}^{m}\frac{c_i}{d_i}F'_2 + |\sigma|pF'_1 \leqslant p(2-\alpha)$$

holds, then $\Omega(f) \in Q_H(p, \alpha)$.

Again, Corollary 3.10 for $m = 2$ together with the result (2.14) yields:

Corollary 3.11 Let $f \in H(p)$ and $H^p(f)$ be given by (2.7) along with $(c_1 - a_1 - b_1) > 1$, $(c_2 - a_2 - b_2) > 1$ and if inequality

$$[a_1b_1 + p(c_1 - a_1 - b_1 - 1)]\,\frac{\Gamma(c_1)\Gamma(c_1 - a_1 - b_1 - 1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)} +$$

$$|\sigma|\,[a_2b_2 + p(c_2 - a_2 - b_2 - 1)]\,\frac{\Gamma(c_2)\Gamma(c_2 - a_2 - b_2 - 1)}{\Gamma(c_2 - a_2)\Gamma(c_2 - b_2)} \leqslant p(2-\alpha)$$

holds, then $H^p(f) \in Q_H(p, \alpha)$.

Further, on using summation formula: ${}_2F_1(a, 1; c; 1) = \frac{(c-1)}{(c-a-1)}$, $(c-a) > 1$, Corollary 3.11 yeilds the following result:

Corollary 3.12 Let $f \in H(p)$ and $L^p(f)$ be given by (2.8) along with $(c_1 - a_1) > 2, (c_2 - a_2) > 2$ and if inequality

$$[a_1 + p(c_1 - a_1 - 2)] \frac{(c_1 - 1)}{(c_1 - a_1 - 2)_2} + |\sigma| [a_2 + p(c_2 - a_2 - 2)] \frac{(c_2 - 1)}{(c_2 - a_2 - 2)_2} \leqslant p(2 - \alpha)$$

holds, then $L^p(f) \in Q_H(p, \alpha)$.

Theorem 3.13 Let $f \in TS_H^*(p, \alpha)$ and $Wf(z)$ be given by (2.1) under the same parameter conditions along with $\sum_{i=1}^{m} \delta_i > 1, \sum_{i=1}^{m} \delta_i' > 1$ and if inequality

$$\frac{1}{\lambda_p} \Psi_1 + |\sigma| \frac{1}{\lambda_p'} \Psi_1' \leqslant 2 \tag{3.8}$$

holds, then $Wf(z) \in S_H^*(p, \alpha)$.

Proof To prove the theorem it is required from Lemma 1.1, to show

$$S_4 := \sum_{k=p+1}^{\infty} \frac{k - p\alpha}{p(1 - \alpha)} |h_k| \theta_k + \sum_{k=p}^{\infty} \frac{k + p\alpha}{p(1 - \alpha)} |g_k| \theta_k' \leqslant 1. \tag{3.9}$$

Since $f \in TS_H^*(p, \alpha)$ we have again by lemma 1.1

$$|h_k| \leqslant \frac{p(1 - \alpha)}{(k - p\alpha)}; \; k \geqslant p \text{ and } |g_k| \leqslant \frac{p(1 - \alpha)}{(k + p\alpha)}; \; k \geqslant p.$$

Hence, with the use of identities (2.10) and (2.11) for $n = 1$, we get

$$S_4 \leqslant \sum_{k=p+1}^{\infty} \theta_k + \sum_{k=p}^{\infty} \theta_k' = \frac{1}{\lambda_p} \Psi_1 - 1 + |\sigma| \frac{1}{\lambda_p'} \Psi_1' \leqslant 1 \text{ if (3.8) holds.}$$

This proves the result. ■

Similarly, if $f \in TK_H(p, \alpha)$ $(TQ_H(p, \alpha))$ with the same hypothesis of Theorem 3.13, inequality (3.8) ensures that $Wf(z) \in K_H(p, \alpha)$ $(Q_H(p, \alpha))$ respectively.

As special cases of Theorem 3.13, we have following results:

Let $f \in TS_H^*(p, \alpha)$ $(TK_H(p, \alpha))$ $(TQ_H(p, \alpha))$ and $\Omega f(z)$ be given by (2.5) with $\sum_{i=1}^{m} (b_i - a_i) > 1, \sum_{i=1}^{m} (d_i - c_i) > 1$, inequality: $F_1 + |\sigma| F_1' \leqslant 2$ implies that $\Omega f(z) \in S_H^*(p, \alpha)$ $(K_H(p, \alpha))$ $(Q_H(p, \alpha))$.

Let $f \in TS_H^*(p, \alpha)$ $(TK_H(p, \alpha))$ $(TQ_H(p, \alpha))$ and $H^p f(z)$ be given by (2.7) with $c_1 > a_1 + b_1$, $c_2 > a_2 + b_2$, if inequality: $\frac{\Gamma(c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)} + |\sigma| \frac{\Gamma(c_2)\Gamma(c_2 - a_2 - b_2)}{\Gamma(c_2 - a_2)\Gamma(c_2 - b_2)} \leqslant 2$ holds, then $H^p f(z) \in S_H^*(p, \alpha)$ $(K_H(p, \alpha))$ $(Q_H(p, \alpha))$.

Let $f \in TS_H^*(p, \alpha)$ $(TK_H(p, \alpha))$ $(TQ_H(p, \alpha))$ and $L^p f(z)$ be given by (2.8) with $c_1 > a_1 + 1$, $c_2 > a_2 + 1$, inequality: $\frac{(c_1 - 1)}{(c_1 - a_1 - 1)} + |\sigma| \frac{(c_2 - 1)}{(c_2 - a_2 - 1)} \leqslant 2$ ensures that $L^p f(z) \in S_H^*(p, \alpha)$ $(K_H(p, \alpha))$ $(Q_H(p, \alpha))$.

References

[1] B.C. Carlson and D.B. Shaffer, *Starlike and prestarlike hyprgeometric functions*, SIAMJ. Math. Anal. **15** (1984), 737–745.

[2] E.M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc.(Ser.2) **46** (1940), 389–408.

[3] H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limlted and John Wiley & Sons) New York, Chichester,Toronto.

[4] H. Bateman, A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Ticomi (Eds.), *Higher Transcendental Functions*, Vol. 1, MaGraw-Hill, New York, 1953.

[5] J. Dziok, H.M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. **103** (1999), 1–13.

[6] L.J. Slater, *Generalized Hypergeometric functions*, Cambridge Univ. Press, London 1966.

[7] O.P. Ahuja and J.M. Jahangiri, *On a linear combination of multivalently harmonic functions*, Kyungpook Math. J. **42** (2002), 61–71.

[8] O.P. Ahuja and J.M. Jahangiri, *Multivalent harmonic starlike functions*, Ann. Univ. Mariae Curie-Sklodowska, Sectio A, **55(1)** (2001), 1–13.

[9] O.P. Ahuja and J.M. Jahangiri, *Errata to Multivalent harmonic starlike function*, Ann.Univ.Mariae Curie-Sklodowska, Vol.LV,1 Sectio A **56(1)** (2002), 105.

[10] O.P. Ahuja and J.M. Jahangiri, *Multivalent harmonic convex functions*, submited for publication.

[11] O.P. Ahuja and S.B. Joshi and A. Swaminathan, *Multivalent Harmonic Convolution Operators Associated With Generalized Hypergeometric Functions*, submitted for publication.

[12] O.P. Ahuja and H. Silverman, *Inequalities Associating Hypergeometric functions with Planer Harmonic Mapping*, J. Ineq. Pure Appl. Math. **5(4)** Art.99, 2004.

[13] O.P. Ahuja, *Harmonic Starlikeness and Convexity of Integral Operators generated by hypergeometric Series*, Integral Transforms and Special Functions (2009), 1–13.

[14] O.P. Ahuja, *Interconnectivity of Hohlov-Type Harmonic Convolution Operators with the Harmonic Starlike Mapping*, to appear.

[15] P. Duren, W. Hengartner and R.S. Laugesen, *The Argument Principle for Harmonic Functions*, Amer. Math. Monthly, **103** (1996), 411–415.

[16] V.S. Kriyakova, *Generalized Fractional Calculas and Applications*, Pitman Research Notes in Math. Series, **301**, Longman, Harlow (U.K.) 1994.

[17] V.S. Kriyakova, M. Saigo, S. Owa, *Study on differential operator and Integral operators in Univalent Function Theory*, Res. Inst. Math. Sci. Kyoto Seminar, March 3-5 (2003), 12–30.

[18] V.S. Kriyakova, M. Saigo and H.M. Srivastava, Fract. Calc. Appl. Anal. **1**, No. 1, 1998, 79–104.

[19] Yu. F. Hovlov, *Convolution operators preserving univalent functions*, Pliska Stud. Math. Bulgar. **10** (1989), 87–92.

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