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WANG YUWEN and CHEN SHUTAO (Harbin)

Non-squareness, *B*-convexity and flatness of Orlicz space with Orlicz norm

Abstract. This paper investigates the properties mentioned in the title under the case of finite atomless measure when the Orlicz spaces are generated by N-functions (for definition of N-functions see [8]). It is shown here that Orlicz spaces of this kind are locally uniformly non-square and therefore are not flat. Another main result is that uniform non-squareness, uniform non-$l_n^{(1)}$ property, B-convexity and reflexivity coincide.

Introduction. Non-squareness, B-convexity and flatness are important geometric properties of Banach spaces, which expose the intrinsic construction of the spaces according to the 'shape' of the unit ball of the spaces. Therefore, it is interesting to investigate those problems in classical Banach spaces, for example, Orlicz spaces.

Let X be a Banach space, $S(X)$ the sphere of X. X is said to be *uniformly non-*$l_n^{(1)}$ $(n \geqslant 2)$ if there exists $\delta > 0$ such that for any $x_1, x_2, \ldots, x_n$ in $S(X)$ with $\|x_1 \pm x_2 \pm \ldots \pm x_n\| \leqslant n(1-\delta)$ for some choice of signs. X is said to be *B-convex* if, for some integer $n \geqslant 2$, X is uniformly non-$l_n^{(1)}$. Particularly, a uniformly non-$l_2^{(1)}$ space is called *uniformly non-square* [5].

Schöffer [11] introduces another definition of uniform non-squareness and other geometric concepts. X is called *uniformly non-square* if there exists $l > 1$ such that for any x, y in $S(X)$ we have $\max\{\|x+y\|, \|x-y\|\} \geqslant l$; X is called *locally uniformly non-square* if for each $x \in S(X)$, there exists $l_x > 1$ such that for any $y \in S(X)$, we have $\max\{\|x+y\|, \|x-y\|\} > l_x$.

The two definitions of uniform non-squareness coincide (see Lemma 1).

X is called *flat* if there exists a curve on $S(X)$ with antipoints and length two.

Non-squareness, B-convexity and flatness of Orlicz spaces equipped with Luxemburg norm have been exactly examined in papers [1]–[4], [9], [12], [13].

Let (G, Σ, μ) be a finite atomless measure space. By $M(u)$ we denote an N-function [8] and by $N(v)$ the complementary N-function to $M(u)$. $M(u)$ is said to *satisfy condition* Δ_2 for large u if there exist $u_0 > 0$ and $K > 2$ such that $M(2u) \leqslant K \cdot M(u)$ for all $u \geqslant u_0$. The functional $I_M(x) = \int_G M(x(t))\,dt$

defined on the set of all measurable functions $x\colon G \mapsto R$ (real line) is a pseudomodular. This functional defines the modular spaces called the *Orlicz space* and usually denoted by L_M. The norms introduced in L_M, the so-called *Luxemburg norm* and *Orlicz norm*, are defined respectively as follows:

$$(1)\qquad \|x\|_{(M)} = \inf\{\lambda > 0\colon I_M(x/\lambda) \leqslant 1\}, \qquad \|x\|_M = \sup_{I_N(y)\leqslant 1} \int_G x(t)y(t)\,dt,$$

where x is a function belonging to L_M.

It is well known that the two norms are equivalent, nevertheless, the corresponding geometric properties between the two norms are perfectly different.

Without any confusion, throughout this paper we denote by L_M the space $(L_M, \|\cdot\|_M)$.

I. Lemmas.

LEMMA 1. *Let X be a Banach space. Then*

(i) *if X contains a subspace isomorphic to c_0 or l_1, then X is not B-convex;*

(ii) *if X contains a bounded sequence not containing any weak Cauchy subsequence, then X contains a subspace isomorphic to l_1;*

(iii) *if X is locally uniformly non-square, then it is not a flat space;*

(iv) *if X is uniformly non-$l_{n-1}^{(1)}$, then it must be uniformly non-$l_n^{(1)}$* $(n \geqslant 3)$;

(v) *there exists $\delta > 0$ such that for any $u, v \in S(X)$ with*

$$(*)\qquad \min\{\|u+v\|, \|u-v\|\} \leqslant 2(1-\delta)$$

if and only if there exists $l > 1$ such that for any $x, y \in S(X)$, we have

$$(**)\qquad \max\{\|x+y\|, \|x-y\|\} \geqslant l.$$

Proof. (i) (see the proof of Theorem 2.2 in [5]). (ii) (see [10]). (iii) (see 17H in [11]).

(iv) If X is uniformly non-$l_{n-1}^{(1)}$ $(n \geqslant 3)$, then by definition there exists $\delta_0 > 0$ such that for any $x_1, x_2, \ldots, x_{n-1}, x_n \in (S(X))$ there exist $n-2$ numbers $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{n-1}$ ($\varepsilon_i = \pm 1$, $i = 2, \ldots, n-1$) with

$$\|x_1 + \varepsilon_2 x_2 + \ldots + \varepsilon_{n-1} x_{n-1}\| \leqslant (n-1)(1-\delta_0).$$

Let

$$\delta = \frac{n-1}{n}\delta_0, \qquad \varepsilon_n = 1;$$

then we have

$$\|x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \ldots + \varepsilon_{n-1} x_{n-1} + \varepsilon_n x_n\|$$
$$\leqslant \|x_1 + \varepsilon_2 x_2 + \ldots + \varepsilon_{n-1} x_{n-1}\| + \|x_n\| \leqslant (n-1) - (n-1)\delta_0 + 1 = n(1-\delta);$$

hence, X is uniformly non-$l_n^{(1)}$ $(n \geqslant 3)$.

(v) *Necessity*. If (**) is not satisfied, we can choose x_n, $y_n \in S(X)$ $(n = 1, 2, \ldots)$ with

$$\|x_n + y_n\| \to 1, \quad \|x_n - y_n\| \to 1 \quad (n \to \infty).$$

Let $u_n = (x_n + y_n)/\|x_n + y_n\|$, $v_n = (x_n - y_n)/\|x_n - y_n\|$ $(n = 1, 2, \ldots)$. By the inequality

$$\begin{aligned}(2) \quad 2/\|x_n+y_n\| - \|x_n-y_n\| \cdot \big|1/\|x_n-y_n\| - 1/\|x_n+y_n\|\big| \\ &= \big\|2x_n/\|x_n+y_n\|\big\| - \big\|(x_n-y_n)(1/\|x_n-y_n\| - 1/\|x_n+y_n\|)\big\| \\ &\leqslant \big\|2x_n/\|x_n+y_n\| + (x_n-y_n)(1/\|x_n-y_n\| - 1/\|x_n+y_n\|)\big\| \\ &= \big\|(x_n+y_n)/\|x_n+y_n\| + (x_n-y_n)/\|x_n-y_n\|\big\| = \|u_n+v_n\| \\ &\leqslant 2/\|x_n+y_n\| + \|x_n-y_n\| \big[\big|1/\|x_n-y_n\| - 1/\|x_n+y_n\|\big|\big]\end{aligned}$$

we immediately obtain

$$(3) \qquad \|u_n + v_n\| \to 2 \quad (n \to \infty).$$

Almost the same proof as that of (3), we have

$$(4) \qquad \|u_n - v_n\| \to 2 \quad (n \to \infty);$$

this contradicts (*).

Sufficiency. If (*) is not satisfied, we can choose u_n, $v_n \in S(X)$ $(n = 1, 2, \ldots)$ with

$$(5) \qquad \|u_n + v_n\| \to 2, \quad \|u_n - v_n\| \to 2 \quad (n \to \infty).$$

Let $x_n = (u_n + v_n)/\|u_n + v_n\|$, $y_n = (u_n - v_n)/\|u_n - v_n\|$ $(n = 1, 2, \ldots)$; similarly as in the case of (3) and (4), we have $\|x_n + y_n\| \to 1$, $\|x_n - y_n\| \to 1$ $(n \to \infty)$ contradicting (**).

LEMMA 2. (a) *Let x be in L_M, $x \neq 0$; then there exists a $k_0 > 0$ such that*

$$\|x\|_M = \inf_{k>0} \frac{1}{k}[1 + I_M(kx)] = \frac{1}{k_0}[1 + I_M(k_0 x)].$$

(b) *L_M is weakly sequentially complete iff no closed subspace of L_M is isomorphic to c_0.*

(c) *Let $M(u)$ satisfy the condition Δ_2 for large u; then for any $\varepsilon > 0$ and $c > 0$, there exists a $\delta > 0$ such that $I_M(x) \leqslant c$ and $I_M(y) \leqslant \delta$ imply $|I_M(x+y) - I_M(y)| \leqslant \varepsilon$.*

Proof. (a) (see Theorem 10.5 in [8] and Theorem 1 in [15]). (b) (see Chapter 10 in [7] or [14]). (c) (see [6]).

LEMMA 3. *If x_n, $y_n \in S(L_M)$ with* $\max\{\|x_n + y_n\|_M, \|x_n - y_n\|_M\} \leqslant 1 + 1/n$ *$(n = 1, 2, \ldots)$, then*

$$\text{(6)} \qquad \frac{k_n+h_n}{k_n h_n} I_M \left[\frac{k_n h_n}{k_n+h_n}(|x_n|-|y_n|) \right] \to 0 \quad (n \to \infty),$$

where k_n, $h_n > 0$ *satisfy*

$$\|x_n+y_n\|_M = \frac{1}{k_n}[1+I_M(k_n(x_n+y_n))], \qquad \|x_n-y_n\|_M = \frac{1}{h_n}[1+I_M(h_n(x_n-y_n))]$$

$$(n = 1, 2, \ldots).$$

Proof. By (6) and the convexity of $M(u)$, we have

$$\text{(7)} \quad 2+2/n > \|x_n+y_n\|_M + \|x_n-y_n\|_M$$
$$= \frac{k_n+h_n}{k_n h_n}\left\{1 + \int_G \left| \frac{h_n}{k_n+h_n} M(k_n(x_n(t)+y_n(t))) + \right.\right.$$
$$\left.\left. + \frac{k_n}{k_n+h_n} M(h_n(x_n(t)-y_n(t))) \right| dt \right\}$$
$$\geqslant \frac{k_n+h_n}{k_n h_n}\left\{1+I_M\left(\frac{k_n h_n}{k_n+h_n} 2x_n\right)\right\} \geqslant \|2x_n\|_M = 2 \quad (n = 1, 2, \ldots).$$

Similarly,

$$\text{(8)} \quad 2+2/n > \|x_n+y_n\|_M + \|x_n-y_n\|_M$$
$$= \frac{k_n+h_n}{k_n h_n}\left\{1 + \int_G \left| \frac{h_n}{k_n+h_n} M(k(x_n(t)+y_n(t))) + \frac{k_n}{k_n+h_n} M(h(x_n(t)-y_n(t))) \right| dt \right\}$$
$$\geqslant \frac{k_n+h_n}{k_n h_n}\left\{1+I_M\left(\frac{k_n h_n}{k_n+h_n} 2y_n\right)\right\} \geqslant \|2y_n\|_M = 2 \quad (n = 1, 2, \ldots).$$

Therefore, by (7), (8) and $M(u)$ satisfying the condition $M(|u|-|v|) \leqslant |M(2u)-M(2v)|$, we have

$$\frac{k_n+h_n}{k_n h_n} I_M \left[\frac{k_n h_n}{k_n+h_n}(|x_n|-|y_n|) \right]$$
$$\leqslant \frac{k_n+h_n}{k_n h_n} \int_G \left| M\left(\frac{k_n h_n}{k_n+h_n} 2x_n(t)\right) - M\left(\frac{k_n h_n}{k_n+h_n} 2y_n(t)\right)\right| dt < 4/n \to 0$$
$$(n \to \infty).$$

LEMMA 4. *If* $N(v)$ *satisfies the condition* Δ_2 *for large* v, *then the set*

$$K = \left\{k \colon \|x\|_M = \frac{1}{k}\{1+I_M(kx)\},\ a \leqslant \|x\|_M \leqslant b\right\}$$

is bounded for any $b \geqslant a > 0$.

Proof. Write $u_0 = M^{-1}(1/2\mu(G))$. By Theorem 4.2 in [8], $N(v)$ satisfies the condition Δ_2 for large v iff there exists $l > 1$ such that

$$M(lu) \geqslant 2lM(u) \quad \text{for all } u \geqslant u_0. \tag{9}$$

For given $b \geqslant a > 0$ and $x \in L_M$ with $a \leqslant \|x\|_M = (1+I_M(kx))/k \leqslant b$, since $\|x\|_{(M)} \geqslant \frac{1}{2}\|x\|_M \geqslant \frac{1}{2}a$, by the definition of $\|\cdot\|_{(M)}$, we have $I_M(3x/a) > 1$, therefore

$$\int_{G(|3x(t)|/a \geqslant u_0)} M(3x(t)/a)\,dt$$
$$= I_M(3x/a) - \int_{G(|3x(t)|/a < u_0)} M(3x(t)/a)\,dt \geqslant I_M(3x/a) - M(u_0)\mu(G) > \tfrac{1}{2}.$$

Now, suppose $k > 3l/a$, so we can select a positive integer i such that $l^i < \frac{1}{3}ak \leqslant l^{i+1}$. By repeatly utilizing (9), we get $M(l^i u) \geqslant 2^i l^i M(u)$ $(u \geqslant u_0)$. Hence

$$b \geqslant \|x\|_M = \frac{1}{k}\{1+I_M(kx)\} \geqslant \frac{1}{k}\int_{G(3/a|x(t)| \geqslant u_0)} M\left(\tfrac{1}{3}ak\frac{3}{a}x(t)\right)dt$$
$$\geqslant \frac{1}{k}\int_{G(3/a|x(t)| \geqslant u_0)} M\left(l^i\frac{3}{a}x(t)\right)dt$$
$$\geqslant \frac{2^i l^i}{k}\int_{G(3/a|x(t)| \geqslant u_0)} M\left(\frac{3}{a}x(t)\right)dt > \frac{2^i l^i}{k}\cdot\tfrac{1}{2}$$
$$> (2^i l^i)/\left(\frac{6}{a}l^{i+1}\right) = a2^i/6l;$$

therefore, $i < \log_2 6lb/a$. Thus,

$$k \leqslant \max\left\{\frac{3}{a}l,\ \frac{3}{a}l^{1+\log_2 6lb/a}\right\} = \frac{3}{a}l^{1+\log_2 6lb/a}.$$

Lemma 5. *Let $x \in L_M$, $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\|x\chi_{G\setminus e}\|_M \geqslant \|x\|_M - \varepsilon$ for any $e \subset G$ with $\mu(e) < \delta$.*

Proof. Since $x \in L_M$, by the definition of $\|\cdot\|_M$, we can select $y \in L_N$ such that $I_N(y) \leqslant 1$ and $\int_G x(t)y(t)\,dt \geqslant \|x\|_M - \varepsilon/2$. By the absolute continuity of the integral, there exists $\delta > 0$ such that for any $e \subset G$, if only $\mu(e) < \delta$, we have $\int_e x(t)y(t)\,dt \leqslant \varepsilon/2$. Therefore,

$$\|x\chi_{G\setminus e}\|_M \geqslant \int_{G\setminus e} x(t)y(t)\,dt \geqslant \|x\|_M - \varepsilon.$$

Lemma 6. *The following properties are equivalent.*

(I) *$N(v)$ satisfies condition Δ_2 for large v;*

(II) *there exist $\varepsilon > 0$ and $v_0 > 0$ such that $N((1+\varepsilon)v) \leqslant 2N(v)$ for all $v \geqslant v_0$;*

(III) *there exist $\delta > 0$ and $u_0 > 0$ such that $M(2u) \geqslant (2+\delta)M(u)$ for all $u \geqslant u_0$;*

(IV) *there exist $l > 1$ and $u_1 > 0$ such that $M(u) \leqslant M(lu)/2l$ for all $u \geqslant u_1$.*

Proof. (I)$\Rightarrow$(II). Let $K \geqslant 2$, $v_0 > 0$ with $N(2v) \leqslant KN(v)$ for all $v \geqslant v_0$. Taking $\varepsilon = 1/(K-1)$, by the convexity of $N(v)$ we have

$$N((1+\varepsilon)v) = N(\varepsilon 2v + (1-\varepsilon)v) \leqslant N\varepsilon(2v) + (1-\varepsilon)N(v)$$
$$\leqslant [1+(K-1)\varepsilon]N(v) = 2N(v) \quad (v \geqslant v_0).$$

(II)$\Rightarrow$(III). Write $N_1(v) = \frac{1}{2}N((1+\varepsilon)v)$ and denote by $M_1(u)$ the complementary N-function to $N_1(v)$. By (II), $N_1(v) \leqslant N(v)$ $(v \geqslant v_0)$, therefore there exists $u_0 > 0$ such that $\frac{1}{2}M(2u/(1+\varepsilon)) = M_1(u) \geqslant M(u)$ $(u \geqslant u_0)$. Taking $\delta = 2\varepsilon$, for all $u \geqslant u_0$, we have

$$M(u) \leqslant \tfrac{1}{2}M(2u/(1+\varepsilon)) \leqslant \tfrac{1}{2}M(2u)/(1+\varepsilon) = M(2u)/(2+\delta).$$

(III)$\Rightarrow$(IV). For given $\delta > 0$ in (III), select a positive integer n such that $n\delta \geqslant 2$. Let $l = 2^n$, $u_1 = u_0$; then for all $u \geqslant u_1$, we have

$$M(lu) = M(2^n u) \geqslant (2+\delta)^n M(u) \geqslant (2^n + n2^{n-1}\delta)M(u) \geqslant 2lM(u).$$

(IV)$\Rightarrow$(I). See Theorem 4.2 in [8].

II. Main theorems.

THEOREM 1. *The following conditions are equivalent.*

(A) *L_M is reflexive;*

(B) *L_M is uniformly non-square;*

(C) *L_M is uniformly non-$l_n^{(1)}$ $(n \geqslant 2)$;*

(D) *L_M is B-convex;*

(E) *noclosed subspace of L_M is isomorphic to c_0 or l_1;*

(F) *$M(u)$ satisfies condition Δ_2 for large u, and there exist $\delta > 0$ and $u_0 > 0$ such that $M(2u) > (2+\delta)M(u)$ for all $u \geqslant u_0$.*

Proof. The implications (B)$\Rightarrow$(C)$\Rightarrow$(D)$\Rightarrow$(E) are evident by (i) and (iv) of Lemma 1. Also (A)$\Leftrightarrow$(F), by Lemma 6. So, to complete the proof it is sufficient to show the implications (E)$\Rightarrow$(A) and (A)$\Rightarrow$(B).

(E)$\Rightarrow$(A). Since noclosed subspace of L_M is isomorphic to c_0, by (b) of Lemma 2 L_M is weakly sequentially complete. Let $\{x_n\}$ be an arbitrary sequence in the unit ball of L_M. Since noclosed subspace of L_M is isomorphic to l_1, by (ii) of Lemma 1 we can choose a weak Cauchy subsequence $\{x_{n_k}\}$ of $\{x_n\}$; therefore, in virtue of the weak sequential completeness of L_M, there exists x_0 in L_M such that $x_{n_k} \xrightarrow{w} x$ $(k \to \infty)$. By the lower semi-continuity of the norm $\| \ \|_M$ we have

$$\|x_0\|_M \leqslant \varliminf_{k \to \infty} \|x_{n_k}\|_M \leqslant 1.$$

It follows from the famous Eberlein–Smulian Theorem that L_M is reflexive.

(A) $\Rightarrow$ (B). If (B) does not hold, we can choose x_n, y_n in $S(L_M)$ such that

$$\max\{\|x_n+y_n\|_M, \|x_n-y_n\|_M\} < 1+1/n \quad (n = 1, 2, \ldots).$$

By (a) of Lemma 2, there exist k_n, $h_n > 0$ such that

$$\|x_n+y_n\|_M = \frac{1}{k_n}\{1+I_M(k_n(x_n+y_n))\},$$
$$\|x_n-y_n\|_M = \frac{1}{h_n}\{1+I_M(h_n(x_n-y_n))\},$$

$n = 1, 2, \ldots$ Therefore, by Lemma 3,

$$\frac{k_n+h_h}{k_n h_n} I_M\left(\frac{k_n h_n}{k_n+h_n}(|x_n|-|y_n|)\right) \to 0 \quad (n \to \infty). \tag{9}$$

Observe that $\frac{1}{2} \leqslant 1-1/n < \|x_n+y_n\|_M \leqslant 2$, $\frac{1}{2} \leqslant 1-1/n < \|x_n-y_n\|_M \leqslant 2$ for all $n \geqslant 2$; by Lemma 4, the set $\{k_n, h_n\}_{n=1}^{\infty}$ is bounded. Without loss of generality let $h_n \geqslant k_n$ $(n = 1, 2, \ldots)$. Write $H_0 = \sup_n h_n$ and $K_0 = \inf_n k_n$.

Since $\|x_n+y_n\|_M \leqslant 2$ $(n = 1, 2, \ldots)$, it is obvious that $K_0 > 0$ and

$$\frac{K_0}{2} \leqslant \frac{k_n h_n}{k_n+h_n} \leqslant \frac{H_0}{2}.$$

Hence, by (9), we obtain

$$\frac{2}{H_0} I_M\left(\frac{K_0}{2}(|x_n|-|y_n|)\right) \to 0 \quad (n \to \infty).$$

Remembering that $M(u)$ satisfies the condition Δ_2 for large u and $\sup_n k_n < \infty$, we have

$$I_M(k_n(|x_n|-|y_n|)) \to 0 \quad (n \to \infty) \tag{10}$$

and $I_M(2k_n x_n) \leqslant c$ $(n = 1, 2, \ldots)$ for some positive constant c.

By (c) of Lemma 2, there exists $\delta > 0$ such that for any $x, y \in L_M$, if only $I_M(x) \leqslant c$ and $I_M(y) \leqslant \delta$, we have

$$I_M(x+y) > I_M(x) - K_0/2H_0. \tag{11}$$

By (10), there exists an integer n_0 such that

$$I_M(k_n(|x_n|-|y_n|)) \leqslant \delta \quad \text{for all } n \geqslant n_0. \tag{12}$$

Let $G_n = \{t \in G: x_n(t)\, y_n(t) \geqslant 0\}$ $(n = 1, 2, \ldots)$. Since $M(u_1)/u_1 \leqslant M(u_2)/u_2$ whenever $u_1 \leqslant u_2$, by (11) and (12), we have

$$\begin{aligned}
2+2/n &> \|x_n+y_n\|_M + \|x_n-y_n\|_M \\
&\geqslant \frac{1}{H_0}+\frac{1}{k_n}+\frac{1}{k_n}\int_{G_n} M\big(k_n(|x_n(t)|+|y_n(t)|)\big)\,dt + \\
&\qquad\qquad + \frac{1}{h_n}\int_{G\setminus G_n} M\big(h_n(|x_n(t)|+|y_n(t)|)\big)\,dt \\
&\geqslant \frac{1}{H_0}+\frac{1}{k_n}+\frac{1}{k_n} I_M\big(k_n(|x_n|+|y_n|)\big) \\
&= \frac{1}{H_0}+\frac{1}{k_n}+\frac{1}{k_n} I_M\big(2k_n|x_n| + k_n(|y_n|-|x_n|)\big) \\
&> \frac{1}{H_0}+\frac{1}{k_n}\{1+I_M(2k_n x_n)\} - \frac{1}{k_n}\cdot\frac{K_0}{2H_0} \\
&\geqslant \|2x_n\|_M + 1/2H_0 = 2+1/2H_0
\end{aligned}$$

for all $n \geqslant n_0$. This contradiction shows condition (B).

THEOREM 2. (α) *L_M is locally uniformly non-square*; (β) *L_M is non-flat.*

Proof. By (iii) of Lemma 1, we only need to show (α). If L_M is not locally uniformly non-square, we can choose x, $y_n \in S(L_M)$ with

$$\max\{\|x+y_n\|_M, \|x-y_n\|_M\} < 1+1/n \quad (n = 1, 2, \ldots).$$

By (a) of Lemma 2, there exist k_n and h_n with

$$\|x+y_n\|_M = \frac{1}{k_n}\{1+I_M\big(k_n(x+y_n)\big)\}, \qquad \|x-y_n\|_M = \frac{1}{h_n}\{1+I_M\big(h_n(x-y_n)\big)\},$$

$n = 1, 2, \ldots$; therefore, by (7), we have

$$2+\frac{2}{n} > \frac{k_n+h_n}{k_n h_n}\left\{1+I_M\left(\frac{k_n h_n}{k_n+h_n}2x\right)\right\} \geqslant 2 \quad (n = 1, 2, \ldots).$$

Since $x(t)$ is a fixed function, it is obvious that

$$b = \sup_n \frac{k_n h_n}{k_n+h_n} < \infty \quad \text{and} \quad a = \inf_n \frac{k_n h_n}{k_n+h_n} > 0.$$

Hence, by Lemma 3, we obtain $I_M\big(a(|x|-|y_n|)\big)/b \to 0$ $(n \to \infty)$. Therefore, $|y_n(t)| \xrightarrow{\mu} |x(t)|$ $(n \to \infty)$ (where $\xrightarrow{\mu}$ denotes the convergence in measure).

Without losing the generality, let $h_n \geqslant k_n$ $(n = 1, 2, \ldots)$, $\lim_{n\to\infty} k_n = k_0$,

$\lim_{n\to\infty} h_n = h_0$ and $|y_n(t)| \to |x(t)|$ $(n \to \infty)$ a.e. on G (otherwise, we may select subsequences with those properties). It is obvious that $k_0 > 0$, $h_0 > 0$. Let us consider a few cases.

(A′) $k_0 = h_0 = +\infty$. In this case, $k_n h_n/(k_n + h_n) \to +\infty$ $(n \to \infty)$, a contradiction.

(B′) $k_0 \leqslant h_0 < +\infty$. By Lemma 5, there exists $\delta > 0$ such that for any $e \subset G$ with $\mu(e) < \delta$, we have

$$\|2x\chi_{G\setminus e}\|_M \leqslant 2 - 1/2h_0. \tag{12′}$$

Choosing $G_0 \subset G$ such that $\mu(G/G_0) < \delta$ and $|y_0(t)| \to |x(t)|$ $(n \to \infty)$ uniformly on G_0, by $M(u_1)/u_1 \leqslant M(u_2)/u_2$ $(u_1 \leqslant u_2)$ we have

$$\begin{aligned} 2 + 2/n &> \|x + y_n\|_M + \|x - y_n\|_M \\ &\geqslant \frac{1}{h_n} + \frac{1}{k_n} + \frac{1}{k_n} \int\limits_{G_0(x(t)y_n(t) \geqslant 0)} M\big(k_n(x(t) + y_n(t))\big)\,dt + \\ &\qquad + \frac{1}{h_n} \int\limits_{G_0(x(t)y_n(t) < 0)} M\big(h_n(x(t) - y_n(t))\big)\,dt \\ &\geqslant \frac{1}{h_n} + \frac{1}{k_n}\Big\{1 + \int\limits_{G_0} M\big(k_n(|x(t)| + |y_n(t)|)\big)\,dt\Big\}. \end{aligned}$$

Let $n \to \infty$; by (12′) we obtain a contradiction:

$$2 \geqslant \frac{1}{h_0} + \|2x\chi_{G_0}\|_M \geqslant \frac{1}{h_0} + 2 - \frac{1}{2h_0} = 2 + \frac{1}{2h_0}.$$

(C′) $k_0 < h_0 = +\infty$. For any $\eta > 0$, write $G_\eta = G(|x(t)| \geqslant \eta)$, $G_n = G(x(t)\,y_n(t) < 0)$ $(n = 1, 2, \ldots)$; then

$$\mu(G_n \cap G_\eta) \to 0 \quad (n \to \infty). \tag{13}$$

In fact,

$$\begin{aligned} 2 \geqslant \|x - y_n\|_M &\geqslant \frac{1}{h_n} \int\limits_{G_n \cap G_\eta} M\big(h_n(|x(t)| + |y_n(t)|)\big)\,dt \geqslant \frac{1}{h_n} \int\limits_{G_n \cap G_\eta} M\big(h_n x(t)\big)\,dt \\ &\geqslant \frac{1}{h_n} M(h_n \eta)\,\mu(G_n \cap G_\eta). \end{aligned}$$

Hence, we have

$$\mu(G_n \cap G_\eta) \leqslant \frac{2h_n}{M(h_n \eta)} \to 0 \quad (n \to \infty).$$

For any $\varepsilon > 0$, by Lemma 5, there exists $\delta > 0$ such that $\|2x\chi_{G\setminus e}\|_M > 2-\varepsilon$ whenever $e \subset G$ satisfies $\mu(e) < \delta$.

By (13), there exists a subsequence $\{G_m\}$ of $\{G_n\}$ such that, for each m, we have

$$\mu(G_m \cap G_\varepsilon) < \delta/2^{m+1}.$$

Choose $G_0 \subset G$ such that $\mu(G_0) < \frac{1}{2}\delta$ and such that $|y_n(t)| \to |x(t)|$ $(n \to \infty)$ uniformly on $G \setminus G_0$. Then $\mu(G') < \delta$, where

$$G' = G_0 \cup \bigcup_{m=1}^{\infty} (G_m \cap G_\varepsilon) = G_0 \cup \Big[\Big(\bigcup_{m=1}^{\infty} G_m\Big) \cap G_\varepsilon\Big].$$

Noticing that

$$G \setminus G' \subset \Big(G \setminus \bigcup_{m=1}^{\infty} G_m\Big) \cup \Big[\Big(\bigcup_{m=1}^{\infty} G_m\Big) \setminus (G_0 \cup G_\varepsilon)\Big],$$

we obtain

$$1+\frac{1}{m} > \|x+y_m\|_M \geqslant \frac{1}{k_m}\Big[1+\int\limits_{G\setminus \bigcup_{m=1}^{\infty} G_m} M\big(k_m(|x(t)|+|y_m(t)|)\big)\,dt\Big]$$

$$\geqslant \frac{1}{k_m}\Big[1+\int\limits_{G\setminus G'} M\big(k_m(|x(t)|+|y_m(t)|)\big)\,dt - $$

$$- \int\limits_{(\bigcup_{m=1}^{\infty} G_m)\setminus(G_0\cup G_\varepsilon)} M\big(k_m(|x(t)|+|y_m(t)|)\big)\,dt\Big]$$

$$\geqslant \frac{1}{k_m}\Big[1+\int\limits_{G\setminus G'} M\big(k_m(|x(t)|+|y_m(t)|)\big)\,dt - $$

$$- \int\limits_{G\setminus(G_0\cup G_\varepsilon)} M\big(k_m(|x(t)|+|y_m(t)|)\big)\,dt\Big]$$

$(n = 1, 2, \ldots)$. Let $m \to \infty$; we get

$$(14) \qquad 1 \geqslant \frac{1}{k_0}\Big[1+\int\limits_{G\setminus G'} M\big(k_0 2x(t)\big)\,dt - \int\limits_{G\setminus(G_0\cup G_\varepsilon)} M\big(k_0 2x(t)\big)\,dt\Big]$$

$$\geqslant \|2x\chi_{G\setminus G'}\|_M - \frac{1}{k_0} M(2k_0\varepsilon)\,\mu(G) \geqslant 2-\varepsilon+\frac{1}{k_0} M(2k_0\varepsilon)\,\mu(G).$$

Since ε is arbitrary, (14) gives a contradiction, completing the proof.

References

[1] Chen Shutao, *Non-squareness of Orlicz spaces* (in Chinese), Chinese Ann. Math. 6A (1985), 619–624.

[2] M. Denker, R. Kombrink, *On B-convex Orlicz spaces*, Lect. Notes Math. 79 (1979), 87–95.

[3] H. Hudzik, *Flat Musielak–Orlicz spaces under Luxemburg norm*, Bull. Pol. Acad. Sci. 32 (1984), 203–208.

[4] —, *Uniformly non-$l_n^{(1)}$ Orlicz spaces with Luxemburg norm*, Studia Math. (to appear).

[5] R. C. James, *Uniformly non-square Banach spaces*, Ann. Math. 80 (1964), 542–550.

[6] A. Kamińska, *On uniform convexity of Orlicz spaces*, Indag. Math. 44 (1982), 27–36.

[7] L. B. Kantorovich, G. P. Akilov, *Functional Analysis*, 1977.

[8] M. A. Krasnoselskiĭ, Ya. B. Rutickiĭ, *Convex functions and Orlicz spaces*, Groningen 1961.

[9] A. J. Pach, M. A. Smith, B. Turett, *Flat Orlicz spaces*, Proc. Amer. Soc. 81 (1981), 528–530.

[10] H. P. Rosenthal, *Some recent discoveries in the isomorphic theory of Banach spaces*, Bull. Amer. Math. 84 (1978), 803–831.

[11] J. J. Schäffer, *Geometry of sphere in normed spaces*, Marcel Dekker, Inc. New York and Basel 1970.

[12] K. Sundaresan, *Uniformly non-square Orlicz spaces*, Nieaw. Arch. Wish. 14 (1966), 31–39.

[13] Wang Tingfu, *Uniformly non-$l_n^{(1)}$ condition of Orlicz spaces*, Chinese Science Exploration 5 (1985), 125–126.

[14] Wang Yuwen, *Weakly sequential completeness of Orlicz spaces* (in Chinese), Chinese Northeast J. Math. 1 (1985), 241–246.

[15] Wu Congxin, Zhao Shanzhong, Chen Junao, *Formulae of Orlicz norm and the condition of rotundity on Orlicz spaces* (in Chinese), Acta of Harbin Institute of Technology 2 (1978), 1–12.

DEPARTMENT OF MATHEMATICS
HARBIN TEACHERS UNIVERSITY
HARBIN, CHINA