COMMENTATIONES MATHEMATICAE Vol. 50, No. 1 (2010), 69-72

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Baire measurable solutions of a generalized Gołąb–Schinzel equation

Abstract. J. Brzdęk [1] characterized Baire measurable solutions $f : X \to \mathbb{K}$ of the functional equation

$$f(x + f(x)^n y) = f(x)f(y)$$

under the assumption that X is a Fréchet space over the field $\mathbb{K}$ of real or complex numbers and n is a positive integer. We prove that his result holds even if X is a linear topological space over $\mathbb{K}$; i.e. completeness and metrizability are not necessary.

2000 Mathematics Subject Classification: 39B52.

Key words and phrases: generalized Gołąb–Schinzel equation, net, finer net, Baire measurability.

For the first time, in connection with examining of subgroups of the centroaffine group of the plane, the functional equation

$$f(x + f(x)y) = f(x)f(y) \tag{1}$$

has been studied by S. Gołąb and A. Schinzel [3] in the class of continuous real functions. In 1965 C.G. Popa [7] proved that every Lebesgue measurable real solution of (1) is continuous or equal to zero almost everywhere. Following this idea J. Brzdęk [2] showed that the same is true for each Christensen measurable solution of the generalized Gołąb–Schinzel equation

$$f(x + f(x)^n y) = f(x)f(y) \tag{2}$$

mapping a real or complex separable Fréchet space into the field of real or complex numbers, respectively, where n is a positive integer.

J. Brzdęk [1] proved also an analogous result for Baire measurable solutions of the equation

$$f(x + f(x)^n y) = tf(x)f(y) \tag{3}$$

mapping a Fréchet space over the field $\mathbb{K}$ of real or complex numbers into $\mathbb{K}$, where n is a positive integer and $t \in \mathbb{K}\setminus\{0\}$. To prove this fact he used the open mapping theorem (see, for example, [5, 11.4]). We will show that it is enough to assume that X is a linear topological space over the field of real or complex numbers; i.e. X need not be complete metrizable. Thereby we must "go around"the open mapping theorem.

Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the set of all positive integers, reals and complex numbers, respectively. Moreover, we use some basic facts concerning nets, which can be found in [4, Chapter I (3.10), p.14]. Here, we recall only that a net $\{x_{\sigma'} : \sigma' \in \Sigma'\}$ in a topological space X (where Σ' is directed by the relation $\leqslant_{\Sigma'}$) is finer than a net $\{x_\sigma : \sigma \in \Sigma\}$ in X (where Σ is directed by $\leqslant_\Sigma$), if there exists a function $\varphi : \Sigma' \to \Sigma$ fulfilling the following two conditions:

- ✓ for every $\sigma_0 \in \Sigma$ there is $\sigma_0' \in \Sigma'$ such that $\sigma' \geqslant_{\Sigma'} \sigma_0'$ implies $\varphi(\sigma') \geqslant_\Sigma \sigma_0$;
- ✓ for every $\sigma' \in \Sigma'$ we have $x_{\varphi(\sigma')} = x_{\sigma'}$.

To generalize the result of J. Brzdęk [1] we need the following

Proposition 1 (cf. [1, Lemma 5]) *Let X be a linear–topological space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $V \subset \mathbb{K}$. If $g : X \to \mathbb{K}$ is a nontrivial continuous linear functional and $g^{-1}(V)$ has a subset of second category and with the Baire property, then the set V has a subset of second category and with the Baire property.*

Proof Let $Y := \ker g$. Then Y is a closed linear subspace of X and there exists a point $x_0 \in X \setminus Y$ such that $g^{-1}(V) = Vx_0 + Y$. Since Y is a linear–topological space (with the induced topology), $Y \times \mathbb{K}$ is a linear–topological space with the product topology.

Define a mapping

$$h : Y \times \mathbb{K} \ni (y, r) \to y + rx_0 \in Y + \mathbb{K}x_0.$$

It is easy to see that h is a continuous linear bijection. We prove that $H = h^{-1}$ is continuous; i.e. for every net $\{r_\sigma : \sigma \in \Sigma\} \subset \mathbb{K}$ and $\{y_\sigma : \sigma \in \Sigma\} \subset Y$ such that $\lim_{\sigma\in\Sigma}(r_\sigma x_0 + y_\sigma) = 0$ we have $\lim_{\sigma\in\Sigma}(r_\sigma, y_\sigma) = (0, 0)$.

To this end we prove that for each net $S = \{r_\sigma : \sigma \in \Sigma\}$, there is a net $S' = \{r_{\sigma'} : \sigma' \in \Sigma'\}$ finer than S, such that

$$\text{either } S' \text{ is convergent or } \lim_{\sigma'\in\Sigma'} \frac{1}{r_{\sigma'}} = 0. \tag{4}$$

First consider the case, when there exists a net S' finer than S such that $r_{\sigma'} \neq 0$ for each $\sigma' \in \Sigma'$. If S' is bounded, then $S' \subset \{z \in \mathbb{K} : |z| \leqslant M\}$ for some $M > 0$ and $\{z \in \mathbb{K} : |z| \leqslant M\}$ is a compact topological space (with the induced topology). Hence there is a convergent net $\{r_{\sigma''} : \sigma'' \in \Sigma''\}$ finer than S'. So assume that S' is unbounded. Then, for each $M > 0$, there is $\sigma' \in \Sigma'$ such that $\frac{1}{|r_{\sigma'}|} < \frac{1}{M}$. Denote

$\varepsilon = \frac{1}{M}$ and $a_{\sigma'} = \frac{1}{r_{\sigma'}}$. Thus, for each $\varepsilon > 0$, there is $\sigma' \in \Sigma'$ such that $|a_{\sigma'}| < \varepsilon$. Define

$$\Sigma'' = \{(\sigma', \varepsilon) : \ \varepsilon > 0, \ \sigma' \in \Sigma', \ a_{\sigma'} \in \{z \in \mathbb{K} : |z| < \varepsilon\}\}.$$

The set Σ'' is directed by the relation $\leqslant_{\Sigma''}$ given by

$$(\sigma_1, \varepsilon_1) \leqslant_{\Sigma''} (\sigma_2, \varepsilon_2) \ \Leftrightarrow \ \sigma_1 \leqslant_{\Sigma'} \sigma_2 \text{ and } \varepsilon_2 < \varepsilon_1.$$

Since the function $\varphi((\sigma', \varepsilon)) = \sigma'$ is a nondecreasing surjection, the net $\{a_{\sigma''} : \sigma'' \in \Sigma''\}$, where $a_{\sigma''} = a_{\sigma'}$ for $\sigma'' = (\sigma', \varepsilon)$, is finer than $\{a_{\sigma'} : \sigma' \in \Sigma'\}$. Moreover, for every $\varepsilon > 0$ there is $\sigma' \in \Sigma'$ such that $a_{\sigma'} \in \{z \in \mathbb{K} : |z| < \varepsilon\}$ and, for $\sigma'' \geqslant_{\Sigma''} (\sigma', \varepsilon)$, $a_{\sigma''} \in \{z \in \mathbb{K} : |z| < \varepsilon\}$. Hence $\lim_{\sigma'' \in \Sigma''} a_{\sigma''} = 0$.

Next consider the case, when for every net S' finer than S there exists $\sigma' \in \Sigma'$ with $r_{\sigma'} = 0$. Denote $\widetilde{\Sigma} = \{\sigma \in \Sigma : \ r_\sigma = 0\}$. We show that for every $\sigma \in \Sigma$ there is $\widetilde{\sigma} \in \widetilde{\Sigma}$ such that $\widetilde{\sigma} >_\Sigma \sigma$. So take $\sigma \in \Sigma$ and define $\Sigma' = \{\sigma' \in \Sigma : \sigma' >_\Sigma \sigma\}$. Then the set Σ' is directed by the relation $\leqslant_\Sigma$. Moreover, for each $\sigma_0 \in \Sigma$, there exists σ_0' with $\sigma_0' >_\Sigma \sigma$ and $\sigma_0' >_\Sigma \sigma_0$. Hence $\sigma' \geqslant_\Sigma \sigma_0'$ implies $\varphi(\sigma') := \sigma' \geqslant_\Sigma \sigma_0' >_\Sigma \sigma_0$. Thus S' is finer than S. Hence there is $\sigma_0' \in \Sigma'$ with $r_{\sigma_0'} = 0$. Since $\sigma_0' >_\Sigma \sigma$, for each $\sigma \in \Sigma$, there exists $\widetilde{\sigma} \in \widetilde{\Sigma}$ such that $\widetilde{\sigma} >_\Sigma \sigma$. In this way we obtain that for each $\sigma_0 \in \Sigma$ there is $\widetilde{\sigma}_0 \in \widetilde{\Sigma}$ such that $\widetilde{\sigma} \geqslant_\Sigma \widetilde{\sigma}_0$ implies $\widetilde{\varphi}(\widetilde{\sigma}) := \widetilde{\sigma} \geqslant_\Sigma \widetilde{\sigma}_0 >_\Sigma \sigma_0$. Thus the net $\{r_{\widetilde{\sigma}} : \widetilde{\sigma} \in \widetilde{\Sigma}\}$ is finer than S. So we proved that for every net S, there is a net S' finer than S such that (4) holds.

If $\lim_{\sigma' \in \Sigma'} r_{\sigma'} = r \neq 0$, then we have $\lim_{\sigma' \in \Sigma'} r_{\sigma'} x_0 = r x_0$ and

$$\lim_{\sigma' \in \Sigma'} y_{\sigma'} = \lim_{\sigma' \in \Sigma'} ((r_{\sigma'} x_0 + y_{\sigma'}) - r_{\sigma'} x_0) = -r x_0.$$

In the case where $\lim_{\sigma' \in \Sigma'} \frac{1}{r_{\sigma'}} = 0$ we have

$$\lim_{\sigma' \in \Sigma'} \frac{1}{r_{\sigma'}} y_{\sigma'} = \lim_{\sigma' \in \Sigma'} (\frac{1}{r_{\sigma'}} (r_{\sigma'} x_0 + y_{\sigma'}) - x_0) = -x_0$$

Since Y is a closed linear subspace of X, $x_0 \in Y$. This is a contradiction. Thus every net finer than S is convergent to 0. Hence $\lim_{\sigma \in \Sigma} r_\sigma = 0$ and $\lim_{\sigma \in \Sigma} r_\sigma x_0 = 0$. Consequently we obtain

$$\lim_{\sigma \in \Sigma} y_\sigma = \lim_{\sigma \in \Sigma} ((r_\sigma x_0 + y_\sigma) - r_\sigma x_0) = 0.$$

So we have $\lim_{\sigma \in \Sigma} (r_\sigma, y_\sigma) = (0, 0)$, what ends the proof of continuity of H.

Since h is a homeomorphism, $Y \times V$ possesses a subset of second category and with the Baire property in $Y \times \mathbb{K}$, whence so does V in $\mathbb{K}$ (see [6, Theorem 15.2 and 15.4]), what ends the proof. ∎

Using Proposition 1 instead of [1, Lemma 5], we can prove [1, Theorem 1] under the assumption that X is a linear–topological space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Consequently, the following Theorem holds:

Theorem 2 (cf. [1, Corollary 2]) Let X be a linear–topological space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $f : X \to \mathbb{K}$ be a Baire measurable solution of (3), where $n \in \mathbb{N}$ and $t \in \mathbb{K} \setminus \{0\}$. Then f is continuous or the set $\{x \in X : f(x) \neq 0\}$ is of the first category.

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(Received: 28.01.2008)