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On some dense subspaces of topological linear spaces. II

1. Introduction. We are mainly concerned with the existence and properties of some special dense subspaces of F-spaces and of more general topological linear spaces. The subspaces considered include $\varkappa$-subspaces, spaces which have property (K) and Baire spaces (see Section 2 for some terminological explanations). In the F-space case, each of these classes of subspaces is strictly smaller than the next one. This is still so under various additional assumptions, which is a central theme of this paper(*).

The results of Sections 4, 6 and 7 complement those of Burzyk, Kliś, Labuda and the second author ([3], [11], [12]). The results of Section 7, which are concerned with subspaces of codimension 1 (often called *hyperplanes* in the literature), are also related to those of Arias de Reyna [1] and Valdivia [20]. Section 5 contains generalizations, to the case of $\varkappa$-subspaces, of some theorems of Kruse [10] and Pol [16] on the existence of topological linear spaces which admit only very few continuous linear operators.

In addition, we are concerned with extracting m-independent subsequences from linearly independent sequences (Section 3). Generalizations and improvements of some results of [11], [12] and [13] are given. One of the obtained results, Corollary 1(b), proves useful for some constructions in Sections 6 and 7.

The final Section 8 is independent of the preceding ones. It gives a proof (and a generalization) of a result due to Godefroy and Talagrand [6], which is basic for the material of Section 7.

2. Notation, terminology and Proposition (*). Let X be a (Hausdorff) topological linear space. We say that a sequence (x_n) in X is

m-independent if for every sequence $(\lambda_n)\in m = l_\infty$ such that $\sum_{n=1}^{\infty} \lambda_n x_n = 0$ we have $(\lambda_n) = 0$;

(*) Some of this material was presented at the 6th Prague Topological Symposium (1986).

an *l-sequence* if it is linearly independent and the series $\sum_{n=1}^{\infty} x_n$ is subseries convergent.

(See [11], Definition 1, and [5], p. 62, respectively, where a slightly different terminology is used.)

We shall frequently use the following result ([11], Proposition 3; see also Theorem 1 below for an improvement):

(*) *Every l-sequence has an m-independent subsequence.*

We say that the space X has *property* (L) if it contains an l-sequence. It is easily seen that if Z is an *F-space* (i.e., a complete metrizable topological linear space) and $T\colon Z \to X$ is a continuous linear operator with infinite-dimensional range, then $T(Z)$ (and, a fortiori, X) has property (L). In fact, $T(Z)$ then has the following stronger property: it contains a linearly independent sequence (x_n) such that the series $\sum_{n=1}^{\infty} x_n$ is bounded multiplier convergent. Note that, by [2], Theorem 1 (see also [4], Theorem 4), the space m_0 of all sequences in m which have finite range with the topology of pointwise convergence does not have the latter property. Nevertheless, it has property (L), since the standard basis of m_0 is an l-sequence.

Let (x_n) be a sequence in X. We denote by $\langle (x_n) \rangle$ the subspace of X consisting of those $x \in X$ which admit an expansion of the form

$$x = \sum_{n=1}^{\infty} \lambda_n x_n, \quad \text{where } (\lambda_n) \in m.$$

In the case where the series $\sum_{n=1}^{\infty} x_n$ is subseries convergent, we put

$$x_M = \sum_{n=1}^{\infty} 1_M(n)\, x_n \quad \text{for each subset } M \text{ of } N.$$

Following [5] (see also [12]), we call a (linear) subspace Y of X a $\varkappa$-*subspace* provided that for every l-sequence (x_n) in X there is an infinite subset M of N with $x_M \in Y$. (In fact, using (*), it is not hard to see that the word "infinite" can be replaced by "non-empty" in this definition and that one may also require that $x_M \neq 0$.)

Recall that a topological linear space X is said to have *property* (K) if every sequence (x_n) in X with $x_n \to 0$ contains a subsequence (x_{n_k}) such that the series $\sum_{k=1}^{\infty} x_{n_k}$ is convergent. (This completeness-type property was first isolated by Mazur and Orlicz [14], p. 169; see also [12] for other references and relevant information.) Clearly, every $\varkappa$-subspace of an F-space has property (K). The converse fails, which is discussed in Section 6 below.

3. Existence of m-independent subsequences. We start with a simple lemma.

LEMMA 1. *Let X be a topological linear space and let V be a neighbourhood of 0 in X. Then every sequence (x_n) in X with $x_n \to 0$ contains a subsequence (x_{n_k}) such that $\sum_{k=1}^{\infty} \lambda_k x_{n_k} \in V$ whenever $|\lambda_k| \leqslant 1$, $k \in N$, and the series converges.*

Proof. Take a neighbourhood U of 0 in X with $\bar{U} \subset V$ and a sequence (U_n) of balanced neighbourhoods of 0 in X with

$$U_1 + \ldots + U_k \subset U, \quad k \in N.$$

Then it is enough to choose a subsequence (x_{n_k}) with $x_{n_k} \in U_k$.

In the case where $A = \{0\}$ the following result coincides with [13], Proposition 1. In the sequel we shall only need a rather special consequence of it (Corollary 1(b)); see the proofs of Theorems 6 and 8 below.

THEOREM 1. *Let X be a topological linear space, let (x_n) be a linearly independent sequence in X with $x_n \to 0$ and let A be an F_σ-subset of X such that*

$$A \cap \operatorname{lin}\{x_n \colon n \in N\} \subset \{0\}.$$

Then there exists an m-independent subsequence (x_{n_k}) such that

$$A \cap \langle (x_{n_k}) \rangle \subset \{0\}.$$

Proof. First observe that if K is a compact subset of X and C is a closed subset of X with $K \cap C = \emptyset$, then for every $r > 0$ there is a subsequence (z_n) of (x_n) such that

$$\Big(K + \Big\{\sum_{n=1}^{\infty} \lambda_n z_n \colon |\lambda_n| \leqslant r,\ n \in N\Big\}\Big) \cap C = \emptyset.$$

In fact, we can find a neighbourhood V of 0 in X with $(K+V) \cap C = \emptyset$ ([19], Theorem 1.10) and then apply Lemma 1 to $\frac{1}{r}V$.

Let $A = \bigcup_{i=1}^{\infty} C_i$, where the C_i's are closed and $C_1 \subset C_2 \subset \ldots$; we may also assume that $0 \in C_1$. Then, using what we have proved so far, we construct, by induction, $1 = n_1 < n_2 < \ldots$ so that, for every $k \in N$,

$$\sum_{i=1}^{k} \lambda_i x_{n_i} + \sum_{j=k+1}^{\infty} \lambda_j x_{n_j} \notin C_k$$

provided that $\|(\lambda_i)\|_\infty \leqslant k$ and $\max_{1 \leqslant i \leqslant n_k} |\lambda_i| \geqslant 1/k$ (cf. [11], proof of Proposition 1). Then, as easily seen, (x_{n_k}) has the desired properties.

It is worth-while to note that Theorem 1 fails without the assumption that $x_n \to 0$ even when A is closed and X is a Banach space. This is shown by the following

EXAMPLE. Let X be an infinite-dimensional Banach space and choose (x_n) to be a normalized basic sequence in X. Put

$$A = \{\sum_{k=1}^{\infty} 2^{-k} x_{n_k}: n_1 < n_2 < \ldots\}.$$

Then, as easily seen, A is closed. Moreover,

$$A \cap \operatorname{lin}\{x_n: n \in N\} = \emptyset, \quad \text{but} \quad \sum_{k=1}^{\infty} 2^{-k} x_{n_k} \in A \cap \langle (x_{n_k}) \rangle.$$

COROLLARY 1. *Let X be a topological linear space, let (x_n) be a sequence in X and let (y_n) be a linearly independent sequence in X with $y_n \to 0$.*

(a) *If* $\operatorname{lin}\{x_n: n \in N\} \cap \operatorname{lin}\{y_n: n \in N\} = \{0\}$, *then there exists an m-independent subsequence* (y_{n_k}) *such that* $\operatorname{lin}\{x_n: n \in N\} \cap \langle (y_{n_k}) \rangle = \{0\}$.

(b) *If the series* $\sum_{n=1}^{\infty} x_n$ *is bounded multiplier convergent and* $\langle (x_n) \rangle \cap \operatorname{lin}\{y_n: n \in N\} = \{0\}$, *then there exists an m-independent subsequence* (y_{n_k}) *such that* $\langle (x_n) \rangle \cap \langle (y_{n_k}) \rangle = \{0\}$.

Proof. Clearly, $\operatorname{lin}\{x_n: n \in N\}$ is σ-compact. Also, $\langle (x_n) \rangle$ is σ-compact as the union of the compact sets

$$\{\sum_{n=1}^{\infty} \lambda_n x_n: \|(\lambda_n)\|_\infty \leqslant k\}, \qquad k \in N,$$

provided that $\sum_{n=1}^{\infty} x_n$ is bounded multiplier convergent. Hence both the assertions follow directly from Theorem 1.

We note that the corollary above can be used to give an alternative proof of Proposition of [12].

4. Relation between l-sequences and $\varkappa$-subspaces. Most of the material of this section complements that of [12], Section 2. Namely, we show that a $\varkappa$-subspace of X is, in various respects, large in X. Some of these results are applied in the next section. We also present two simple results on spaces with property (L) (Proposition 1 and Corollary 3).

LEMMA 2. *Let Y be a $\varkappa$-subspace of a topological linear space X. If (x_n) is an l-sequence in X, then*

(a) *Every $x \in X \setminus \{0\}$ can be represented in the form*

$$x = y + \lambda x_M,$$

where $y \in Y$, $\lambda \geqslant 1$ and M is an infinite subset of N.

(b) *There is a sequence* (M_k) *of disjoint infinite subsets of* N *such that* $x_{M_k} \in Y$ *for every* $k \in N$ *and* (x_{M_k}) *is an l-sequence in* X.

(c) *There is a family* $\{M_t: t \in R\}$ *of almost disjoint infinite subsets of* N *such that* $\{x_{M_t}: t \in R\}$ *is a linearly independent family of elements of* Y.

Proof. (a) Let $x \in X \setminus \{0\}$; then, by omitting a finite number of terms in (x_n), we may assume that $x \notin \operatorname{lin}\{x_n: n \in N\}$. It follows that $(2^{-n}x - x_n)$ is an l-sequence in X, whence there is an infinite set $M \subset N$ such that

$$\Big(\sum_{n \in M} 2^{-n}\Big)x - x_M \in Y.$$

This yields the assertion.

In view of (*), in the rest of the proof we may and do assume that the sequence (x_n) is, in addition, m-independent. Further, for every infinite set $L \subset N$, let L' denote any infinite subset of L with $x_{L'} \in Y$. Now, let (L_k) be a sequence of disjoint infinite subsets of N and let $\{L_t: t \in R\}$ be a family of almost disjoint infinite subsets of N. Then the sets $M_k = L'_k$ and $M_t = L'_t$ are easily seen to satisfy (b) and (c), respectively. (Cf. [5], Lemma, and [12], Corollary 2.)

THEOREM 2. *Let* Y *be a* $\varkappa$*-subspace of a topological linear space* X. *If* Z *is a subspace of* X *with property* (L), *then*

(a) $Y + Z = X$;

(b) $\dim(Y \cap Z) \geqslant 2^{\aleph_0}$.

Proof. These assertions follow from Lemma 2(a) and (c), respectively.

The next result is also given in [8], Remarks.

COROLLARY 2. *If* Y *is a* $\varkappa$*-subspace of a topological linear space* X *and* Y *has property* (L), *then* $Y = X$.

THEOREM 3. *If* X *has property* (L) *and* Y *is a* $\varkappa$*-subspace of* X, *then*

(a) $\operatorname{codim} Y \leqslant 2^{\aleph_0}$;

(b) Y *is sequentially dense in* X.

Proof. Let (x_n) be an l-sequence in X. Then (a) follows directly from Theorem 2(a) applied to $Z = \operatorname{lin}\{x_M: M \subset N\}$.

(b) By Lemma 2(b), there is an l-sequence (y_n) in X such that $y_n \in Y$ for every $n \in N$. Fix $x \in X \setminus \{0\}$. Applying Lemma 2(a), we find $y \in Y$, $\lambda \geqslant 1$ and $M \subset N$ such that $x = y + \lambda y_M$. Then

$$v_n = y + \lambda \sum_{k=1}^{n} 1_M(k)\, y_k \to x \quad \text{as } n \to \infty$$

and $v_n \in Y$ for every $n \in N$.

Theorem 3, with a different proof and with "dense" in place of "sequentially dense" in (b), was given in [12], Theorem 2.

Proposition 1. *If X is a topological linear space with property* (L) *and Y is a closed subspace of X, then either Y or X/Y has property* (L).

Proof. Let Q: $X \to X/Y$ be the quotient map and let (x_n) be an l-sequence in X. Then $\sum_{n=1}^{\infty} Q(x_n)$ is subseries convergent in X/Y. Suppose X/Y does not have property (L). Then we must have

$$\dim \operatorname{lin}\{Q(x_n): n \in N\} < \aleph_0.$$

It follows that there exist $u_1, \ldots, u_k$ in X and a sequence (y_n) in Y such that $\{Q(u_1), \ldots, Q(u_k)\}$ is linearly independent and

$$x_n = y_n + \sum_{i=1}^{k} \lambda_{ni} u_i, \qquad n \in N,$$

where the λ_{ni}'s are scalars. Put $z_n = x_n - y_n$. Since

$$Q| \operatorname{lin}\{u_1, \ldots, u_k\}$$

is an isomorphism and the series $\sum_{n=1}^{\infty} Q(z_n)$ $(= \sum_{n=1}^{\infty} Q(x_n))$ is subseries convergent, so is the series $\sum_{n=1}^{\infty} z_n$. It follows that $\sum_{n=1}^{\infty} y_n$ is subseries convergent in Y. Clearly, (y_n) contains a linearly independent subsequence and therefore Y has property (L).

In the sequel we shall only need the following simple consequence of Proposition 1 (see Remark 2 of Section 6).

Corollary 3. *If X_1 and X_2 are topological linear spaces such that $X_1 \times X_2$ has property* (L), *then either X_1 or X_2 has property* (L).

5. Operators and $\varkappa$-subspaces. We start with a lemma. Note that its first assertion does not characterize $\varkappa$-subspaces; see [2], Theorem 3a.

Lemma 3. *Let Z be an F-space and let Y be a proper $\varkappa$-subspace of a topological linear space X. For every continuous linear operator T: $Z \to X$ we have*

(a) $\dim T(Z) < \aleph_0$ *provided* $T(Z) \subset Y$;

(b) $\dim(Y \cap T(Z)) \geqslant 2^{\aleph_0}$ *provided* $\dim T(Z) \geqslant \aleph_0$.

Proof. Since $T(Z)$ has property (L) whenever $\dim T(Z) \geqslant \aleph_0$, the assertion of the lemma follows immediately from Corollary 2 and Theorem 2(b), respectively.

A special case of our next result has been previously obtained by Pol

([16], Theorem 2.1); see also van Mill [15] for spaces which are even poorer in continuous linear operators.

THEOREM 4 ([1]). *Let X_1 and X_2 be F-spaces, let Y_1 be a $\varkappa$-subspace of X_1 and let Y_2 be a proper $\varkappa$-subspace of X_2. If S: $Y_1 \to Y_2$ is a continuous linear operator, then either $S(Y_1)$ or $S^{-1}(0)$ is finite-dimensional.*

Proof. By Theorem 3(b), Y_1 is dense in X_1. It follows that S extends to a continuous linear operator T: $X_1 \to X_2$. If $T(X_1) \subset Y_2$, then $T(X_1)$ and, a fortiori, $S(Y_1)$ is finite-dimensional, by Lemma 3(a).

Consider the case where $T(X_1) \not\subset Y_2$. We claim that $T^{-1}(0)$ is then finite-dimensional. Otherwise, in view of Theorem 2(a), we would have

$$Y_1 + T^{-1}(0) = X_1, \quad \text{whence} \quad T(X_1) \subset Y_2,$$

a contradiction.

COROLLARY 4 (cf. [16], Theorem 2.1(iii)). *Let Y be a proper $\varkappa$-subspace of an F-space and let Y be linearly homeomorphic to the product $Z_1 \times Z_2$ of two topological linear spaces. Then either Z_1 or Z_2 is finite-dimensional.*

Finally, Lemma 3 above and Theorem 2 of [11] yield the following generalization of a result due to Kruse ([10], Theorem A).

COROLLARY 5. *Let X be an F-space with* $\dim X = 2^{\aleph_0}$. *Then there are* $2^{2^{\aleph_0}}$ *(dense) subspaces Y of X such that assertions* (a) *and* (b) *of Lemma* 3 *hold for every continuous linear operator T: $Z \to X$, where Z is an arbitrary F-space.*

6. Dense subspaces with property (K) which are not $\varkappa$-subspaces. The existence of such subspaces for a class of F-spaces is due to Burzyk ([12], Example). His proof yields the following more general result, which is also an easy consequence of Corollary 2 above.

PROPOSITION 2. *Let X and Z be topological linear spaces and let Z have property* (L). *If X_1 and Z_1 are subspaces of X and Z, respectively, such that $X_1 \times Z_1$ is a $\varkappa$-subspace of $X \times Z$, then $X_1 = X$.*

In particular, in the situation of [12], Theorem 4, $X_1 \times X_2$ is not a $\varkappa$-subspace of $X \times X$.

It is still unknown whether every infinite-dimensional F-space contains a proper dense subspace with property (K), be it a $\varkappa$-subspace or not. (It is so if property (K) is weakened to that of being a Baire space; cf. [3], Theorem 3 and its proof.) The answer to this problem is known to be affirmative if

([1]) A version of this result was established by the authors jointly with I. Labuda during the 11th Winter School on Abstract Analysis, Železná Ruda, Czechoslovakia, 1983.

$\dim X = 2^{\aleph_0}$ ([11], Theorem 2). Another partial solution is contained in the following theorem.

THEOREM 5. *Let X be an F-space such that there exists an infinite-dimensional closed subspace Z with $\dim X/Z = 2^{\aleph_0}$. Then X contains a dense subspace Y such that*

(i) *Y has property* (K);

(ii) *Y has property* (L);

(iii) $\operatorname{codim} Y = 1$.

In particular, Y is not a $\varkappa$-subspace of X.

Note that, in view of Theorem 8 and Remark 5 below, (i) is not implied by the conjunction of (ii) and (iii).

Proof (cf. [5], proof of Theorem). Let W be a (dense) proper $\varkappa$-subspace of X/Z ([11], Theorem 2). We may and do assume that $\operatorname{codim} W = 1$. Let $Q: X \to X/Z$ be the quotient map and put $Y = Q^{-1}(W)$. As easily seen, (iii) holds. Moreover, since $Z \subset Y$, (ii) holds, which, in view of Corollary 2, yields the final part of the assertion. As Q is open, it follows that Y is dense in X. Finally, it is easy to check that, W having property (K), Y also has this property.

Remark 1 (cf. [5], Remark). The assumption of Theorem 5 is satisfied provided that X is an F-space whose topological dual X' is infinite-dimensional. Indeed, equip X' with its (Hausdorff) weak* topology. Then, in view of [9], Proposition 2.2, and [18], Chapter II, Proposition 7, there exists a biorthogonal system (f_n, x_n) with $f_n \in X'$ and $x_n \in X$. Choose $M \subset N$ so that M and $N \setminus M$ are both infinite. Define a continuous linear operator T: $X \to R^M$ by $T(x) = (f_n(x))_{n \in M}$ and put $Z = T^{-1}(0)$. Then

$$\{x_n:\ n \in N \setminus M\} \subset Z$$

and $\aleph_0 \leqslant \dim X/Z \leqslant \dim R^M = 2^{\aleph_0}$. It follows that $\dim X/Z = 2^{\aleph_0}$ (see, e.g., [11], Corollary 2).

In particular, the assumption of Theorem 5 is satisfied if X is an infinite-dimensional Banach (or Fréchet) space. On the other hand, this assumption fails for some F-spaces with dimension $> 2^{\aleph_0}$, which is a recent result of Popov [17] (answering Problem 1 of [5]). The case where $\dim X = 2^{\aleph_0}$ (and X is separable) is apparently open.

Our next result yields still another class of examples of spaces with property (K) which are not $\varkappa$-subspaces of their completions.

THEOREM 6. *Every F-space X with $\dim X = 2^{\aleph_0}$ contains a dense subspace Y such that*

(i) *Y has property* (K);
(ii) *Y does not have property* (L);
(iii) *Y is not a $\varkappa$-subspace of X.*

Proof (cf. [12], proof of Theorem 4([2])). Fix an m-independent sequence (x_n) in X such that $\sum_{n=1}^{\infty} x_n$ is bounded multiplier convergent; the existence of such sequences follows easily from $(*)$ because X is an F-space of infinite dimension (cf. [11], Proposition 1 and Lemma 4). We may and do assume that $\operatorname{codim}\langle(x_n)\rangle \geqslant \aleph_0$ ([3]). Indeed, we have

$$\langle(x_{2n})\rangle \cap \langle(x_{2n-1})\rangle = \{0\}.$$

Arrange the family of all m-independent sequences (y_n) in X such that $\sum_{n=1}^{\infty} y_n$ is bounded multiplier convergent and $\langle(x_n)\rangle \cap \langle(y_n)\rangle = \{0\}$ into a transfinite sequence $((y_n^\alpha))_{\alpha<\varphi}$ where φ is the initial ordinal of cardinality $2^{\aleph_0}$. (This family is non-empty by Corollary 1(b).) Moreover, let $(U_\alpha)_{\alpha<\varphi}$ be a base for the topology of X.

We shall construct inductively two increasing sequences $(X_\alpha)_{\alpha<\varphi}$ and $(Y_\alpha)_{\alpha<\varphi}$ of subspaces of X such that for all $\alpha < \varphi$

(1) $\langle(x_n)\rangle \cap \operatorname{lin}(X_\alpha \cup Y_\alpha) = \{0\}$;
(2) $\dim \operatorname{lin}(X_\alpha \cup Y_\alpha) \leqslant \aleph_0 + \operatorname{card}\alpha$;
(3) $X_\alpha \cap Y_\alpha = \{0\}$;
(4) $Y_\alpha \cap U_\alpha \neq \emptyset$;
(5) there exist infinite sets $M_\alpha, L_\alpha \subset N$ with $y_{M_\alpha}^\alpha \in X_\alpha$ and $y_{L_\alpha}^\alpha \in Y_\alpha$.

Suppose the construction has been carried out for all $\beta < \alpha$. Put

$$X^\alpha = \bigcup_{\beta<\alpha} X_\beta \quad \text{and} \quad Y^\alpha = \bigcup_{\beta<\alpha} Y_\beta.$$

Clearly, X^α and Y^α are subspaces of X. Moreover, in view of (2) and (3), we have

$$\dim \operatorname{lin}(X^\alpha \cup Y^\alpha) \leqslant \aleph_0 + \operatorname{card}\alpha \quad \text{and} \quad X^\alpha \cap Y^\alpha = \{0\}.$$

Using Corollary 1 of [11], we can find an infinite set $M_\alpha \subset N$ such that

$$x \notin \operatorname{lin}(X^\alpha \cup Y^\alpha \cup \langle(x_n)\rangle), \quad \text{where } x = y_{M_\alpha}^\alpha.$$

([2]) We note that condition (3) thereof should read: $\dim X_i^\alpha \leqslant \operatorname{card}\alpha$ for $i = 1, 2$.

([3]) Actually, this is always so. In fact, by [19], Theorem 1.22, and the Baire category theorem, $\operatorname{codim}\langle(x_n)\rangle > \aleph_0$.

Put $X_\alpha = \mathrm{lin}(X^\alpha \cup \{x\})$. By the same argument, we can find an infinite set $L_\alpha \subset N$ such that

$$y \notin \mathrm{lin}\big(X_\alpha \cup Y^\alpha \cup \langle\!\langle (x_n) \rangle\!\rangle\big), \quad \text{where } y = y^\alpha_{L_\alpha}.$$

Finally, since $\mathrm{lin}\big(X_\alpha \cup Y^\alpha \cup \langle\!\langle (x_n) \rangle\!\rangle \cup \{y\}\big) \neq X$, we can find

$$u \in U_\alpha \setminus \mathrm{lin}\big(X_\alpha \cup Y^\alpha \cup \langle\!\langle (x_n) \rangle\!\rangle \cup \{y\}\big).$$

Put $Y_\alpha = \mathrm{lin}(Y^\alpha \cup \{y, u\})$. As easily seen, X_α and Y_α satisfy (1)–(5).

Put $Y = \bigcup_{\alpha < \varphi} Y_\alpha$. Clearly, Y is a subspace of X. In view of (4), Y is dense in X. We infer from (1) that $\langle\!\langle (x_n) \rangle\!\rangle \cap Y = \{0\}$. This yields (iii) and, in view of (5) and (*), (i). Suppose (ii) fails and fix an l-sequence (y_n) in Y. Then, by passing to a subsequence and using (*), we may assume that (y_n) is m-independent and $\sum_{n=1}^{\infty} y_n$ is bounded multiplier convergent. In view of Corollary 1(b), we have $(y_{n_k}) = (y^\alpha_n)$ for some $n_1 < n_2 < \ldots$ and $\alpha < \varphi$. From this and (5) we get $X_\alpha \cap Y \neq \{0\}$, which contradicts (3).

Remarks. 2. Under the additional assumption that $X = Z \times Z$, where Z is an F-space, Theorem 6 also follows from Theorem 4 of [12] and Theorem 3(b), Proposition 2 and Corollaries 2 and 3 above.

3. Theorem 6 partially solves a problem due to J. Burzyk and the second author (1980). The problem is concerned with the existence, in an arbitrary infinite-dimensional F-space X, of a dense subspace with property (K) which is not a $\varkappa$-subspace and does not contain any infinite-dimensional closed subspace of X. Without the denseness condition the answer is also positive in case $\dim X > 2^{\aleph_0}$. Indeed, it is then enough to apply Theorem 6 to a closed subspace of X with dimension $2^{\aleph_0}$.

7. Subspaces of codimension 1 without property (K). We shall present two results on the existence of such subspaces. Since a metrizable topological linear space with property (K) is a Baire space ([3], Theorem 2), the first of these results is, in some respects, weaker than the following one due to Arias de Reyna ([1], Theorem 1) and Valdivia ([20], Theorem 1): Every infinite-dimensional separable Baire topological linear space contains a (dense) subspace of codimension 1 which is not a Baire space. This last result was, however, obtained under Martin's axiom, and it is not known whether it holds in ZFC.

THEOREM 7. *Every topological linear space X with property* (L) *contains a dense subspace Y such that*

(i) *Y does not have property* (K);

(ii) $\mathrm{codim}\, Y = 1$.

Proof. Denote by S the linear subspace of R^N (over R) generated by the set $\{1_M: M \subset N\}$. Let μ: $2^N \to R$ be an additive set function such that $\mu(M) = 0$ if and only if $M \subset N$ is finite. (The existence of such set functions was proved in [6], Proposition 5; see also Proposition 3 of Section 8 below.) Denote by I_μ the unique linear functional on S with $I_\mu(M) = \mu(M)$ for all $M \subset N$.

Let (x_n) be an m-independent l-sequence in X (see $(*)$) and put

$$T(s) = \sum_{n=1}^{\infty} s(n)\, x_n \quad \text{for } s \in S.$$

Then T: $S \to X$ is a linear isomorphism. Define

$$g\colon T(S) \to R \quad \text{by} \quad g(T(s)) = I_\mu(s) \quad \text{for } s \in S.$$

Then g is a linear functional with $g(x_n) = 0$ for all $n \in N$ and $g(x_M) \neq 0$ for all infinite $M \subset N$. Extend g to a linear functional f on the whole of X and put $Y = f^{-1}(0)$. Clearly, (i) and (ii) hold. The denseness of Y in X is seen as $x_n \in Y$ and $x_N \notin Y$.

We shall need the following simple lemma.

Lemma 4 (cf. [7], Corollary 4.2). *Let Y and W be topological linear spaces. If Y is metrizable and has property* (K) *and there exists a surjective open continuous linear operator T: $Y \to W$, then W also has property* (K).

Proof. Let $w_n \in W$ and $w_n \to 0$. For every neighbourhood U of 0 in Y, we have $w_n \in T(U)$ for n large enough. Accordingly, we can choose a subsequence (w_{n_k}) of (w_n) and a sequence (y_k) in Y such that $T(y_k) = w_{n_k}$ and $y_k \to 0$. The assertion readily follows from this.

Theorem 8. *Let X be an F-space such that there exists a closed subspace Z with $\dim X/Z = 2^{\aleph_0}$. Then X contains a* (*dense*) *subspace Y such that*

(i) *Y does not have property* (K);

(ii) *Y contains a dense subspace Y_0 with property* (K);

(iii) $\operatorname{codim} Y = 1$.

In particular, Y is a Baire space.

Proof. We first note that the additional assertion follows from (ii). Indeed, as Y_0 is a Baire space ([3], Theorem 2), so is Y, by [7], Theorem 1.15.

We next prove the rest of the result under the additional assumption that $Z = \{0\}$. Let (x_n) be an m-independent sequence such that $\sum_{n=1}^{\infty} x_n$ is bounded multiplier convergent in X and let Y_0 be a dense subspace of X which has property (K) and satisfies

$$\langle (x_n) \rangle \cap Y_0 = \{0\}$$

(see the proof of Theorem 6). Let g be a linear functional on $\langle (x_n) \rangle$ such that $g(x_M) = 0$ if and only if $M \subset N$ is finite (see the proof of Theorem 7). Extend g to a linear functional f on the whole of X with $f | Y_0 = 0$ and put $Y = f^{-1}(0)$. Then (i) and (iii), clearly, hold. Since $Y \supset Y_0$, (ii) also holds.

In the general case, by what we have proved so far, there exists a subspace W of X/Z such that (i)–(iii) hold. Let Q: $X \to X/Z$ be the quotient map and put $Y = Q^{-1}(W)$. Then (ii) and (iii) are easily seen to hold and (i) follows from Lemma 4.

Remarks. 4. Theorem 8 is a partial strengthening of [3], Theorem 3.

5. The proof of Theorem 8 can be modified so that the following additional condition holds:

(iv) *Y has property* (L).

Indeed, it is enough to take g with $g| \langle (x_{2n-1}) \rangle = 0$ and $g(x_M) = 0$ if and only if $M \subset 2N$ is finite.

8. Appendix. We shall give a new proof of the result due to Godefroy and Talagrand which has been used in the proofs of Theorems 7 and 8 above. This proof is, in contrast to the original one, purely algebraic. Also, it can easily be adapted to yield a more general result (Proposition 3′ below).

PROPOSITION 3 ([6], Proposition 5). *There exists an additive set function μ: $2^N \to R$ such that $\mu(M) = 0$ if and only if $M \subset N$ is finite.*

Proof. Denote by Q the field of rational numbers and let S_Q be the linear subspace of Q^N generated by the set $\{1_M: M \subset N\}$. Let B be a subset of the latter set such that

$$B \cup \{1_{\{n\}}: n \in N\}$$

is a Hamel basis for S_Q. Let, further, $\{\lambda_b: b \in B\}$ be a set of real numbers which is independent over Q. Then there exists a unique Q-linear operator T: $S_Q \to R$ such that

$$T(b) = \lambda_b \quad \text{and} \quad T(1_{\{n\}}) = 0$$

for $b \in B$ and $n \in N$. Put $\mu(M) = T(1_M)$ for $M \subset N$. We claim that μ is as desired. Indeed, we have

$$1_M = \sum_{i=1}^{m} r_i b_i + \sum_{j=1}^{p} s_j 1_{\{n_j\}}$$

for some r_i, $s_j \in Q$ and $b_i \in B$ and $n_j \in N$. If now $\mu(M) = 0$, then $\sum_{i=1}^{m} r_i \lambda_{b_i} = 0$, whence $r_i = 0$, $i = 1, \ldots, m$. It follows that M is finite.

PROPOSITION 3'. *If $\mathfrak{M}$ is an algebra of subsets of a set Ω with* card $\mathfrak{M} \leqslant 2^{\aleph_0}$, *then there exists an additive set function μ: $\mathfrak{M} \to R$ such that $\mu(M) \neq 0$ for every non-empty set $M \in \mathfrak{M}$.*

The latter result yields the former when applied, via the Stone representation theorem, to the Boolean algebra 2^N modulo the ideal of finite sets.

Added in proof. For a result related to Theorem 1 see Z. Lipecki, *Residual sets of compact operators and of vector-valued measures*, in preparation.

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