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GRZEGORZ GROMADZKI (Toruń)

Abelian groups of automorphisms of compact non-orientable Klein surfaces without boundary

1. Introduction. Let X be a compact Klein surface without boundary of algebraic genus $g \geqslant 2$. Singerman [6] showed that the order of a group of automorphisms of X is bounded above by $84(g-1)$. On the other hand, Bujalance showed that every finite group can be represented as a group of automorphisms of a compact Klein surface without boundary [2], [3].

Thus this bound may be considered as a particular case of finding the minimum genus of surfaces of genus $p \geqslant 3$ for which a given group G is a group of automorphisms. The case of cyclic group was considered by Bujalance [2]. In this paper we consider the above problem for finite abelian groups. The corresponding problem for compact orientable Klein surfaces was solved by Maclachlan [5].

2. Preliminaries. By a *compact Klein surface without boundary* we mean a compact surface without boundary together with a dianalytic structure [1]. It is well known that a compact Klein surface may be expressed as C^+/Γ, where C^+ is upper half complex plane with hyperbolic structure and Γ is a certain non-Euclidean crystalographic group (NEC group).

An *NEC group* is a discrete subgroup Γ of the group $\mathscr{G}$ of isometries of the non-Euclidean plane C^+ (including those which reverse orientation-reflections and glide-reflections) with compact quotient space C^+/Γ.

Let $\mathscr{G}^+$ denote the subgroup of index 2 in $\mathscr{G}$ consisting of orientation preserving isometries. An NEC group contained in $\mathscr{G}^+$ is called *Fuchsian group*, otherwise it is called a *proper NEC group*. Given an NEC group Γ let $\Gamma^+ = \Gamma \cap \mathscr{G}^+$ be the canonical Fuchsian subgroup of Γ. Macbeath and Wilkie [4], [7] associated to every NEC group a signature that has the form

$$(1) \qquad (g, \pm, [m_1, \ldots, m_r], \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})$$

and determines the algebraic structure of the group.

The numbers m_i $(m_i \geqslant 2,\ r \geqslant 0)$ are the *periods*, the brackets $(n_{i1}, \ldots, n_{is_i})$ $(k \geqslant 0,\ s_i \geqslant 0,\ n_{ij} \geqslant 2)$ are the *period cycles*, and $g \geqslant 0$ is called the *orbit genus*.

Sometimes we will denote the period cycles shortly by $C_1, \ldots, C_k$ and in that way the signature just defined can be written in the form

$$(g, \pm, [m_1, \ldots, m_r], \{C_1, \ldots, C_k\}).$$

The group with signature (1) has a presentation with the following generators

(i) $x_1, \ldots, x_r$,
(ii) $e_1, \ldots, e_k$,
(iii) $c_{10}, \ldots, c_{1s_1}, \ldots, c_{k0}, \ldots, c_{ks_k}$,
(iv) $a_1, b_1, \ldots, a_g, b_g$ (if the sign is $+$), $d_1, \ldots, d_g$ (if the sign is $-$),

subject to the relations

(i) $x_i^{m_i} = 1$, $i = 1, \ldots, r$,
(ii) $c_{i0}^2 = c_{ij}^2 = (c_{i,j-1}c_{ij})^{n_{ij}} = 1$, $i = 1, \ldots, k$; $j = 1, \ldots, s_i$,
(iii) $e_i^{-1}c_{i0}e_ic_{is_i} = 1$, $i = 1, \ldots, k$,
(iv) $x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1$ (if the sign is $+$), $x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_g^2 = 1$ (if the sign is $-$).

Hereafter the letters x, a, b, c, d, e will be used for these canonical generators of the group only. Let $\alpha\colon \Gamma \to Z_2$ be a homomorphism which maps all x, a, b, e onto $+1$ and all c, d onto -1. We denote $\operatorname{Ker}\alpha$ by Γ^+. A group with the above presentation is called in this paper a *group with signature*.

Every NEC group Γ has an associated fundamental region whose area depends only on the group and not the region chosen. It is given by

$$|\Gamma| = 2\pi\Big(\eta g + k - 2 + \sum_{i=1}^{r}(1 - 1/m_i) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - 1/n_{ij})/2\Big), \tag{2}$$

where $\eta = 1$ if the sign is $-$ and $\eta = 2$ if the sign is $+$.

Conversely, if for a group with signature Γ the right-hand side expression of (2) is greater than zero, then Γ can be realized as an NEC group with signature (1) [8].

If Λ is a subgroup of finite index in an NEC group Γ, then it is an NEC group and

$$[\Gamma : \Lambda] = |\Lambda|/|\Gamma|.$$

An NEC group Γ_p is called the *group of a non-orientable surface* if it has signature $(p, -, [-], \{-\})$ where $[-]$ indicates that the signature has no periods and $\{-\}$ that the signature has no period cycles.

A homomorphism Θ of a proper NEC group Γ into a finite group G is called *non-orientable surface-kernel homomorphism* if Ker Θ is a group of surface and $\Theta(\Gamma^+) = G$.

The following theorems will be basic for our considerations.

THEOREM 2.1 (Singerman [6]). *A necessary and sufficient condition for a finite group G to be a group of automorphisms of a non-orientable Klein surface of genus $p \geqslant 3$ is that there exists a proper NEC group Γ and a non-orientable surface-kernel homomorphism $\Theta\colon \Gamma \to G$.*

THEOREM 2.2 (Bujalance [2]). *A homomorphism Θ of a proper NEC group with signature* (1) *into finite group G is a non-orientable surface-kernel homomorphism if and only if $\Theta(c_{ij})$ has order 2, $\Theta(x_i)$ has order m_i, $\Theta(c_{i,j-1}c_{ij})$ has order n_{ij} and $\Theta(\Gamma^+) = G$.*

For a finite group G, let $\mathscr{K}_G$ denote the class of all proper NEC groups Γ for which there exists a non-orientable surface-kernel homomorphism $\Theta\colon \Gamma \to G$.

If G is an abelian group then $\Theta(c_{i,j-1}c_{ij})^2 = 1$. Thus all $n_{ij} = 2$. Moreover, there are no period cycles of the length 1, otherwise $1 = \Theta(ec_{i0}e^{-1}c_{i1}) = \Theta(c_{i0}c_{i1})$ what is impossible since all n_{ij} are greater than 1. Similarly, if G is of odd order, then the set of period cycles is empty. Thus we have the following corollary:

COROLLARY 2.3. *Let A be a finite abelian group. Then any $\Gamma \in \mathscr{K}_A$ has a signature $(g, \pm, [m_1, \ldots, m_r], \{(2^{s_1}), \ldots, (2^{s_k}), (-)^l\})$, where $s_1, \ldots, s_k \geqslant 2$, (2^s) denotes $(2, \overset{s}{\ldots}, 2)$ and $(-)^l$ an empty period cycle repeated l-times. Moreover, if A is of odd order, then Γ has signature $(g, -, [m_1, \ldots, m_r], \{-\})$.*

By Theorem 2.1 we have that if G is a group of automorphisms of a non-orientable Klein surface of genus $p \geqslant 3$, then $G = \Gamma/\Gamma_p$ for some proper NEC group Γ. Thus $2\pi(p-2)/|G| = |\Gamma|$. In the case of abelian group A we have

$$(p-2)/|A| = g-2+\sum_{i=1}^{r}(1-1/m_i)+k+l+(s_1+\ldots+s_k)/4. \tag{3}$$

We want to find the minimal p for a fixed A and thus we have to minimize the right-hand side expression on the class $\mathscr{K}_A$. Let us denote this minimum for a group A by m_A.

3. Some reductions. An abelian group A is uniquely determined by its *invariants* $m_1, \ldots, m_s$, where $m_i | m_{i+1}$. When A is given in the form $A = Z_{m_1} \oplus \ldots \oplus Z_{m_s}$ we say that it is in canonical form and the generators of the cyclic groups will be called *canonical generators*.

For given homomorphism Θ of an NEC group Γ into abelian group A, denote by $\boldsymbol{x}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$ elements of A corresponding under the homomorphism Θ to the canonical generators of Γ.

LEMMA 3.1. *Let A be a non-cyclic abelian group different from $Z_2 \oplus Z_{2m}$ $(m \geqslant 1)$ and $Z_2 \oplus Z_2 \oplus Z_2$. Let Γ be a group with signature* (1) *for which $\Gamma^+ \neq \Gamma$ and let $\Theta\colon \Gamma \to A$ be a homomorphism such that $\Theta(x_i)$ has order m_i, $\Theta(c_{ij})$ has order 2, $\Theta(c_{i,j-1}c_{ij})$ has order n_{ij} and $\Theta(\Gamma^+) = A$. Then Γ is an NEC group and Θ is a non-orientable surface-kernel homomorphism.*

Proof. Assume that an abelian group A and a homomorphism $\Theta\colon \Gamma \to A$ satisfy hypothesis of the lemma. Arguing as in the proof of Corollary 2.3 we show that Γ has signature of the following type: $(g, \pm, [m_1, \ldots, m_r], \{(2^{s_1}), \ldots, (2^{s_k}), (-)^l\})$. Since $\Gamma^+ \neq \Gamma$, we have that $g > 0$ and the sign is $-$ when the set of period cycles is empty. Suppose that the group Γ is not an NEC group. Then the right-hand side of (3) for Γ is less than 0 and it is easy to check that the following is the complete list of such groups.

$$\begin{array}{ll} (0, +, [-], \{(-)\}), & (0, +, [m], \{(-)\}), \\ (0, +, [-], \{(-), (-)\}), & (0, +, [2], \{(2, 2)\}), \\ (0, +, [-], \{(2, 2, 2)\}), & (0, +, [-], \{(2, 2, 2, 2)\}), \\ (1, -, [-], \{(-)\}), & (1, -, [2], \{-\}), \\ (2, -, [-], \{-\}), & \end{array}$$

$$\begin{array}{l} (0, +, [2, 2], \{(-)\}), \\ (0, +, [-], \{(2, 2)\}), \\ (1, -, [-], \{-\}), \\ (1, -, [2, 2], \{-\}). \end{array}$$

Take Γ with signature $(0, +, [2, 2], \{(-)\})$, for example. Γ is generated by x_1, x_2, c, e subject to the relations $x_1^2 = x_2^2 = c^2 = e^{-1}cec = x_1x_2e = 1$. Now for given homomorphism $\Theta\colon \Gamma \to A$ satisfying the hypothesis, A is generated by $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{e}$. Since $\boldsymbol{e} = -(\boldsymbol{x}_1 + \boldsymbol{x}_2)$, we have that the third generator is redundant and hence A is generated by two elements of order 2. Thus A is not the group in question, a contradiction. The remining cases can be proved similarly.

For the rest of this section, let A be an abelian group satisfying the hypothesis of the previous lemma. For any $\Gamma \in \mathcal{K}_A$ denote $|\Gamma|/2\pi$ by $\mu(\Gamma)$.

LEMMA 3.2. *Let $\Gamma \in \mathcal{K}_A$ be a group with non-empty set of period cycles. Then there exists $\Gamma' \in \mathcal{K}_A$ with orbit genus zero such that $\mu(\Gamma') \leqslant \mu(\Gamma)$.*

Proof. Assume that Γ has signature $(g, +, [m_1, \ldots, m_r], \{C_1, \ldots, C_k\})$

and let Θ be a corresponding non-orientable surface-kernel homomorphism.

Consider the group Γ' with signature $(0, +, [n_1, \ldots, n_{r+2g}], \{C_1, \ldots, C_k\})$, where

$$n_i = \begin{cases} m_i, & i = 1, \ldots, r, \\ |\Theta(a_{i-r})|, & i = r+1, \ldots, r+g, \\ |\Theta(b_{i-(r+g)})|, & i = r+g+1, \ldots, r+2g, \end{cases}$$

and the following homomorphism Θ': $\Gamma' \to A$

$$\Theta'(x_i') = \begin{cases} \Theta(x_i), & i = 1, \ldots, r, \\ \Theta(a_{i-r}), & i = r+1, \ldots, r+g, \\ \Theta(b_{i-(r+g)}), & i = r+g+1, \ldots, r+2g, \end{cases}$$

$$\Theta'(c_{ij}') = \Theta(c_{ij}), \quad i = 1, \ldots, k;\ j = 0, \ldots, s_i,$$

$$\Theta'(e_i') = \begin{cases} \Theta(e_i), & i = 1, \ldots, k-1, \\ -(\Theta'(x_1') + \ldots + \Theta'(x_{r+2g}') + \Theta(e_1) + \ldots + \Theta(e_{k-1})), & i = k. \end{cases}$$

By Lemma 3.1, $\Gamma \in \mathscr{K}_A$. Moreover,

$$\mu(\Gamma') = \mu(\Gamma) - (1/n_{r+1} + \ldots + 1/n_{r+2g}) \leqslant \mu(\Gamma).$$

Now suppose that Γ has signature $(g, -, [m_1, \ldots, m_r], \{C_1, \ldots, C_k\})$, and let Θ be the corresponding homomorphism.

Consider the group Γ' with signature $(0, +, [n_1, \ldots, n_{r+g}], \{C_1, \ldots, C_k\})$, where

$$n_i = \begin{cases} m_i, & i = 1, \ldots, r, \\ |\Theta(d_{i-r} c_{10})|, & i = r+1, \ldots, r+g. \end{cases}$$

If some n_i are equal to 1, then we regard that this period does not appear in the signature.

Let Θ': $\Gamma' \to A$ be the homomorphism given by

$$\Theta'(x_i') = \begin{cases} \Theta(x_i), & i = 1, \ldots, r, \\ \Theta(d_{i-r} c_{10}), & i = r+1, \ldots, r+g, \end{cases}$$

$$\Theta'(c_{ij}') = \Theta(c_{ij}), \quad i = 1, \ldots, k;\ j = 0, \ldots, s_i,$$

$$\Theta'(e_i') = \begin{cases} \Theta(e_i), & i = 1, \ldots, k-1, \\ -(\Theta'(x_1) + \ldots + \Theta'(x_{r+g}) + \Theta(e_1) + \ldots + \Theta(e_{k-1})), & i = k. \end{cases}$$

Now, since $2(d_i + c_{10}) = 2d_i$, $(d_i + c_{10}) + (d_j + c_{10}) = d_i + d_j$, $(d_i + c_{10}) + (c_{10} + c_{kl}) = d_i + c_{kl}$, we have that $\Theta'(\Gamma^+) = A$ and thus, by Lemma 3.1, $\Gamma' \in \mathscr{K}_A$. Moreover, $\mu(\Gamma') = \mu(\Gamma) - (1/n_{r+1} + \ldots + 1/n_{r+g}) \leqslant \mu(\Gamma)$.

LEMMA 3.3. *Assume that* $\Gamma \in \mathscr{K}_A$ *has signature* $(0, +, [m_1, \ldots, m_r],$

$\{C_1, \ldots, C_k\}$). *Then there exists* $\Gamma' \in \mathscr{K}_A$ *with signature* $(0, +, [n_1, \ldots, n_s], \{C_1, \ldots, C_l\})$ *such that* $C_1 = (2^s)$, $C_2 = \ldots = C_l = (-)$ *and* $\mu(\Gamma') \leqslant \mu(\Gamma)$.

Proof. By Corollary 2.3 we have that $C_1 = (2^{s_1}), \ldots, C_n = (2^{s_n})$, $C_{n+1} = \ldots = C_k = (-)$, where $s_1, \ldots, s_n \geqslant 2$.

Firstly, note that we can assume that, for any $i \leqslant n$, we have $c_{ij} = c_{ij'}$ for $j < j'$ only if $j = 0$ and $j' = s_i$. If this is not the case for some i_0, let $0 < j_1 < \ldots < j_{s-1} < s_{i_0}$ be integers such that $c_{i_0,0}, c_{i_0,j_1}, \ldots, c_{i_0,j_{s-1}}$ are all different elements among $c_{i_0,0}, c_{i_0,1}, \ldots, c_{i_0 s_{i_0}}$. Consider the group Γ' with signature $(0, +, [m_1, \ldots, m_r], \{C'_1, \ldots, C'_k\})$, where $C'_i = C_i$ for $i \neq i_0$ and $C'_{i_0} = (2^s)$.

Let $\Theta'\colon \Gamma' \to A$ be a homomorphism defined by $\Theta'(x'_i) = \Theta(x_i)$ for $i = 1, \ldots, r$, $\Theta'(e'_i) = \Theta(e_i)$ for $i = 1, \ldots, k$ and

$$\Theta'(c'_{it}) = \begin{cases} \Theta(c_{it}), & i \neq i_0,\ t = 0, \ldots, s_i, \\ \Theta(c_{i_0,0}), & i = i_0,\ t = 0, s, \\ \Theta(c_{i_0,j_t}), & i = i_0,\ t = 1, \ldots, s-1. \end{cases}$$

Clearly, $\Gamma' \in \mathscr{K}_A$ and $\mu(\Gamma') \leqslant \mu(\Gamma)$; hence, repeating the procedure for other period cycles if necessary, we obtain the assertion.

Secondly, we can assume that for any two non-empty period cycles C_i, C_u we have $c_{ij} \neq c_{ut}$ for any $j = 0, \ldots, s_i$ and $t = 0, \ldots, s_i$. If this is not the case, let Γ' be a group with signature $(0, +, [m_1, \ldots, m_r], \{C'_1, \ldots, C'_k\})$, where $C'_i = C_i$ for $i \neq u$ and $C_u = (2^{s_u - 1})$ and let $\Theta'\colon \Gamma' \to A$ be a homomorphism defined by $\Theta'(x'_i) = \Theta(x_i)$ for $i = 1, \ldots, r$, $\Theta'(e'_i) = \Theta(e_i)$ for $i = 1, \ldots, k$ and

$$\Theta'(c'_{ij}) = \begin{cases} \Theta(c_{ij}), & i \neq u,\ j = 0, \ldots, s_i, \\ \Theta(c_{uj}), & i = u,\ j = 0, \ldots, t-1, \\ \Theta(c_{u,j+1}), & i = u,\ j = t, \ldots, s_u - 1. \end{cases}$$

Clearly, $\Gamma' \in \mathscr{K}_A$ and $\mu(\Gamma') \leqslant \mu(\Gamma)$, and hence, repeating the procedure for other period cycles if necessary, we obtain the assertion.

We will prove the lemma by induction on n, the number of non-empty period cycles. Clearly, it is true for $n = 1$. For $n > 1$ consider the group with signature $(0, +, [m_1, \ldots, m_r], \{C'_1, \ldots, C'_{n-1}, (-)^{k-n+1}\})$, where $C'_i = C_i$ for $i \leqslant n-2$ and $C'_{n-1} = (2^{s_{n-1} + s_n})$.

Let $\Theta'\colon \Gamma' \to A$ be a homomorphism given by $\Theta'(x'_i) = \Theta(x_i)$ for $i = 1, \ldots, r$, $\Theta'(e'_i) = \Theta(e_i)$ for $i = 1, \ldots, k$ and

$$\Theta'(c'_{ij}) = \begin{cases} \Theta(c_{ij}), & i = 1, \ldots, n-2,\ j = 0, \ldots, s_i, \\ \Theta(c_{n-1,j}), & i = n-1, \quad j = 0, \ldots, s_{n-1} - 1, \\ \Theta(c_{n,j-s_{n-1}}), & i = n-1, \quad j = s_{n-1}, \ldots, s_{n-1} + s_n - 1, \\ \Theta(c_{n-1,0}), & i = n-1, \quad j = s_{n-1} + s_n. \end{cases}$$

Two previous remarks guarantee that Θ' is a non-orientable surface-kernel homomorphism and thus $\Gamma' \in \mathscr{K}_A$. Clearly, $\mu(\Gamma') = \mu(\Gamma)$; hence the lemma.

LEMMA 3.4. *Assume that $\Gamma \in \mathscr{K}_A$ has signature $(g, -, [m_1, \ldots, m_r], \{-\})$, $g \geqslant 1$. Then there is $1 \leqslant i \leqslant g$ such that $d_i \in \Theta(\langle x_1, \ldots, x_r, d_1, \ldots, \hat{d}_i, \ldots, d_g\rangle^+)$, where $\langle x_1, \ldots, x_r, \hat{d}_1, \ldots, d_i, \ldots, d_g\rangle^+$ is the subgroup of Γ^+ generated by the elements in brackets with d_i omitted.*

Proof. Since $\Theta(\Gamma^+) = A$, $d_1 = w$ for some $w \in \Gamma^+$

Consider two cases.

(1) Exponent sum of d_1 in w is even. Then since $d_1^2 = (d_2^2 \ldots d_g^2 x_1 \ldots x_r)^{-1}$, we obtain that $d_1 \in \Theta(\langle x_1, \ldots, x_r, d_2, \ldots, d_g\rangle^+)$.

(2) Exponent sum of d_1 in w is odd. Then there exists $2 \leqslant i \leqslant g$ such that exponent sum of d_i in w is odd (otherwise $w \notin \Gamma^+$). Thus we have that $d_i = v$ for $v = wd_1^{-1} d_i \in \Gamma^+$ and exponent sum of d_i in v is even. As in the previous case we obtain that $d_i \in \Theta(\langle x_1, \ldots, x_r, d_1, \ldots, \hat{d}_i, \ldots, d_g\rangle^+)$.

PROPOSITION 3.5. *Let A be an abelian group of even order and assume that $\Gamma \in \mathscr{K}_A$ has signature $(g, -, [m_1, \ldots, m_r], \{-\})$. Then there exists $\Gamma' \in \mathscr{K}_A$ with signature $(0, +, [n_1, \ldots, n_s], \{(-)\})$ such that $\mu(\Gamma') \leqslant \mu(\Gamma)$.*

Proof. Note first that since $\Gamma^+ \neq \Gamma$, $g \geqslant 1$. By Lemma 3.4, there is $i \leqslant g$ such that $d_i \in \Theta(\langle x_1, \ldots, x_r, d_1, \ldots, \hat{d}_i, \ldots, d_g\rangle^+)$. Consider the group Γ' with signature $(0, +, [n_1, \ldots, n_{r+g-1}], \{(-)\})$, where

$$n_j = \begin{cases} m_j, & j = 1, \ldots, r, \\ |\Theta(d_{j-r})|, & j = r+1, \ldots, r+i-1, \\ |\Theta(d_{j-r+1})|, & j = r+i, \ldots, r+g-1, \end{cases}$$

and let $\Theta' \colon \Gamma' \to A$ be a homomorphism given by

$$\Theta'(x_j') = \begin{cases} \Theta(x_j), & j = 1, \ldots, r, \\ \Theta(d_{j-r}), & j = r+1, \ldots, r+i-1, \\ \Theta(d_{j-r+1}), & j = r+i, \ldots, r+g-1, \end{cases}$$

$\Theta'(e') = -(\Theta(x_1) + \ldots + \Theta(x_{r+g-1}))$ and let $\Theta'(c')$ be any element of order 2. Since A is generated by $x_1, \ldots, x_r, d_1, \ldots, \hat{d}_i, \ldots, d_g$ we have that $\Gamma' \in \mathscr{K}_A$. Moreover, $\mu(\Gamma') = \mu(\Gamma) - (1/n_{r+1} + \ldots + 1/n_{r+g-1}) \leqslant \mu(\Gamma)$.

PROPOSITION 3.6. *Let A be an abelian group of odd order and let $\Gamma \in \mathscr{K}_A$. Then there exists $\Gamma' \in \mathscr{K}_A$ with a signature $(1, -, [n_1, \ldots, n_s], \{-\})$ such that $\mu(\Gamma') \leqslant \mu(\Gamma)$.*

Proof. Note first that by 2.3 Γ has signature $(g, -, [m_1, \ldots, m_r], \{-\})$. Let $i \leqslant g$ be an integer such that $d_i \in \Theta(\langle x_1, \ldots, x_r, d_1, \ldots, \hat{d}_i, \ldots, d_g\rangle^+)$.

Consider the group Γ' with signature $(1, -, [n_1, \ldots, n_s], \{-\})$, where n_j are defined as in the previous proposition and let Θ': $\Gamma' \to A$ be a homomorphism defined on all x' as in the previous proposition and let $\Theta'(d)$ be the element of A satisfying $2 \cdot \Theta'(d) = -(\Theta'(x'_1) + \ldots + \Theta'(x'_{r+g-1}))$. Clearly, $\Gamma' \in \mathscr{K}_A$ and as in the previous proposition $\mu(\Gamma') \leqslant \mu(\Gamma)$.

Now let for given abelian group A of even order $\mathscr{K}_A^0$ denote the subclass of $\mathscr{K}_A$ consisting of all Γ having signatures of the following type $(0, [m_1, \ldots, m_r], \{(2^s), (-)^t\})$.

LEMMA 3.7. *Let $\Gamma \in \mathscr{K}_A^0$. Then there exists $\Gamma' \in \mathscr{K}_A$ with signature $(0, +, [-], \{(2^s), (-)^t\})$ or $(0, +, [m_1, \ldots, m_r], \{(-)^t\})$ such that $\mu(\Gamma') \leqslant \mu(\Gamma)$.*

Proof. Assume that Γ has signature $(0, +, [m_1, \ldots, m_r], \{(2^s), (-)^t\})$. If $s = 0$ or $r = 0$ then the assertion clearly holds (Γ has signature of the desired form). Suppose thus that $s \geqslant 2$ and $r > 0$. Let Θ be the corresponding homomorphism.

Suppose $m_k = 2$ for some k. Then two cases are possible.

(1) $x_k \in \langle c_{10}, \ldots, c_{1s}, c_2, \ldots, c_{t+1}\rangle^+$. Consider the group Γ' with signature $(0, +, [m_1, \ldots, \hat{m}_k, \ldots, m_r], \{(2^s), (-)^t\})$ and a homomorphism Θ': $\Gamma' \to A$ defined by

$$\Theta'(x'_i) = \begin{cases} \Theta(x_i), & i = 1, \ldots, k-1, \\ \Theta(x_{i+1}), & i = k, \ldots, r-1, \end{cases}$$

$$\Theta'(e'_i) = \begin{cases} \Theta(e_i), & i = 1, \ldots, t, \\ -(\Theta'(x'_1) + \ldots + \Theta'(x'_{r-1}) + \Theta(e_1) + \ldots + \Theta(e_t)), & i = t+1, \end{cases}$$

$\Theta'(c'_{1j}) = \Theta(c_{1j})$ for $j = 0, \ldots, s$ and $\Theta'(c'_l) = \Theta(c_l)$ for $l = 2, \ldots, t+1$. Clearly, $\Gamma' \in \mathscr{K}_A^0$ and obviously, $\mu(\Gamma') \leqslant \mu(\Gamma)$.

(2) $x_k \notin \langle c_{10}, \ldots, c_{1s}, c_2, \ldots, c_t\rangle^+$. Consider the group Γ' with signature $(0, +, [m_1, \ldots, \hat{m}_k, \ldots, m_r], \{(2^{s+1}), (-)^t\})$ and a homomorphism Θ': $\Gamma' \to A$ defined, on all x' and e' as in the previous case, and by

$$\Theta'(c'_{1j}) = \begin{cases} \Theta(c_{1j}), & i = 0, \ldots, s-1, \\ \Theta(c_{10} x_k), & i = s, \\ \Theta(c_{10}), & i = s+1, \end{cases}$$

$$\Theta'(c'_l) = \Theta(c_l), \quad l = 2, \ldots, t+1.$$

Clearly, $\Gamma' \in \mathscr{K}_A^0$ and $\mu(\Gamma') = \mu(\Gamma) - \frac{1}{4} \leqslant \mu(\Gamma)$.

Thus we can assume that if Γ has non-empty period cycle then no period is equal to 2.

Now suppose that $m_k = 3$ for some k. Consider in this case the group Γ' with signature $(0, +, [n_1, \ldots, n_r], \{(2^{s-1}), (-)^t\})$, where we understand that

(2^{s-1}) is an empty period cycle if $s-1=1$, $n_i = m_i$ for $i \neq k$ and $n_k = 6$. Let $\Theta'\colon \Gamma' \to A$ be a homomorphism defined by

$$\Theta'(x_i') = \begin{cases} \Theta(x_i), & i \neq k, \\ \Theta(x_k c_{10} c_{1,s-1}), & i = k, \end{cases}$$

$$\Theta'(e_i') = \begin{cases} \Theta(e_i), & i = 1, \ldots, t, \\ -(\Theta'(x_1') + \ldots + \Theta'(x_r') + \Theta(e_1) + \ldots + \Theta(e_t)), & i = t+1, \end{cases}$$

$$\Theta'(c_{1j}') = \begin{cases} \Theta(c_{1j}), & j = 0, \ldots, s-2, \\ \Theta(c_{10}), & j = s-1, \end{cases}$$

$$\Theta'(c_l') = \Theta(c_l), \quad l = 2, \ldots, t+1.$$

Clearly, $\Gamma' \in \mathscr{K}_A^0$ and since $1-\frac{1}{6} < (1-\frac{1}{3}) + \frac{1}{4}$, we have $\mu(\Gamma') \leqslant \mu(\Gamma)$.

Thus we can assume that if Γ has non-empty period cycle then all periods are $\geqslant 4$ and as in the proof of 3.3 we argue that $c_{1j} = c_{1j'}$ for $j < j'$ only if $j = 0$ and $j' = s$.

Let us consider two cases now.

(1) $s > r$. Let Γ' be the group with signature $(0, +, [-], \{(2^{s-r}), (-)^{r+t}\})$, where as earlier we regard (2^1) as an empty period cycle and let $\Theta'\colon \Gamma' \to A$ be a homomorphism defined by

$$\Theta'(e_i') = \begin{cases} \Theta(x_i), & i = 1, \ldots, r, \\ \Theta(e_{i-r}), & i = r+1, \ldots, r+t, \\ -(\Theta'(e_1') + \ldots + \Theta'(e_{r+t}')), & i = r+t+1, \end{cases}$$

$$\Theta'(c_{1j}') = \begin{cases} \Theta(c_{1j}), & j = 0, \ldots, s-r-1, \\ \Theta(c_{10}), & j = s-r, \end{cases}$$

$$\Theta'(c_l') = \begin{cases} \Theta(c_{1,s-r-1+l}), & l = 1, \ldots, r, \\ \Theta(c_{l-r+1}), & l = r+1, \ldots, r+t. \end{cases}$$

Γ' has the desired signature and since $1 \leqslant 1 - 1/m_i + \frac{1}{4}$ we have that $\mu(\Gamma') \leqslant \mu(\Gamma)$.

(2) $s \leqslant r$. Let Γ' be the group with signature $(0, +, [m_1, \ldots, m_{r-s+1}], \{(-)^{s+t}\})$ and let $\Theta'\colon \Gamma' \to A$ be a homomorphism defined by

$$\Theta'(x_i') = \Theta(x_i), \quad i = 1, \ldots, r-s+1,$$

$$\Theta'(e_i') = \begin{cases} \Theta(x_{r-s+1+i}), & i = 1, \ldots, s-1, \\ \Theta(e_{i-s+1}), & i = s, \ldots, s+t-1, \\ -(\Theta(x_1) + \ldots + \Theta(x_{r-s+1}) + \Theta'(e_1') + \ldots + \Theta'(e_{s+t-1}')), & i = s+t, \end{cases}$$

$$\Theta'(c_l') = \begin{cases} \Theta(c_{1,l-1}), & l = 1, \ldots, s, \\ \Theta(c_{l-s+1}), & l = s+1, \ldots, s+t. \end{cases}$$

Now, $\Gamma' \in \mathscr{K}_A^0$ is of desired signature and as in the first case we show that $\mu(\Gamma') \leqslant \mu(\Gamma)$.

Now for a given abelian group A of even order let $\mathscr{K}_A^R$ denote the subclass of $\mathscr{K}_A^0$ consisting of all Γ having signatures of one of the following types $(0, +, [-], \{(2^s), (-)^t\})$ or $(0, +, [m_1, \ldots, m_r], \{(-)^t\})$. We will call any group with such signature *reduced*.

Now we may summarize the results of this section.

PROPOSITION 3.8. *Let A be an abelian group of even order satisfying the hypothesis of Lemma* 3.1. *Then $m_A = \min\{\mu(\Gamma)\colon \Gamma \in \mathscr{K}_A^R\}$* (see the end of Section 2 for the definition of m_A).

PROPOSITION 3.9. *Let A be an abelian group of odd order. Then $m_A = \min\{\mu(\Gamma)\colon \Gamma \in \mathscr{K}_A\}$ and Γ has signature $(1, -, [m_1, \ldots, m_r], \{-\})$.*

4. Some elementary algebra.

THEOREM 4.1 (Maclachlan [5]). *In an abelian group A let $\xi_1, \ldots, \xi_r$ be elements such that*

$$\xi_1 + \xi_2 + \ldots + \xi_r = 0, \qquad m_1 \cdot \xi_1 = \ldots = m_r \cdot \xi_r = 0.$$

Then there are elements $\eta_1, \ldots, \eta_t$ of A, which generate the same subgroup of A as $\xi_1, \ldots, \xi_r$, satisfying

$$\eta_1 + \eta_2 + \ldots + \eta_t = 0, \qquad n_1 \cdot \eta_1 = \ldots = n_t \cdot \eta_t = 0,$$

where

$$\sum_{i=1}^{t} (1 - 1/n_i) \leqslant \sum_{j=1}^{r} (1 - 1/m_i)$$

and in addition the divisibility conditions holds: $n_1 | n_2 | \ldots | n_t$.

THEOREM 4.2 (Maclachlan [5]). *If*

$$A = Z_{m_1} \oplus \ldots \oplus Z_{m_k}, \qquad \text{where } m_i | m_{i+1},$$

and

$$A' = Z_{n_1} \oplus \ldots \oplus Z_{n_k}, \qquad \text{where } n_i | n_{i+1},$$

and there exists a homomorphism of A onto A', then $n_i | m_i$ for all i.

5. Following Maclachlan ideas. Let A be an abelian group of even order. Let $X = \{\xi_1, \ldots, \xi_r\}$ $(r \geqslant 0)$ be an ordered r-tuple of elements of A and let $K = \{\alpha_1, \ldots, \alpha_s\}$ $(s \geqslant 0)$ be an unordered s-tuple of trivial or of order 2 elements. Let $m_1, \ldots, m_r$ be the orders of $\xi_1, \ldots, \xi_r$. A couple $(X\colon K)$ is said to be *generating couple* if $X \cup K$ generates A. For given generating couple

$(X:K) = (\xi_1, \ldots, \xi_r \colon \alpha_1, \ldots, \alpha_s)$ let

$$\mu(X:K) = \begin{cases} s-1+\sum_{i=1}^{r-s}(1-1/m_i) & \text{if } s \leqslant r, \\ r-1+(s-r+1)/4 & \text{if } s > r. \end{cases}$$

For given reduced NEC group Γ in $\mathscr{K}_A$ let

$$X(\Gamma) = \{x_1, \ldots, x_r, e_1, \ldots, e_{t-1}\}, \quad K(\Gamma) = \{c_1 c_2, \ldots, c_1 c_t\}$$

or

$$X(\Gamma) = \{e_1, \ldots, e_t\},$$
$$K(\Gamma) = \{c_{10}c_{11}, \ldots, c_{10}c_{1,s-1}, c_{10}c_2, \ldots, c_{10}c_{t+1}\}$$

according as Γ has signature $(0, +, [m_1, \ldots, m_r], \{(-)^t\})$ or $(0, +, [-], \{(2^s), (-)^t\})$. Since $\Theta(\Gamma^+) = A$, $(X(\Gamma) \colon K(\Gamma))$ is a generating couple. It is also clear that $\mu(\Gamma) = \mu(X(\Gamma) \colon K(\Gamma))$.

Conversely, if A is an abelian group of even order with non-cyclic maximal 2-subgroup and satisfying hypothesis of Lemma 3.1 and $(X:K) = (\xi_1, \ldots, \xi_r \colon \alpha_1, \ldots, \alpha_s)$ is a generating couple, then there exists at least one reduced NEC group $\Gamma = \Gamma(X:K)$ in $\mathscr{K}_A$ such that $\mu(\Gamma(X:K)) = \mu(X:K)$. This is the group with signature $(0, +, [m_1, \ldots, m_{r-s}], \{(-)^{s+1}\})$ if $s \leqslant r$ and $(0, +, [-]; \{(2^{s-r+1}), (-)^r\})$ if $s > r$.

In order to define the corresponding homomorphism Θ consider two cases.

Case 1. There exists an element β of order 2 in A such that $\beta \neq \alpha_i$ for $i = 1, \ldots, s$.

If $s \leqslant r$, let Θ be the homomorphism defined by

$$\begin{aligned} \Theta(x_i) &= \xi_i && \text{for } i = 1, \ldots, r-s, \\ \Theta(e_i) &= \xi_{r-s+i} && \text{for } i = 1, \ldots, s \text{ and } -(\xi_1 + \ldots + \xi_r) \text{ for } i = s+1, \\ \Theta(c_l) &= \beta\alpha_l && \text{for } l = 1, \ldots, s \text{ and } \beta \text{ for } l = s+1. \end{aligned}$$

If $s > r$ note first that $s \geqslant 2$ (otherwise A is cyclic). Renumerate elements of K in such way that $\alpha_1, \ldots, \alpha_t$ are all distinct elements of order 2 in K. Since there are at least 3 different elements of order 2 in A, we can assume that $t \geqslant 2$. (If this is not the case, we can take two of these elements instead of α_1, α_2 if $t = 0$ and β instead of α_2 if $t = 1$.) Then the new tuple K' gives us the generating couple $(X:K')$ such that $\mu(X:K') = \mu((X:K))$. Now let

$$\begin{aligned} \Theta(e_i) &= \xi_i && \text{for } i = 1, \ldots, r \text{ and } -(\xi_1 + \ldots + \xi_r) \text{ for } i = r+1, \\ \Theta(c_{1j}) &= \beta\alpha_{[j-1]+1} && \text{for } j = 1, \ldots, s-r \text{ and } \beta \text{ for } j = 0, s-r+1, \\ \Theta(c_l) &= \alpha_{[s-r-2+l]+1} && \text{for } l = 2, \ldots, r+1, \end{aligned}$$

where $[i]$ means reduction mod t. Clearly, Θ is the homomorphism we have looked for.

Case 2. For every element β of order 2 in A there is $i \leqslant s$ such that $\beta = \alpha_i$. Renumerate if necessary elements of K in such way that $\alpha_1, \ldots, \alpha_t$ are all distinct elements of order 2 in K. Since A has non-cyclic maximal 2-subgroup, $t \geqslant 3$. Let $s > r$.

If $t > s-r$, let

$$\begin{aligned} \Theta(e_i) &= \xi_i && \text{for } i = 1, \ldots, r \text{ and } -(\xi_1 + \ldots + \xi_r) \text{ for } i = r+1, \\ \Theta(c_{1j}) &= \alpha_1 \alpha_{j+1} && \text{for } j = 1, \ldots, s-r \text{ and } \alpha_1 \text{ for } j = 0, s-r+1, \\ \Theta(c_l) &= \alpha_1 \alpha_{s-r+l} && \text{for } l = 2, \ldots, t-(s-r) \\ & && \qquad \text{and } \alpha_t \text{ for } l = t-(s-r)+1, \ldots, r+1. \end{aligned}$$

If $t \leqslant s-r$, let Θ be defined on all e as in the previous case and let $\Theta(c_{1j}) = \alpha_1 \alpha_{j+1}$ for $j = 1, \ldots, t-1$, α_t for $t \leqslant j \leqslant s-r$ and $j-t$ even, α_{t-1} for $t \leqslant j \leqslant s-r$ and $j-t$ odd, α_1 for $j = 0$, $s-r+1$, $\Theta(c_l) = \alpha_{s-r-1+l}$ for $l = 2, \ldots, r+1$.

It is easy to see that in both cases Θ is the homomorphism we have looked for. The homomorphism Θ in case $s \leqslant r$ can be defined similarly.

Now let $A = A_2 \oplus A'$, where A_2 is the maximal 2-subgroup of A and let $A_2 = Z_{m_1} \oplus \ldots \oplus Z_{m_k}$ be the canonical decomposition. Suppose that $m_1 = \ldots = m_l = 2$ and $m_{l+1} \neq 2$.

Definition 5.1. A generating couple $(X:K) = (\xi_1, \ldots, \xi_r{:}\ \alpha_1, \ldots, \alpha_s)$ is said to be

(a) *reduced* if $\alpha_1, \ldots, \alpha_s$ are non-trivial and $A = \langle \alpha_1 \rangle \oplus \ldots \oplus \langle \alpha_s \rangle \oplus \langle X \rangle$,

(b) *divisibly reduced* if it is reduced and $|\xi_i| \,|\, |\xi_{i+1}|$ for $i = 1, \ldots, r-1$,

(c) *totally reduced* if it is divisibly reduced and $\xi_1, \ldots, \xi_r$ is a canonical set of generators for the subgroup of A it generates.

Lemma 5.2. *If $(X:K)$ is any generating couple for A, then there exists $(X':K')$ which is reduced and $\mu(X':K') \leqslant \mu(X:K)$.*

Proof. Let $X = \{\xi_1, \ldots, \xi_r\}$ and $K = \{\alpha_1, \ldots, \alpha_s\}$. If $(X:K)$ is not reduced, then $\alpha_i \in \langle \xi_1, \ldots, \xi_r, \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_s \rangle$ for some i. Thus if $K' = \{\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_s\}$, $(X:K')$ is a generating couple, Moreover, if $s \leqslant r$ then $\mu(X:K') = \mu(X:K) - 1/m_{r-(s-1)} < \mu(X:K)$, if $s = r+1$ then $\mu(X:K') = \mu(X:K) - \frac{1}{2} < \mu(X:K)$ and finally if $s > r+1$ then $s-1 > r$ and consequently $\mu(X:K') = \mu(X:K) - \frac{1}{4} \leqslant \mu(X:K)$. Thus the lemma follows by induction on s.

Lemma 5.3. *If $(X:K)$ is any generating couple for A, then there exists $(X':K')$ which is divisibly reduced and $\mu(X':K') \leqslant \mu(X:K)$.*

Proof. Let $X = \{\xi_1, \ldots, \xi_r\}$, $K = \{\alpha_1, \ldots, \alpha_s\}$. By the previous lemma we can assume that $(X:K)$ is reduced.

If $s > r$, let us take the set $\eta_1, \ldots, \eta_t$ of canonical generators for the subgroup of A generated by X. Clearly, $t \leqslant r$ and thus $\mu(X':K) = t-1+(s-t+1)/4 \leqslant r-1+(s-r+1)/4 = \mu(X:K)$, where $X' = \{\eta_1, \ldots, \eta_t\}$.

If $s \leqslant r$, let us consider $(\xi_1, \ldots, \xi_{r-s}\colon \xi_{r-s+1}, \ldots, \xi_r)$ as a generating couple in the sense of Maclachlan [5] for the subgroup of A generated by X. By Lemma 6.2 [5], there is a set of elements $\eta_1, \ldots, \eta_{t+s}$ of orders n_i that generates the same subgroup of A such that $n_i | n_{i+1}$ for every i and $\sum_{i=1}^{t} (1-1/n_i) \leqslant \sum_{i=1}^{r-s} (1-1/m_i)$. Let $X' = \{\eta_1, \ldots, \eta_{t+s}\}$. Since clearly $\mu(X':K) \leqslant \mu(X:K)$, $(X':K)$ is the couple we have looked for.

LEMMA 5.4. *Let $(X\colon K)$ be any divisibly reduced generating couple for A and let $(X^*:K^*)$ be totally reduced generating couple such that $|K^*| = |K|$. Then $\mu(X^*:K^*) \leqslant \mu(X:K)$.*

Proof. Let $X = \{\xi_1, \ldots, \xi_r\}$, $K = \{\alpha_1, \ldots, \alpha_s\}$ and $X^* = \{\xi_1^*, \ldots, \xi_{r'}^*\}$, $K^* = \{\alpha_1^*, \ldots, \alpha_s^*\}$.

We have to consider two cases: I. $r' < r$, II. $r' = r$.

Case I. Consider three subcases.

(i) $r < s$. Then $\mu(X^*:K^*) = r'-1+(s-r'+1)/4 < r-1+(s-r+1)/4 = \mu(X:K)$.

(ii) $r \geqslant s$ and $s > r'$. Then $\mu(X^*:K^*) = r'-1+(s-r'+1)/4 < r'-1+s-r' = s-1 \leqslant s-1+\sum_{i=1}^{r-s} (1-1/m_i) = \mu(X:K)$.

(iii) $s \leqslant r'$. If $s = r'$ then

$$\mu(^*:K^*) = r'-1 < r-1+\sum_{i=1}^{r-s} (1-1/m_i) = \mu(X:K).$$

Thus let $s < r' \leqslant r$. Since both couples are reduced, $A = \langle X^* \rangle \oplus \langle K^* \rangle = \langle X \rangle \oplus \langle K \rangle$. Now since $|K^*| = |K|$, $\langle X^* \rangle \cong \langle X \rangle$. Let $m_1, \ldots, m_r$ and $n_1, \ldots, n_{r'}$ be the orders of $\xi_1, \ldots, \xi_r$ and $\xi_1^*, \ldots, \xi_{r'}^*$ respectively. We want to employ Theorem 4.2. So let $A' = Z_{m_1} \oplus \ldots \oplus Z_{m_r}$ and $A'' = Z_{d_1} \oplus \ldots \oplus Z_{d_r}$, where $d_1 = \ldots = d_{r-r'} = 1$ and $d_{r-r'+1} = n_i$ for $i = 1, \ldots, r'$.

We have an epimorphism of A' onto A''. Thus, by Theorem 4.2, $d_i | m_i$ for $i = 1, \ldots, r$. Thus

$$\mu(X^*\colon K^*) = s-1+\sum_{i=1}^{r'-s} (1-1/n_i) = s-1+\sum_{i=1}^{r-s} (1-1/d_i)$$

$$\leqslant s-1+\sum_{i=1}^{r-s} (1-1/m_i) = \mu(X:K).$$

Case II. Consider two subcases.

(i) $s > r$. Then clearly $\mu(X^*: K^*) = \mu(X:K)$.
(ii) $s \leqslant r$. Then the proof is contained in Case I (iii).

COROLLARY 5.5. *Let A be an abelian group with non-cyclic maximal 2-subgroup and satisfying hypothesis of Lemma 3.1. Then* $m_A = \min\{\mu(X:K)$: *$(X:K)$ is totally reduced generating couple for A*$\}$.

LEMMA 5.6. *Let A be an abelian group with cyclic maximal 2-subgroup satisfying hypothesis of Lemma 3.1. Then* $m_A = \min\{\mu(\Gamma)$: *$\Gamma \in \mathscr{K}_A$ and Γ has signature* $(0, +, [m_1, \ldots, m_r], \{(-)\})\}$.

Proof. By Proposition 3.8, it is sufficient to show that for any reduced group Γ in $\mathscr{K}_A$ there exists $\Gamma' \in \mathscr{K}_A$ with desired signature such that $\mu(\Gamma') \leqslant \mu(\Gamma)$. Clearly, such group has no non-empty period cycles.

Let Γ has signature $(0, +, [m_1, \ldots, m_r], \{(-)^s\})$ and let Θ be the corresponding homomorphism. Let β be the only element of order 2 in A. Consider the group Γ' with signature $(0, +, [n_1, \ldots, n_{r+s-1}], \{(-)\})$, where

$$n_i = \begin{cases} m_i, & i = 1, \ldots, r, \\ |\Theta(e_{i-r})|, & i = r+1, \ldots, r+s-1, \end{cases}$$

and the homomorphism Θ: $\Gamma \to A$ defined by

$$\Theta'(x_i') = \begin{cases} \Theta(x_i), & i = 1, \ldots, r, \\ \Theta(e_{i-r}), & i = r+1, \ldots, r+s-1, \end{cases}$$

$\Theta'(e') = \Theta(e_s)$ and $\Theta'(c') = \beta$.

Since A has only one element of order 2, we have $\Theta(c_1) = \ldots = \Theta(c_s)$ and hence $\Gamma' \in \mathscr{K}_A$. Now since $1 - 1/n_i < 1$ for $i = r+1, \ldots, r+s-1$, $\mu(\Gamma') \leqslant \mu(\Gamma)$. Hereby the lemma is proved.

6. Minimum genus. We have reduced the problem of finding the minimum genus of surfaces of genus $p \geqslant 3$ for which a given abelian group A is a group of automorphism to the problem of calculating m_A. In the light of the previous considerations it is natural to split the problem into following cases.

(i) A satisfies hypothesis of Lemma 3.1 and has non-cyclic maximal 2-subgroup.

(ii) A satisfies hypothesis of Lemma 3.1 and has cyclic maximal 2-subgroup.

(iii) A is of odd order.

(iv) A does not satisfy hypothesis of Lemma 3.1.

PROPOSITION 6.1. *Assume that an abelian group A satisfies hypothesis of Lemma 3.1 and has non-cyclic maximal 2 subgroup. Let $A = Z_2^s \oplus A'$, where s is as big as possible and $A' = Z_{m_1} \oplus \ldots \oplus Z_{m_k}$ is the canonical decomposition. Suppose that $m_1, \ldots, m_l$ are odd and $m_{l+1}, \ldots, m_k$ are even. Then*

$$m_A = \begin{cases} s-1+\sum_{i=1}^{k-s}(1-1/m_i) & \text{if } s \leqslant k-l, \\ k-1 & \text{if } s-(k-l) = 2l, \\ k-1+(s-k-l+1)/4 & \text{if } s-(k-l) > 2l, \\ (k+s-l)/2-1+\sum_{i=1}^{(k+l-s)/2}(1-1/m_i) & \text{if } 0 < s-(k-l) < 2l \\ & \text{and } s-(k-l) \text{ is even}, \\ (k+s-l-1)/2-1+\sum_{i=1}^{(k+l-s-1)/2}(1-1/m_i)+(1-1/2m_{(k+l-s+1)/2}) & \\ \qquad \text{if } 0 < s-(k-l) < 2l \text{ and } s-(k-l) \text{ is odd}. & \end{cases}$$

Proof. By Corollary 5.5, we have to minimize $\mu(X:K)$ over all totally reduced generating couples $(X:K)$. Although for given abelian group A, totally reduced generating couple $(X:K)$ is not uniquely determined by the number of elements in K, the sequence $(n_1, \ldots, n_r\colon u)$, where $n_i = |\xi_i|$, $u = |K|$ and $\mu(X:K)$ are. We will refer to any such sequence as to the sequence corresponding to u.

Let $s \leqslant k-l$. We will show that if $(X:K)$ is any totally reduced generating couple and $(n_1, \ldots, n_r\colon u)$ is the sequence corresponding to $u = |K|$ in which $n_1, \ldots, n_t$ are odd and $n_{t+1}, \ldots, n_r$ are even and $u \leqslant r-t$, then for any $v \leqslant u$ and for any totally reduced generating couple $(X':K')$ with $|K'| = v$, we have $\mu(X:K) \leqslant \mu(X':K')$. It can be done by induction on u. The assertion clearly holds for $u = 0$. Suppose that the assertion holds for some $u < s$. We will prove it for $u+1$. Let $(X:K)$ be any totally reduced generating couple with $|K| = u+1$ and let $(n_1, \ldots, n_r\colon u+1)$ be the corresponding sequence where $n_1, \ldots, n_t$ are odd and $n_{t+1}, \ldots, n_r$ are even and $u+1 \leqslant r-t$.

Now let $v \leqslant u+1$ and let $(X':K')$ be totally reduced generating couple with $|K'| = v$.

If $v = u+1$ then clearly $\mu(X':K') = \mu(X:K)$.

If $v = u$, let us consider two cases:

(i) $t \geqslant 1$. Then the sequence corresponding to u is the following one $(n_1, \ldots, n_{t-1}, 2n_t, n_{t+1}, \ldots, n_r\colon u)$. Thus

$$\begin{aligned}\mu(X:K) &= ((u+1)-1)+\sum_{i=1}^{r-(u+1)}(1-1/n_i)\\ &= u-1+\sum_{i=1}^{t-1}(1-1/n_i)+(1-1/2n_t)+\sum_{i=t+1}^{r-u}(1-1/n_i)\\ &\qquad +1-(1-1/2n_t)+(1-1/n_t)-(1-1/n_{r-u})\\ &= \mu(X':K')+1/n_{r-u}-1/2n_t.\end{aligned}$$

Now since $u+1 \leqslant r-t$, $r-u \geqslant t+1$. Thus n_{r-u} is even. Since n_t is odd and in addition $n_t | n_{r-u}$, $n_{r-u}/n_t \geqslant 2$. Consequently, $1/n_{r-u}-1/2n_t \leqslant 0$. Hence $\mu(X:K) \leqslant \mu(X':K')$.

(ii) $t = 0$. Then the sequence corresponding to u is the following one $(2, n_1, \ldots, n_r\colon u)$. Thus

$$\begin{aligned}\mu(X:K) &= ((u+1)-1)+\sum_{i=1}^{r-(u+1)}(1-1/n_i)\\ &= u-1+(1-\tfrac{1}{2})+\sum_{i=1}^{r-u}(1-1/n_i)\\ &\qquad +1-(1-\tfrac{1}{2})-(1-1/n_{r-u})\\ &= \mu(X':K')-1+\tfrac{1}{2}+1/n_{r-u} \leqslant \mu(X':K').\end{aligned}$$

Finally if $v < u$ then let $(X''\colon K'')$ be totally reduced generating couple with $|K''| = u$. Then by the previous case $\mu(X:K) \leqslant \mu(X''\colon K'')$ and by induction hypothesis $\mu(X''\colon K'') \leqslant \mu(X':K')$.

Thus m_A is attained for $(X:K)$ with $|K| = s$ and it is easy to see that it is equal to the value declared in the proposition.

Let $s-(k-l) > 2l$. Note first that if $(X:K)$ is any totally reduced generating couple in which $|K| = u \leqslant s$ and $(n_1, \ldots, n_r\colon u)$ is the corresponding sequence with $n_1, \ldots, n_t$ odd and $n_{t+1}, \ldots, n_r$ even and $u-(r-t) > 2t$, then for any $v \leqslant t$ the sequence corresponding to $u-v$ is $(n_1, \ldots, n_{t-v}, +2n_{t-v+1}, \ldots, 2n_t, n_{t+1}, \ldots, n_r\colon u-v)$. Thus $\mu(X':K') = r-1+ +(u-v-r+1)/4 \leqslant r-1+(u-r+1)/4 = \mu(X:K)$ for any totally reduced generating couple $(X':K')$ with $|K'| = u-v$.

On the other hand, let $(X:K)$ be any totally reduced generating couple with $u = |K|$ and let $(n_1, \ldots, n_r\colon u)$ be the corresponding sequence in which $n_1, \ldots, n_r$ are even. Then for any $v \leqslant u$ the sequence corresponding to $u-v$ is $(2, \overset{v}{\ldots}, 2, n_1, \ldots, n_r\colon u-v)$. We will show that for any totally reduced generating couple $(X':K')$ with $|K'| = u-v$ we have $\mu(X:K) \leqslant \mu(X':K')$. By induction, it is sufficient to prove it for $v = 1$.

If $u \leqslant r$ then $u-1 \leqslant r+1$ and thus

$$\begin{aligned}\mu(X:K) &= u-1+\sum_{i=1}^{r-u}(1-1/n_i)\\ &= ((u-1)-1)+(1-\tfrac{1}{2})+\sum_{i=1}^{r-u+1}(1-1/n_i)\\ &\qquad\qquad +1-(1-\tfrac{1}{2})-(1-1/m_{r-u+1})\\ &= \mu(X':K')-1+\tfrac{1}{2}+1/m_{r-u+1} \leqslant \mu(X':K').\end{aligned}$$

The cases $u = r+1$, $r+2$ and $u > r+2$ are similar.

So in the case $s-(k-l) > 2l$, m_A is attained for $(X:K)$ with $|K| = s-(k-l)$ and, finding the sequence corresponding to $s-(k-l)$, one can show that it is equal to the value declared in the proposition.

For $s-(k-l) = 2l$ the proof is essentially the same as in the previous case.

Now let $0 < s-(k-l) < 2l$ and let $s-(k-l)$ be even. We will show first that if $(X:K)$ is totally reduced generating couple with $u = |K|$ and $(n_1, \ldots, n_r: u)$ is the corresponding sequence in which $n_1, \ldots, n_t$ are odd, $n_{t+1}, \ldots, n_r$ are even and $0 < u-(r-t) < 2t$, then for any $v \leqslant (u-(r-t))/2$ and totally reduced generating couple $(X':K')$ with $|K'| = u-v$, $\mu(X':K') \leqslant \mu(X:K)$. Note first that $(u-v)-(r-(t-v)) = u-(r-t)-2v \leqslant 2t-2v = 2(t-v)$ and $v < t$. Thus it is sufficient to prove the assertion for $v = 1$.

The sequence corresponding to $u-1$ is $(n_1, \ldots, n_{t-1}, 2n_t, n_{t+1}, \ldots, n_r: u-1)$. Let $(X':K')$ be any totally reduced generating couple with $|K'| = u-1$. If $u \leqslant r$ then $u-1 < r$ and

$$\begin{aligned}\mu(X:K) &= u-1+\sum_{i=1}^{r-u}(1-1/n_i)\\ &= ((u-1)-1)+\sum_{i=1}^{r-u}(1-1/n_i)+(1-1/k_{r-u+1})+1-(1-1/k_{r-u+1})\\ &= \mu(X':K')+1/k_{r-u+1} \geqslant \mu(X:K),\end{aligned}$$

where $k_{r-u+1} = 2n_t$ if $r-u+1 = t$ and n_{r-u+1} if $r-u+1 < t$.

If $u > r+1$ then $u-1 > r = r$ and thus $\mu(X':K') = r-1+((u-1)-r+1)/4 < r-1+(u-r+1)/4 = \mu(X:K)$.

Finally, if $u = r+1$ then $u-1 = r$ and $\mu(X:K) = r-1+(u-r+1)/4 > r-1 = \mu(X':K')$.

On the other hand, if $(X:K)$ is totally reduced generating couple with u

$=|K|$ and $(n_1, \ldots, n_r\colon u)$ is the corresponding sequence in which $u<r$, $n_1, \ldots, n_{r-u}$ are odd and $n_{r-u+1}, \ldots, n_r$ are even, then for any $v \leqslant u$ the sequence corresponding to $u-v$ is $(n_1, \ldots, n_{r-u-v}, 2n_{r-u+v+1}, \ldots, 2n_{r-u}, n_{r-u+1}, \ldots, n_r\colon u-v)$ or $(2, \ldots, 2, 2n_1, \ldots, 2n_{r-u}, n_{r-u+1}, \ldots, 2n_r\colon u-v)$ according as $v \leqslant r-u$ or $v>r-u$. Then as in the previous case one can show that $\mu(X\colon K) \leqslant \mu(X'\colon K')$ and thus m_A is attained for $(X\colon K)$ with $|K| = s-(s-(k-l))/2$ and is equal to the value declared in the proposition.

Finally, let $0<s-(k-l)<2l$ and let $s-(k-l)$ be odd. Then as in the previous case we show that if $(X\colon K)$ is totally reduced generating couple with the corresponding sequence $(n_1, \ldots, n_r\colon u)$ in which $n_1, \ldots, n_t$ are odd, $n_{t+1}, \ldots, n_r$ are even and $u=(r-t)+1$, then for any totally reduced generating couple $(X'\colon K')$ such that $|K'|>u$ or $|K'|<u-1$, $\mu(X\colon K) \leqslant \mu(X'\colon K')$. On the other hand, the sequence corresponding to $u-1$ is the following one $(n_1, \ldots, n_{t-1}, 2n_t, n_{t+1}, \ldots, n_r\colon u-1)$. Let $(X'\colon K')$ be totally reduced generating couple with $|K'|=u-1$. Then

$$\begin{aligned}
\mu(X\colon K) &= u-1+\sum_{i=1}^{r-u}(1-1/n_i) \\
&= ((u-1)-1)+\sum_{i=1}^{r-u}(1-1/n_i)+(1-1/2n_{r-u+1}) \\
&\qquad\qquad -(1-1/2n_{r-u+1})+1 \\
&= \mu(X'\colon K')+1/2n_{r-u+1} \geqslant \mu(X'\colon K').
\end{aligned}$$

Thus m_A is attained for $(X\colon K)$ with $|K| = s-(s-(k-l)-1)/2 = (s+k-l-1)/2$ and it is easy to see that it is equal to the value declared in the proposition.

PROPOSITION 6.2. *Assume that an abelian group A satisfies hypothesis of Lemma* 3.1 *and has cyclic maximal* 2-*subgroup. Let $A = Z_{m_1} \oplus \ldots \oplus Z_{m_r}$ be the canonical decomposition. Then* $m_A = -1+\sum_{i=1}^{r}(1-1/m_i)$.

Proof. By Lemma 5.6, we have to minimize $\mu(\Gamma)$ over all groups in $\mathscr{K}_A$ which have signatures of type $(0, +, [n_1, \ldots, n_s], \{(-)\})$. Every NEC group in $\mathscr{K}_A$ with such signature determines generating couple $(X(\Gamma)\colon K(\Gamma)) = (x_1, \ldots, x_s\colon \emptyset)$ such that $\mu(\Gamma) = \mu(X(\Gamma)\colon K(\Gamma))$. Conversely, given any generating couple $(X\colon K) = (\xi_1, \ldots, \xi_s\colon \emptyset)$ determines an NEC group $\Gamma(X\colon K)$ in $\mathscr{K}_A$ with signature $(0, +, [n_1, \ldots, n_s], \{(-)\})$, where $n_i = |\xi_i|$ such that $\mu(\Gamma(X\colon K)) = \mu(X\colon K)$ (the corresponding non-orientable surface-kernel homomorphism is given by $\Theta(x_i) = \xi_i$, $\Theta(e) = -(\xi_1+\ldots+\xi_s)$ and $\Theta(c)$ is the only element of order 2 in A).

Thus in order to find m_A for the group in question it is sufficient to minimize $\mu(X:K)$ over all generating couples $(X:K)$ with empty K. Hereby the assertion follows from Theorem 4.1.

PROPOSITION 6.3. *Let A be an abelian group of odd order and let $A = Z_{m_1} \oplus \ldots \oplus Z_{m_r}$ be the canonical decomposition. Then* $m_A = -1 + \sum_{i=1}^{r} (1-1/m_i)$.

Proof. By Proposition 3.9, we have to minimize $\mu(\Gamma)$ over all groups in $\mathscr{K}_A$ which have signatures of type $(1, -, [n_1, \ldots, n_s], \{-\})$. For any NEC group Γ in $\mathscr{K}_A$ with such signature $(X(\Gamma):K(\Gamma)) = (x_1, \ldots, x_s\colon \emptyset)$ is generating couple for A such that $\mu(\Gamma) = \mu(X(\Gamma):K(\Gamma))$. Conversely, for any generating couple $(X:K) = (\xi_1, \ldots, \xi_s\colon \emptyset)$ there is at least one NEC group $\Gamma(X:K)$ in $\mathscr{K}_A$ such that $\mu(X:K) = \mu(\Gamma(X:K))$. This is the group with signature $(1, -, [n_1, \ldots, n_s], \{-\})$, where n_i is the order of ξ_i (the corresponding non-orientable surface-kernel homomorphism Θ is given by $\Theta(x_i) = \xi_i$ and $\Theta(d)$ is an element satisfying $2\Theta(d) = -(\xi_1 + \ldots + \xi_r)$).

Thus as in the previous proposition we obtain the assertion.

PROPOSITION 6.4. *Assume that an abelian group does not satisfy hypothesis of Lemma 3.1. Then $m_A = (m-1)/2m$ if $A = Z_2 \oplus Z_{2m}$ and $m_A = \frac{1}{4}$ if $A = Z_2 \oplus Z_2 \oplus Z_2$ or $A = Z_2 \oplus Z_2$.*

Proof. Let $A = Z_2 \oplus Z_2$. Then it is easy to observe that for any NEC group Γ in $\mathscr{K}_A$, $\mu(\Gamma)$ is an integral multiplicity of $\frac{1}{4}$. On the other hand, an NEC group Γ with signature $(0, +, [2], \{(2, 2, 2)\})$ belongs to $\mathscr{K}_A$ – the corresponding homomorphism Θ is defined in the following way. Let ξ_1, ξ_2 be generators of A. Then $\Theta(x) = \Theta(e) = \Theta(c_0) = \Theta(c_3) = \xi_1$, $\Theta(c_1) = \xi_2$ and $\Theta(c_2) = \xi_1 + \xi_2$. Moreover, $\mu(\Gamma) = \frac{1}{4}$. Thus $m_A = \frac{1}{4}$.

In the same way one can show that for the group $A = Z_2 \oplus Z_2 \oplus Z_2$, m_A is also equal to $\frac{1}{4}$.

Now let $A = Z_2 \oplus Z_{2m}$, where $m > 1$. Consider an NEC group Γ with signature $(0, +, [2, 2m], \{(-)\})$. Let ξ_1, ξ_2 be canonical generators of A and let $\Theta\colon \Gamma \to A$ be the homomorphism defined by $\Theta(x_1) = \xi_1$, $\Theta(x_2) = \xi_2$, $\Theta(e) = -(\xi_1 + \xi_2)$ and $\Theta(c) = \xi_1$.

Clearly, Θ is non-orientable surface-kernel homomorphism and thus $\Gamma \in \mathscr{K}_A$. $\mu(\Gamma) = (m-1)/2m$. We will show that this is m_A.

Let $\Gamma \in \mathscr{K}_A$ and let Θ be the corresponding homomorphism.

(i) If Γ has more than 2 period cycles, then $\mu(\Gamma) \geqslant 1$.

(ii) If Γ has 2 non-empty period cycles, then $\mu(\Gamma) \geqslant 1$.

(iii) If Γ has 2 period cycles one of which is non-empty, then $\mu(\Gamma) \geqslant \frac{1}{2}$.

(iv) If Γ has 2 empty period cycles, then there is at least one period or the genus is greater than 0 (otherwise $\mu(\Gamma) = 0$). In both cases $\mu(\Gamma) \geqslant \frac{1}{2}$.

(v) Assume that Γ has 1 non-empty period cycle. Then either the genus is greater than 0 or Γ has some periods (otherwise since $e = 1$, A has only elements of order 2 what is a contradiction since $m > 1$).

If Γ has genus greater than 0 then $\mu(\Gamma) \geqslant \frac{1}{2}$. Thus suppose that the genus is 0.

If Γ has more than one period then $\mu(\Gamma) \geqslant \frac{1}{2}$. Assume then that has 1 period only. Clearly this period must be m or $2m$.

If the period is m then the period cycle must "provide" 2 elements of order 2. Consequently $s \geqslant 3$ and thus $\mu(\Gamma) \geqslant -2+1+\frac{3}{4}+(1-1/m) = (3m-4)/4m \geqslant (m-1)/2m$.

If the period is $2m$ then $\mu(\Gamma) \geqslant -2+1+\frac{1}{2}+(1-1/2m) = (m-1)/2m$.

(vi) Assume that Γ has 1 empty period cycle. Then:

If Γ has 1 period, then the genus is greater than 0 (otherwise $\mu(\Gamma) < 0$). Thus let the genus be greater than 0 and let k be the period. Then $\mu(\Gamma) \geqslant (k-1)/k \geqslant (m-1)/2m$.

If Γ has 2 periods and the genus is greater than 0, then $\mu(\Gamma) \geqslant 1$. So let the genus be 0 and let k, l be the periods. A is generated by $\boldsymbol{x}_1, \boldsymbol{x}_2$. Thus $k = 2$ and $l = 2m$. Consequently, $\mu(\Gamma) = (m-1)/2m$.

If Γ has more than 2 periods, then $\mu(\Gamma) \geqslant \frac{1}{2}$.

(vii) Assume that Γ has no period cycles. Then the genus is greater than 0 and the sign is $-$ (otherwise Γ is a Fuchsian group).

If the genus is greater than 2, then $\mu(\Gamma) \geqslant 1$.

If the genus is 2, then Γ has some period (otherwise $\mu(\Gamma) = 0$) and thus $\mu(\Gamma) \geqslant \frac{1}{2}$.

If the genus is 1, then Γ has at least 2 periods (otherwise $\mu(\Gamma) < 0$). If Γ has more than 2 periods, then $\mu(\Gamma) \geqslant \frac{1}{2}$. Assume thus that Γ has 2 periods k, l. Since A is generated by $\boldsymbol{d}^2, \boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{d}^2 = -(\boldsymbol{x}_1+\boldsymbol{x}_2)$ so $k = 2$ and $l = 2m$. Consequently, $\mu(\Gamma) = (m-1)/2m$.

Hereby we showed that $m_A = (m-1)/2m$.

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INSTYTUT MATEMATYKI WSP
BYDGOSZCZ, POLAND