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Baire classification and multivalued maps

A set Y with two topologies τ_1 and τ_2 is called a *bitopological space* [8], [14]. For a set $M \subset Y$ by $\bar{M}^{(i)}$ we denote the τ_i-closure of M.

In (Y, τ_1, τ_2) the topology τ_2 is *perfectly normal* with respect to τ_1 if each τ_2-closed set $M \subset Y$ is of the form $M = \bigcap_{n=1}^{\infty} W_n$, where W_n are τ_1-open sets such that $\bar{W}_{n+1}^{(2)} \subset W_n$ for $n \geqslant 1$ [6]. Equivalently, τ_2 is perfectly normal with respect to τ_1 if each τ_2-open set W is of the form $W = \bigcup_{n=1}^{\infty} W_n$, where $W_n \in \tau_2$ and $\bar{W}_n^{(1)} \subset W_{n+1}$.

This property is not symmetrical. For instance, if Y is the set of real numbers, $\tau_1 = \{(a, \infty): a \in Y\} \cup \{\emptyset, Y\}$ and τ_2 is the natural topology on Y, then τ_1 is perfectly normal with respect to τ_2 but converse does not hold.

In the case $\tau_1 = \tau_2$ we have a perfectly normal topological space [5].

In the sequel by $\mathscr{C}(Y, \tau_i)$ and $\mathscr{K}(Y, \tau_i)$ we shall denote the class of all non-empty τ_i-closed or τ_i-compact subsets of Y, respectively.

Let X be a topological space. If $F\colon X \to Y$ is a multivalued map, then for any sets $A \subset X$ and $B \subset Y$ we denote [3]:

$$F(A) = \bigcup \{F(x): x \in A\},$$

$$F^{+}(B) = \{x \in X: F(x) \subset B\},$$

$$F^{-}(B) = \{x \in X: F(x) \cap B \neq \emptyset\}.$$

For any countable ordinal number α, a multivalued map $F\colon X \to Y$ is said to be of *τ_i-lower* or *τ_i-upper Baire class* α if for each τ_i-open set $V \subset Y$ the set $F^{-}(V)$ or $F^{+}(V)$, respectively, is of the additive class α in X. We shall use $\mathrm{LB}_\alpha(\tau_i)$ and $\mathrm{UB}_\alpha(\tau_i)$ to denote the τ_i-lower and τ_i-upper Baire classes α of multivalued maps. Thus $\mathrm{LB}_0(\tau_i)$ and $\mathrm{UB}_0(\tau_i)$ are classes of τ_i-lower and τ_i-upper semicontinuous maps, respectively.

Now let $F_n, F\colon X \to Y$ be multivalued maps such that $F_n(x), F(x) \in \mathscr{C}(Y, \tau_i)$ for $n \geqslant 1$, $x \in X$. We write $F \in \tau_i\text{-}\lim_{n\to\infty} F_n$ if for each $x \in X$ the sequence $\{F_n(x): n \geqslant 1\}$ converges to $F(x)$ in the Vietoris topology on $\mathscr{C}(Y, \tau_i)$.

I

THEOREM 1.1. ([6], Theorem 2.1). *Let X be a topological space and let (Y, τ_1, τ_2) be a bitopological space such that $\tau_1 \subset \tau_2$ and τ_2 is perfectly normal with respect to τ_1. Suppose that F_n, $F\colon X \to Y$ are multivalued maps such that $F_n(x)$, $F(x) \in \mathscr{C}(Y, \tau_1)$ for each $n \geqslant 1$, $x \in X$, and $F \in \tau_2\text{-}\lim_{n\to\infty} F_n$. Then*

(a) *For every τ_2-closed set $M \subset Y$.*

$$F^+(M) = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} F^+_{n+k}(W_n).$$

where W_n are τ_1-open sets such that $M = \bigcap_{n=1}^{\infty} W_n$ and $\bar{W}^{(2)}_{n+1} \subset W_n$ for $n \geqslant 1$.

(b) *If $F_n(x)$, $F(x) \in \mathscr{K}(Y, \tau_2)$, then*

$$F^-(M) = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} F^-_{n+k}(W_n).$$

This theorem implies the following results:

COROLLARY 1.2. *Let X be a topological space and let (Y, τ_1, τ_2) be a bitopological space such that $\tau_1 \subset \tau_2$ and τ_2 is perfectly normal with respect to τ_1. Suppose that F_n, $F\colon X \to Y$ are multivalued maps such that $F_n(x)$, $F(x) \in \mathscr{C}(Y, \tau_1)$ for $n \geqslant 1$, $x \in X$ and $F \in \tau_2\text{-}\lim_{n\to\infty} F_n$.*

(a) *If $F_n \in \mathrm{UB}_\alpha(\tau_1)$ for $n \geqslant 1$, then $F \in \mathrm{LB}_{\alpha+1}(\tau_2)$.*

(b) *If $F_n(x)$, $F(x) \in \mathscr{K}(Y, \tau_2)$ for $n \geqslant 1$, $x \in X$ and $F_n \in \mathrm{LB}_\alpha(\tau_1)$, then $F \in \mathrm{UB}_{\alpha+1}(\tau_2)$.*

When $\tau_1 = \tau_2$, Corollary 1.2 coincides with the result of Garg [7], Theorem 3.1.

COROLLARY 1.3. *Let X be a topological space and let (Y, τ_1, τ_2) be a bitopological space such that $\tau_1 \subset \tau_2$ and τ_2 is perfectly normal with respect to τ_1. If $F\colon X \to Y$ is a multivalued map such that $F(x) \in \mathscr{C}(Y, \tau_1)$ for $x \in X$, then the following is satisfied:*

(a) *If $F \in \mathrm{UB}_\alpha(\tau_1)$, then $F \in \mathrm{LB}_{\alpha+1}(\tau_2)$.*

(b) *If $F \in \mathrm{LB}_\alpha(\tau_1)$ and $F(x) \in \mathscr{K}(Y, \tau_2)$ for $x \in X$, then $F \in \mathrm{UB}_{\alpha+1}(\tau_2)$.*

If $\tau_1 = \tau_2$ and Y is a compact metric space, then Corollary 1.3 gives the theorem of Kuratowski [12].

By (E, τ_w, τ_s) we denote a separable Banach space with the weak topology τ_w and the topology τ_s determined by the norm on E.

THEOREM 1.4 ([6], Theorem 5.1). *In the bitopological space (E, τ_w, τ_s), τ_s is perfectly normal with respect to τ_w.*

COROLLARY 1.5. *Let X be a topological space and let F_n, $F\colon X \to E$ be multivalued maps such that $F_n(x)$, $F(x) \in \mathscr{C}(E, \tau_w)$ for $n \geqslant 1$, $x \in X$ and $F = \tau_s\text{-}\lim_{n\to\infty} F_n$.*

(a) *If* $F_n \in \mathrm{UB}_\alpha(\tau_w)$, *then* $F \in \mathrm{LB}_{\alpha+1}(\tau_s)$.

(b) *If* $F_n \in \mathrm{LB}_\alpha(\tau_w)$ *and* $F_n(x), F(x) \in \mathscr{K}(E, \tau_s)$, *then* $F \in \mathrm{UB}_{\alpha+1}(\tau_s)$.

COROLLARY 1.6. *Suppose that* X *is a topological space and* $F\colon X \to E$ *is a multivalued map such that* $F(x) \in \mathscr{C}(E, \tau_w)$ *for* $x \in X$. *Then*

(a) *If* $F \in \mathrm{UB}_\alpha(\tau_w)$, *then* $F \in \mathrm{LB}_{\alpha+1}(\tau_s)$.

(b) *If* $F \in \mathrm{LB}_\alpha(\tau_w)$ *and* $F(x) \in \mathscr{K}(E, \tau_s)$ *for* $x \in X$, *then* $F \in \mathrm{UB}_{\alpha+1}(\tau_s)$.

Any singlevalued map $f\colon X \to Y$ can be considered as a multivalued map F defined by $F(x) = \{f(x)\}$. In this case for each set $D \subset Y$ we have $F^-(D) = F^+(D) = f^{-1}(D)$. Moreover, $\mathrm{UB}_\alpha = \mathrm{LB}_\alpha = \mathrm{B}_\alpha$, i.e., it is the Baire class α of singlevalued maps [2], [11].

Therefore from Corollaries 1.5 and 1.6 we obtain the following

COROLLARY 1.7. ([1], Theorems 3 and 2). *Let* X *be a topological space.*

(a) *If* $f_n\colon X \to E$ *are maps of the weak class* α *and* $f = \tau_s\text{-}\lim_{n\to\infty} f_n$, *then* f *is in* $\mathrm{B}_{\alpha+1}(\tau_s)$.

(b) *If a map* $f\colon X \to E$ *is in* $\mathrm{B}_\alpha(\tau_w)$, *then* $f \in \mathrm{B}_{\alpha+1}(\tau_s)$.

II

In this section we consider multivalued maps of two variables. For a map $F\colon X \times Y \to Z$ by F_x and F^y we denote the maps defined by $F_x(y) = F(x, y) = F^y(x)$ for $x \in X$, $y \in Y$.

The paper of Engelking [4] contains the following

THEOREM. *In a metric space the union of a locally finite family of sets of an additive (multiplicative) class* α *is the set of the same class.*

Let us note that the proof of that theorem gives more.

THEOREM 2.1. *In a perfect space having a* σ*-locally finite base the union of a locally finite family of sets of an additive (multiplicative) class* α *is the set of the same class.*

COROLLARY 2.2. *In a perfect space possessing a* σ*-locally finite base the union of a* σ*-locally finite family of sets of an additive class* α *is the set of the same class.*

THEOREM 2.3. *Suppose that* X *is a metric space,* Y *is a perfect space possessing a* σ*-locally finite base and* (Z, τ_1, τ_2) *is a bitopological space in which* τ_2 *is perfectly normal with respect to* τ_1. *If* $F\colon X \times Y \to Z$ *is a multivalued map such that* $F_x \in \mathrm{UB}_\alpha(\tau_1)$ *for* $x \in X$ *and* $F^y \in \mathrm{UB}_0(\tau_1) \cap \mathrm{LB}_0(\tau_2)$ *for* $y \in Y$, *then* $F \in \mathrm{LB}_{\alpha+1}(\tau_2)$.

Proof. Any τ_2-closed set $M \subset Z$ is of the form $M = \bigcap_{n=1}^{\infty} W_n$, where $W_n \in \tau_1$ and $\overline{W}_{n+1}^{(2)} \subset W_n$ for $n \geqslant 1$. Let $\{\mathrm{B}_s\colon s \in S\}$ be a σ-locally finite base for X, let $S_n = \{s \in S\colon \operatorname{diam} \mathrm{B}_s < 1/n\}$ and $\{z_s\colon z_s \in \mathrm{B}_s,\ s \in S\}$. We shall show that

$$F^+(M) = \bigcap_{n=1}^{\infty} \bigcup_{s\in S_n} (\mathrm{B}_s \times Y) \cap (X \times F^+_{z_s}(W_n)). \tag{1}$$

Let us take a point $(x, y) \in F^+(M)$. Then $F(x, y) \subset W_n$ for each $n \geqslant 1$. Because $F^y \in \mathrm{UB}_0(\tau_1)$, there exists a neighbourhood $\mathrm{B}_{s(n)}$ of x such that $\operatorname{diam} \mathrm{B}_{s(n)} < 1/n$ and $F^y(\mathrm{B}_{s(n)}) \subset W_n$. Hence $F(z_{s_n}, y) \subset W_n$ and in consequence

$$F^+(M) \subset \bigcap_{n=1}^{\infty} \bigcup_{s\in S_n} (\mathrm{B}_s \times Y) \cap (X \times F^+_{z_s}(W_n)).$$

Now let $(x, y) \in \bigcap_{n=1}^{\infty} \bigcup_{s\in S_n} (\mathrm{B}_s \times Y) \cap (X \times F^+_{z_s}(W_n))$ and $(x, y) \notin F^+(M)$. There exists a sequence $\{z_{s_n}\colon s_n \in S_n,\ n \geqslant 1\}$ such that

$$x = \lim_{n\to\infty} z_{s_n}, \tag{2}$$

$$F(z_{s_n}, y) \subset W_n \quad \text{for } n \geqslant 1. \tag{3}$$

On the other hand, for some m we have $F(x, y) \cap (Z \setminus \bar{W}_m^{(2)}) \neq \emptyset$. The condition $F^y \in \mathrm{LB}_0(\tau_2)$ and (2) imply that there exists n_0 such that $F(z_{s_n}, y) \cap (Z \setminus \bar{W}_m^{(2)}) \neq \emptyset$ for $n \geqslant n_0$. So for $n \geqslant \max\{n_0, m\}$ we obtain $F(z_{s_n}, y) \cap (Z \setminus \bar{W}_n^{(2)}) \neq \emptyset$, what is a contradiction to (3). Thus (1) is proved. For any fixed $n \geqslant 1$,

$$\{(\mathrm{B}_s \times Y) \cap (X \times F^+_{z_s}(W_n))\colon s \in S\}$$

is a σ-locally finite family of sets of the additive class α in the space $X \times Y$. According to Corollary 2.2 the set $\bigcup_{s\in S_n} (\mathrm{B}_s \times Y) \cap (X \times F^+_{z_s}(W_n))$ is of the same class. Thus (1) implies that $F^+(M)$ is of the multiplicative class $\alpha+1$ and $F \in \mathrm{LB}_{\alpha+1}(\tau_2)$.

THEOREM 2.4. *Suppose that X is a metric space, Y is a perfect space with a σ-locally finite base and (Z, τ_1, τ_2) is a bitopological space such that τ_2 is perfectly normal with respect to τ_1. If $F\colon X \times Y \to Z$ is a multivalued map such that $F(x, y) \in \mathscr{K}(Z, \tau_2)$ for $(x, y) \in X \times Y$, $F_x \in \mathrm{LB}_\alpha(\tau_1)$ for $x \in X$, and $F^y \in \mathrm{LB}_0(\tau_1) \cap \mathrm{UB}_0(\tau_2)$ for $y \in Y$, then $F \in \mathrm{UB}_{\alpha+1}(\tau_2)$.*

Proof. Let $\{\mathrm{B}_s\colon s \in S\}$ be a σ-locally finite base for X, $D = \{z_s\colon z_s \in \mathrm{B}_s,\ s \in S\}$ and $S_n = \{s \in S\colon \operatorname{diam} B_s < 1/n\}$. A τ_2-closed set $M \subset Z$ is of the form $M = \bigcap_{n=1}^{\infty} W_n$, where $W_n \in \tau_1$ and $\bar{W}_{n+1}^{(2)} \subset W_n$ for $n \geqslant 1$.

We shall prove the expression

$$F^-(M) = \bigcap_{n=1}^{\infty} \bigcup_{s\in S_n} (\mathrm{B}_s \times Y) \cap (X \times F^-_{z_s}(W_n)). \tag{4}$$

If $(x, y) \in F^-(M)$, then $F(x, y) \cap W_n \neq \emptyset$ for each n. It follows from the condition $F^y \in \mathrm{LB}_0(\tau_1)$ that there exists a neighbourhood B_s of x such that $\operatorname{diam} \mathrm{B}_s < 1/n$ and $F(x', y) \cap W_n \neq \emptyset$ for $x' \in \mathrm{B}_s$. Thus $F(z_s, y) \cap W_n \neq \emptyset$ for some $s \in S_n$ and

$$(5) \qquad (x, y) \in \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} (B_s \times Y) \cap (X \times F_{z_s}^{-}(W_n)).$$

Now let (x, y) be a point satisfying (5) and let us suppose that $F(x, y) \cap M = \emptyset$. According to (5) there exists a sequence $\{z_{s_n}: s_n \in S_n\}$ such that

$$(6) \qquad x = \lim_{n \to \infty} z_{s_n}, \qquad F(z_{s_n}, y) \cap W_n \neq \emptyset \quad \text{for } n \geqslant 1.$$

On the other hand, $F(x, y) \subset Z \setminus M = \bigcup_{n=1}^{\infty} (Z \setminus \bar{W}_n^{(2)})$. The τ_2-compactness of $F(x, y)$ implies the inclusion $F(x, y) \subset Z \setminus \bar{W}_m^{(2)}$ for some $m \geqslant 1$. Because F^y is τ_2-upper semicontinuous we can take n_0 such that $F(z_{s_n}, y) \subset Z \setminus \bar{W}_m^{(2)}$ for $n \geqslant n_0$. Hence $F(z_{s_n}, y) \subset Z \setminus \bar{W}_m^{(2)} \subset Z \setminus W_n$ for $n \geqslant \max\{n_0, m\}$, what is a contradiction to (6). So (4) is proved. The rest of the proof is analogous as in Theorem 2.3.

If $\tau_1 = \tau_2$, X, Y and Z are metric spaces and $f: X \times Y \to Z$ is a singlevalued map, then each of Theorems 2.3 and 2.4 gives the theorems of Montgomery [13] and Kuratowski [10] (for separable space [9]). Moreover, applying Theorem 1.4, we obtain the following

COROLLARY 2.5. *Assume that X is a metric space, Y is a perfect space with a σ-locally finite base. If $f: X \times Y \to E$ is a singlevalued map such that f^y is continuous for $y \in Y$ and f_x is of the weak class α for $x \in X$, then f is of the class $\alpha + 1$.*

Let u_α and l_α denote the set of all real functions f such that for each real number r the set $\{x: f(x) < r\}$ or $\{x: f(x) > r\}$ is of the additive class α [15].

THEOREM 2.6. *Let X be a metric space and let Y be a perfect space with a σ-locally finite base. If $f: X \times Y \to R$ is a real function such that $f_x \in u_\alpha$ ($f_x \in l_\alpha$) for $x \in X$ and f^y is continuous for $y \in Y$, then $f \in l_{\alpha+1}$ ($f \in u_{\alpha+1}$).*

Proof. Let us put $\tau_1 = \{(a, \infty): a \in R\} \cup \{\emptyset, R\}$, $\tau_2 = \{\emptyset, R\} \cup \{(-\infty, a): a \in R\}$. Then in the bitopological space (R, τ_1, τ_2) the topology τ_i is perfectly normal with respect to τ_j, $i \neq j$, $i, j = 1, 2$. So the conclusion follows from Theorem 2.3 or 2.4.

III

For a multivalued map $F: X \to Y$, the graph of F is denoted by $\Gamma(F)$, i.e., $\Gamma(F) = \{(x, y) \in X \times Y: y \in F(x)\}$.

We present here characterizations of the graphs of multivalued maps belonging to UB_α or LB_α.

At first we formulate some properties of bitopological spaces.

A bitopological space is called *pairwise Hausdorff* [8] if for each distinct points $x, y \in Y$ there exist disjoint subsets $U \in \tau_i$, $V \in \tau_j$ such that $x \in U$, $y \in V$ for $i, j = 1, 2$; $i \neq j$.

In (Y, τ_1, τ_2) the topology τ_i is perfect with respect to τ_j if each τ_i-open set is F_σ in (Y, τ_j) for $i \neq j$ [14].

The bitopological space (E, τ_w, τ_s) is pairwise Hausdorff and τ_s is perfect with respect to τ_w [6].

THEOREM 3.1. *Let (X, τ) be a perfect space with a σ-locally finite base and let (Y, τ_1, τ_2) be a pairwise Hausdorff bitopological space in which τ_2 is perfect and has a σ-locally finite base. If $F\colon X \to Y$ is a multivalued map such that $F(x) \in \mathscr{K}(Y, \tau_1)$ for $x \in X$ and $F \in \mathrm{UB}_\alpha(\tau_1)$, then $\Gamma(F)$ is of the multiplicative class α in $(X \times Y, \tau \times \tau_2)$.*

Proof. Let us denote by $\mathscr{B} = \{V_s\colon s \in S\}$ a σ-locally finite base of the topology τ_2. If $(x, y) \notin \Gamma(F)$, then $y \notin F(x)$. Since $F(x)$ is τ_1-compact and (Y, τ_1, τ_2) is pairwise Hausdorff, there exist sets $U(x, y) \in \tau_1$ and $V_s \in \mathscr{B}$ such that

$$F(x) \subset U(x, y), \quad y \in V_s \quad \text{and} \quad U(x, y) \cap V_s = \emptyset. \tag{7}$$

Let U_s be the union of all sets $U(x, y)$ satisfying (7). Thus we obtain $X \times Y \setminus \Gamma(F) \subset \bigcup_{s \in S} F^+(U_s) \times V_s$. The converse inclusion is evident, so we have

$$X \times Y \setminus \Gamma(F) = \bigcup_{s \in S} F^+(U_s) \times V_s, \tag{8}$$

where U_s and V_s satisfy (7).

$\{F^+(U_s) \times V_s\colon s \in S\}$ is a σ-locally finite family of sets of the additive class α in $(X \times Y, \tau \times \tau_2)$. Applying Corollary 2.2 to (8) we have that $\Gamma(F)$ is of the multiplicative class α in $(X \times Y, \tau \times \tau_2)$.

Let us note that for a map $F \in \mathrm{UB}_0(\tau_1)$ with τ_1-compact values it is sufficient to assume (X, τ) any topological space and (Y, τ_1, τ_2) pairwise Hausdorff [6], Theorem 4.1.

If $\tau_1 = \tau_2$, all spaces are metrizable and F is a singlevalued map, then Theorem 3.1 coincides with the result of Kuratowski [10], p. 541, and Montgomery [13], Theorem 4.

COROLLARY 3.2. *Let (X, τ) be a perfect space with a σ-locally finite base. If $F\colon X \to E$ is a multivalued map such that $F(x) \in \mathscr{K}(E, \tau_w)$ for $x \in X$ and $F \in \mathrm{UB}_\alpha(\tau_w)$, then $\Gamma(F)$ is of the multiplicative class α in $(X \times E, \tau \times \tau_s)$.*

THEOREM 3.3. *Let (X, τ) be a perfect space with a σ-locally finite base and let (Y, τ_1, τ_2) be a bitopological space such that $\tau_1 \subset \tau_2$, τ_2 is perfect with respect to τ_1 and τ_2 has a σ-locally finite base. If $F\colon X \to Y$ is a multivalued map, $F(x) \in \mathscr{C}(Y, \tau_1)$ for $x \in X$ and $F \in \mathrm{LB}_\alpha(\tau_1)$, then $\Gamma(F)$ is of the multiplicative class $\alpha + 1$ in $(X \times Y, \tau \times \tau_2)$.*

Proof. Let us denote by $\{V_{sn}\colon s \in S_n, n \geq 1\}$ a σ-locally finite base of the topology τ_2 (for each $n \geqslant 1$, the family $\{V_{sn}: s \in S_n\}$ is locally finite). If $(x, y) \notin \Gamma(F)$, then $y \in Y \setminus F(x) \in \tau_1$. Thus there exist V_{sn} and τ_1-open set U_{sn} such that $y \in V_{sn} \subset U_{sn} \subset Y \setminus F(x)$. So we have

$$(x, y) \in \bigcup_{n=1}^{\infty} \bigcup_{s \in S_n} F^+(Y \setminus U_{sn}) \times V_{sn}.$$

It is easy to verify that

$$X \times Y \setminus \Gamma(F) = \bigcup_{n=1}^{\infty} \bigcup_{s \in S_n} F^+(Y \setminus U_{sn}) \times V_{sn}. \tag{9}$$

For each fixed n, $\{F^+(Y \setminus U_{sn}) \times V_{sn}: s \in S_n\}$ is a locally finite family of sets of the multiplicative class α in $(X \times Y, \tau \times \tau_2)$. According to Theorem 2.1 the union

$$\bigcup_{s \in S_n} F^+(Y \setminus U_{sn}) \times V_{sn}$$

is of the same class. Therefore, (9) implies that $\Gamma(F)$ is of the additive class $\alpha + 1$ in $(X \times Y, \tau \times \tau_2)$.

COROLLARY 3.4. *Let* (X, τ) *be a perfect space. If* $F: X \to E$ *is a multivalued map such that* $F(x) \in \mathscr{C}(E, \tau_w)$ *for* $x \in X$ *and* $F \in \mathrm{LB}_\alpha(\tau_w)$, *then* $\Gamma(F)$ *is of the multiplicative class* $\alpha + 1$ *in* $(X \times E, \tau \times \tau_s)$.

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