ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XVII (1973)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XVII (1973)

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On a topological characterization of simple closed curve on the plane

1. Introduction. By a *simple closed curve* we mean a set homeomorphic with a unit circle $x^2+y^2 = 1$ or, equivalently, a set of points which is a union of two arcs axb and ayb such that $axb \cap ayb = \{a\} \cup \{b\}$.

A set X is said to be *hereditarily locally connected* provided that each connected subset of X is locally connected.

A set X satisfying the condition

$$(\alpha) \qquad X = \bigcup_{i=1}^{\infty} [a, a_i], \textit{ where } [a, a_i] \textit{ denotes the arc with end-points } a \textit{ and } a_i, \ [a, a_i] \subset [a, a_{i+1}], \ a_i \neq a_{i+1} \textit{ for each } i \textit{ and } a = \lim_{i\to\infty} a_i,$$

may still be rather complicated topologically. However, we shall show that when X is a plane set (i.e., X is lying in the plane) and a hereditarily locally connected one, X must be a simple closed curve (Theorem 2).

This result is simultaneously a partial solution of the unsolved problem proposed by Professor B. Knaster at his topological seminar in Wrocław: is X a simple closed curve if X is a plane point set satisfying condition (α) and (only) locally connected? The following not simple closed curve C on the plane satisfying condition (α) shows that always the condition of local connectedness is an essential one: C is the union of 1° curve $y = \sin\pi/x$, $0 < x \leqslant 1$, 2° segment $x = 0$, $-1 \leqslant y \leqslant 1$, 3° arc of the circle (with center $(1, 1)$ and radius 1) joining points $(0, 1)$ and $(1, 0)$.

Theorem 1 concerns a general property of the plane. We adopt the definition of a topological limit given in [1], p. 245.

Throughout the paper $\delta(A)$ denotes the diameter of set A, $\bar{A}$ the closure of A in Euclidean metric $\varrho(a, b)$ on the plane.

2. A sequence of arcs. Now we shall prove a theorem which concerns a sequence of arcs on the plane and which will be used in the proof of Theorem 2.

THEOREM 1. *For each convergent sequence of mutually disjoint arcs* $[a_n, b_n]$, *lying on the plane and with end-points convergent to different points* a *and* b *respectively, there exists a convergent sequence of subarcs* $[a'_{n_i}, b'_{n_i}]$ *disjoint with its topological limit and with end-points convergent, also respectively, to points* a *and* b.

Proof. Assume, a contrario, that this theorem is false, then for each sequence of subarcs $[a'_{n_i}, b'_{n_i}] \subset [a_{n_i}, b_{n_i}]$ convergent to its topological limit G with end-points convergent to a and b we have

$$(0) \qquad [a'_{n_i}, b'_{n_i}] \cap G \neq 0 \quad \text{for } n_i = 1, 2, \ldots$$

Therefore there exists a sequence of arcs $[a_k, b_k]$ satisfying

$$(1) \qquad [a_k, b_k] \cap [a_m, b_m] = 0 \quad \text{for } k \neq m$$

convergent to topological limit G with end-points convergent to a and b:

$$(2) \qquad \lim_{k\to\infty} a_k = a \neq b = \lim_{k\to\infty} b_k,$$

such that

$$(3) \qquad [a_k, b_k] \cap G \neq 0 \quad \text{for } k = 1, 2, \ldots$$

By (2) we have $a, b \in G$. Let K_a and K_b be closed discs with centres in a and b, and with diameters such that

$$(4) \qquad K_a \cap K_b = 0,$$

(5) *there exist at least three different arcs* $[a_{k_i}, b_{k_i}]$ *such that sets* $Z_i = [a_{k_i}, b_{k_i}] \cap G - (K_a \cup K_b)$ *are non-empty for* $i = 1, 2, 3$.

and

(6) *sets* Z_i *are lying on the subarcs of arcs* $[a_k, b_k]$ *joining* K_a *and* K_b *for* $i = 1, 2, 3$.

We say that K is a subarc $\subset [a_k, b_k]$ *joining sets* A *and* B if for each subarc $L \subset [a_k, b_k]$ inequality $L \cap A \neq 0 \neq L \cap B$ implies that $L \supset K$.

Let us notice that for discs K_a and K_b with sufficiently small diameters condition (4) and (5) are satisfied: (4) follows from $a \neq b$, (5) from (1) and (3). However, condition (6) need not always be satisfied. The next simple Sierpiński's example (see [2], p. 115) will show it. Let the arc $[a_n, b_n]$ be a union of arc $r = 2^{-n}$, $\pi/2 \leqslant \theta \leqslant 2\pi$ (in polar coordinates) and the segment $0 \leqslant x \leqslant 1$, $y = 2^{-n}$. For $n = 1, 2, \ldots$ we have a sequence of arcs convergent to segment $0 \leqslant x \leqslant 1$, $y = 0$ and with end-points convergent, respectively, to $a = (0, 0)$ and $b = (1, 0)$. It is easy to see that for this sequence there do not exist discs K_a and K_b satisfying (6).

Therefore we consider a sequence of arcs $[a_k, b_k]$ satisfying (1), (2) and (3) such that there do not exist discs K_a, K_b satisfying (6). We may assume that for n sufficiently large discs K_a^n and K_b^n with diameters equal

$2/n$ satisfies (4), (5) and do not satisfy (6). Hence for fixed n directly follows that there exists a sequence of subarcs $[a_s^n, b_s^n]$ joining K_a^n and K_b^n and disjoint with its topological limit $G^n \subset G-(K_a^n \cup K_b^n)$. For each n we choose one arc $[a^n, b^n]$ from a sequence $\{[a_s^n, b_s^n]\}_{s=1}^{\infty}$ and so we get a sequence $\{[a^n, b^n]\}_{n=1}^{\infty}$. It is easy to see that we may choose, in this way, a sequence of arcs disjoint with its topological limit and with end-points convergent respectively to point a and b, which contradicts (0).

It remains to consider the case such that for every sequence of arcs $[a_k, b_k]$ satisfying conditions (1), (2) and (3) there exist discs K_a and K_b satisfying (4), (5) and (6). Suppose now that we have discs K_a and K_b and all arcs (subarcs of arcs $[a_k, b_k]$) joining them. Then (4) and (5) implies

(7) *in the midst of these arcs there are at least three ones having points in common with G,*

hence we have on the plane:

(8) *a figure containing two discs joined by disjoint arcs.*

Now, let us consider an upper semicontinuous decomposition of the plane such that its only non-degenerate elements are discs K_a and K_b. Then there exists a continuous function F from the plane to plane such that F is const on K_a and K_b (see [2], p. 356, Theorem 9).

An image of figure (8) is a set containing two points $F(K_a)$ and $F(K_b)$ joined by sequence of arcs having only end-points in common and besides being disjoint. By (7) in the midst of these arcs, for at least three arcs the intersection of these arcs and G is non-empty, but it contradicts with theorem (see [2], p. 360, Theorem 3). This completes the proof.

Now we shall show that if a sequence of arcs is assumed to be in Euclidean 3-space instead of the plane, the conclusion of Theorem 1 need not follow. We consider well-known hereditarily locally connected Urysohn's continuum see ([4], p. 46, example 4).

EXAMPLE 1. Let $p_1, p_2, \ldots$ be a sequence of prime numbers greater than 2, G be segment $[(0,0,0), (1,0,0)]$ and $[a_n, b_n]$ be the arc containing p_n-2 semicircles joining in turn points:

$$(1/p_n, 0, 0), (2/p_n, 0, 0), \ldots, (1-1/p_n, 0, 0)$$

lying on the semiplane $y \geqslant 0$, $z = y/n$. Obviously a sequence of arcs $[a_n, b_n]$ converges to G, $\lim_{n\to\infty} a_n = (0,0,0)$, $\lim_{n\to\infty} b_n = (1,0,0)$ and $[a_k, b_k] \cap [a_m, b_m] = 0$ for $k \neq m$. However, it easy to see that this sequence does not satisfy the conclusion of Theorem 1.

3. A simple closed curve. This section is devoted to the proof that if a set X on the plane satisfies condition (α) and is hereditarily locally connected, then X is a simple closed curve. First we prove two lemmas.

LEMMA 1. *If in the set X satisfying (α) there does not exist a sequence of points $b_i \in [a_i, a_{i+1}]$ such that*

$$(9) \qquad b = \lim_{i\to\infty} b_i, \quad b \in X \text{ and } b \neq a,$$

then for each point p:

$$(10) \qquad p \in \{X-(a, a_i)\}, \quad p \neq a \text{ and } p \neq a_i$$

a set $\{X-(a, a_i)\}-\{p\}$ is not connected.

Proof. Let p satisfy (10); then from condition (α) we have for $i, k = 1, 2, \ldots$

$$\{X-(a, a_i)\} = [a_i, p] \cup (p, a_{i+k}] \cup \bigcup_{j=i+k}^{\infty} [a_j, a_{j+1}] \cup \{a\}.$$

Letting $K = [a_i, p)$ and $L = (p, a_{i+k}] \cup \bigcup_{j=i+k}^{\infty} [a_j, a_{j+1}] \cup \{a\}$ we get

$$(11) \qquad \{X-(a, a_i)\}-\{p\} = K \cup L, \quad a_i \in K \text{ and } a \in L.$$

From (α) we have

$$(12) \qquad K \cap L = 0.$$

K is obviously closed in $\{X-(a, a_i)\}-\{p\}$ and we shall show that L is closed in $\{X-(a, a_i)\}-\{p\}$ also. In fact, because each sequence of points $b_j \in [a_j, a_{j+1}]$ is convergent to a and so $\bigcup_{j=i+k}^{\infty} [a_j, a_{j+1}] \cup \{a\}$ is closed. Hence L as the union of two closed sets is clearly closed. Then by (11) and (12) the set $\{X-(a, a_i)\}-\{p\}$ is not connected, our lemma is proved.

LEMMA 2. *If on the plane a set X is satisfying condition (α) and is hereditarily locally connected, then for each sequence of points $b_j \in [a_i, a_{i+1}]$ we have $\lim_{i\to\infty} b_i = a$.*

Proof. We shall consider two cases:

Case I. In X there exists a sequence of points $b_i \in [a_i, a_{i+1}]$ such that

$$\lim_{i\to\infty} b_i = b, \quad b \neq a \text{ and } b \in X.$$

Let us consider a sequence of arcs $[a_i, b_i] \subset [a_i, a_{i+1}]$. Since all arcs $[a_i, b_i] \subset X$ and the set X satisfies condition (α) hence there exists a convergent subsequence of these arcs, which are denoted $[a_i, b_i]$ also. Hence by Theorem 1 we get

(13) *in X there exists a convergent sequence of disjoint arcs $[a_i', b_i']$ disjoint with its topological limit and $\lim_{i\to\infty} a_i' = a$, $\lim_{i\to\infty} b_i' = b$.*

It is obvious that for n sufficiently large

$$\delta([a_i', b_i']) > \varrho(a, b)/2. \tag{14}$$

Now we shall show that

(15) *there exists a connected subset $S \subset X$ such that $b \in S$ and S is not locally connected at point b.*

In fact, because X is locally connected, let $T \subset X$ be a connected neighbourhood of the point a and such that

$$\delta(T) < \varrho(a, b)/4. \tag{16}$$

From (13) there exists i_0 such that for $i > i_0$, $a_i' \in T$. Let us define a set

$$S = T \cup \bigcup_{i>i_0}^{\infty} [a_i', b_i'] \cup \{b\}.$$

The set S is a union of T, arcs $[a_i', b_i']$ which are not disjoint with T and a point b such that $b \in \overline{T \cup \bigcup_{i>i_0}^{\infty} [a_i', b_i']}$ and so, clearly, S is connected. From the definition of S and fact that in this case $b \in X$ we get $S \subset X$. A set $S-T$ can be written in the form:

$$S-T = M_i \cup N_i \quad \text{for } i > i_0, \tag{17}$$

where $M_i = \bigcup_{j=i_0+1}^{i} [a_j', b_j'] - T$ and $N_i = \bigcup_{j=i+1}^{\infty} [a_j', b_j'] - T \cup \{b\}$. We note that, for each $i > i_0$, $b_i' \in M_i$ and $b \in N_i$, because arcs $[a_j, b_j]$ are disjoint it follows that

$$M_i \cap N_i = 0 \quad \text{for each } i > i_0. \tag{18}$$

The set M_i is closed in $S-T$. The closure $\overline{N}_i$ of N_i in $S-T$ is such that $\overline{N}_i \subset N_i \cup G$, where G is a topological limit of sequence of $[a_i', b_i']$. But from (13) it follows that $M_i \cap G = 0$ hence $\overline{N}_i = N_i$ and so by (17) and (18) we get

(19) *$S-T$ is not connected.*

Let V be a connected subset of S such that $b \in V$ and for arbitrary i $b_i \in V$; then by (19), (14) and (16) we have

$$\delta(V) > \delta([a_i', b_i']) - \delta(T) > \varrho(a, b)/4$$

but it follows by theorem (see [2], p. 161. Theorem 2) that (15) holds (i.e., S is not locally connected at point b).

But X is hereditarily locally connected and $S \subset X$ is connected, and so S is locally connected, particularly locally connected at point b, which contradicts (15). Case I does not hold.

Now we shall show that the second case does not hold either.

Case II. There does not exist in X a sequence of points $\{b_i\}$ convergent to $b \in X$ and $b \neq a$, but X containing a sequence $\{b_i\}$ such that $\lim_{i\to\infty} b_i = c$, $c \neq a$ and $c \in E^2 - X$, where E^2 denotes a plane.

The set X can be written in the form:

$$X = [a, a_1] \cup Z, \quad \text{where } Z = \bigcup_{i=1}^{\infty} [a_i, a_{i+1}] \cup \{a\}. \tag{20}$$

Since $[a, a_1]$ is closed in X and Z is also closed in X ($\lim_{i\to\infty} b_i = c$, $c \notin X$) we get two closed sets such that $[a, a_1] \cup Z$ is locally connected (from (20) and from fact that X is locally connected) and $[a, a_1] \cap Z$ is locally connected too, because $[a, a_1] \cap Z = \{a\} \cup \{a_1\}$. Then by theorem (see [2], p. 164, Theorem 10)

(21) *Z is locally connected.*

In this case there does not exist a sequence $\{b_i\}$ such that $\lim_{i\to\infty} b_i = b$, $b \neq a$ and $b \in X$, hence by Lemma 1 for each b_i a set $Z - \{b_i\}$ is not connected. Then each connected subset $W \subset X$ which contains points a_i and a it must contain points b_i and hence

$$\delta(W) > \delta([a_i, b_i]) > \varrho(a, b)/2,$$

and by theorem (see [2], p. 161, Theorem 2) we get that a set Z is not locally connected, which contradicts (21) and so case II does not hold. This completes the proof of Lemma 2.

Theorem 2. *A plane set X is a simple closed curve if and only if X is hereditarily locally connected and can be written in form* (α).

Proof. If X is a simple closed curve, then, obviously, X is hereditarily locally connected and can be written in form (α).

Conversely, suppose that a plane set X is hereditarily locally connected and can be written in form (α). Let $X = [a, a_1] \cup Z$ (see (20)); then by Lemma 2 there does not exist in X a sequence of points $b_i \in [a_i, a_{i+1}]$ convergent to $b \in X$ and $b \neq a$, hence by Lemma 1 for each point $p \in Z$ such that $p \neq a_1$ and $p \neq a$ a set $Z - \{p\}$ is not connected and so by theorem (see [2], p. 119, Theorem 1) Z is arc with end-points a and a_1. It is easy to see from (α) that $[a, a_1] \cap Z = \{a\} \cup \{a_1\}$, hence X is a union of two arcs $[a, a_1]$ and Z having only end-points in common and so X is a simple closed curve. This completes the proof.

Now we shall show that if X is assumed to be in Euclidean 3-space instead of the plane the conclusion of Theorem 2 need not follow.

Example 2. Let A_n denote an arc $[a_n, b_n]$ in E^3 from example 1 and let B_{0j} and B_{1j} denote, respectively, semicircles joining points

$$(1/p_{2j}, 0, 0), (1/p_{2j+1}, 0, 0)$$

and

$$(1-1/p_{2j-1}, 0, 0), (1-1/p_{2j}, 0, 0)$$

lying in a semiplane $y \leqslant 0$, $z = 0$.

Let

$$X = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{i=0}^{1} \bigcup_{j=1}^{\infty} B_{ij}.$$

The set X can be written in form (α) if we put $a = (1/3, 0, 0)$ and $a_i = A_i \cap P$, where P denotes a plane in E^3 which is perpendicular to the plane $z = 0$ and such that $a \in P$. The set X is hereditarily locally connected (see [3], p. 323) but is not compact (the closure of X contains a segment $[(0, 0, 0), (1, 0, 0)]$) and so X is not a simple closed curve.

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