ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXVIII (1989)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXVIII (1989)



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# Difference methods for nonlinear parabolic differential-functional systems with initial boundary conditions of the Neumann type

**Introduction.** The problems of finite difference approximation to initial boundary problems for parabolic equations were considered by many authors and under various assumptions. Numerical treatment of the Cauchy problem is found in [1], [2], [8], [10], [24]. In [1], [8] the approximated solutions are assumed to be bounded. In [2], [24] the solutions are allowed to belong to a natural class of fast increasing functions. Difference methods for nonlinear parabolic differential equations with initial boundary conditions of the Dirichlet type were considered in [4]–[6], [11], [14], [15]. An error estimate implying the convergence of difference schemes is obtained in these papers by difference inequalities methods. Numerical treatment of the initial boundary problem of the Neumann type is found in [13], [16]–[19]. The method of lines for nonlinear parabolic equations is considered in [10], [24]–[27]. By using a discretization in the spatial variable, the parabolic equation is replaced by a sequence of initial problems for ordinary differential equations. In [10], [24]–[27] the question of under what conditions the solutions of ordinary equations tend to a solution of the original problem is investigated.

In [12], the author studies the error due to the discretization in time of a nonlinear parabolic problems by a multistep method. Error estimates are obtained if the method is of the order $p > 1$ and is strongly stable. Paper [7] discusses Runge–Kutta methods for stiff differential equations of high dimensions. A second order method is constructed and numerical results of stiff problems originating from linear and nonlinear parabolic equations are presented. Paper [3] deals with a linear parabolic equation and with a method, which is founded on linear multistep methods. The author studies the stability and the convergence for this method.

The methods mentioned above have an extensive bibliography. For further information see the references in [20]–[23], [28] and in the papers cited above.

In the present paper we consider parabolic differential-functional systems with initial boundary conditions of the Neumann type. We consider general one-step methods for this problem. We introduce some general difference operators which enable us to get new difference schemes which are convergent to the exact solution of the differential-functional problem. In the first part of the paper we give some estimates of the difference between solutions of recurrent equations. We apply these estimates in the investigation of the stability of difference methods.

**1. Estimates of the difference between solutions of recurrent equations.** Let us denote by $\mathscr{F}(X, Y)$ the set of all functions defined on $X$ taking values in $Y$; $X$, $Y$ being arbitrary sets. Let $d = (d_0, d_1, \ldots, d_n) \in R^{n+1}$ where $d_i > 0$ for $i = 0, 1, \ldots, n$ and $I_d \subset (0, d]$. For each $h = (h_0, h_1, \ldots, h_n) \in I_d$ define the sets $N_h = \{-n_0(h), \ldots, -1, 0, 1, \ldots, n^*(h)\}$, $N_h^{(0)} = \{-n_0(h), \ldots, -1, 0\}$ and $N_h^* = \{0, 1, \ldots, n^*(h)-1\}$ where $n_0(h)$, $n^*(h)$ are integers and $n_0(h) \geqslant 0$, $n(h) > 0$. In the sequel we write $n_0$ and $n^*$ instead of $n_0(h)$ and $n^*(h)$ for $h \in I_d$. Suppose that $\Omega$ is a given set and $\Omega_h \subset \Omega$ for $h \in I_d$. Let $\Gamma_h \subset \mathscr{F}(N_h \times \Omega_h, R^k)$ and $F_h\colon N_h^* \times \Gamma_h \to \mathscr{F}(\Omega_h, R^k)$ where $h \in I_d$.

Suppose that $\Omega_h^{(0)} \subset \Omega_h$ is a given set and $\Omega_h \setminus \Omega_h^{(0)} \neq \emptyset$, $h \in I_d$. For each $(i, t) \in N_h^* \times \Omega_h$, we define the set $V(i, t)$ in the following way. If $(i, t) \in N_h^* \times (\Omega_h \setminus \Omega_h^{(0)})$, then $V(i, t) = \{(j, \tau) \in N_h \times \Omega_h\colon j \leqslant i\}$. For $(i, t) \in N_h^* \times \Omega_h^{(0)}$ we set $V(i, t) = \{(j, \tau) \in N_h \times \Omega_h\colon j \leqslant i\} \cup \{(j, \tau) \in N_h \times (\Omega_h \setminus \Omega_h^{(0)})\colon j = i+1\}$. The function $F_h$ is said to satisfy *condition* (V) with respect to $\Omega_h^{(0)}$ if for each $(i, t) \in N_h^* \times \Omega_h$ and for $u, v \in \Gamma_h$ such that $u(j, \tau) = v(j, \tau)$ for $(j, \tau) \in V(i, t)$ we have $F_h(i, u)(t) = F_h(i, v)(t)$.

Assume that $i \in N_h$, $z \in \mathscr{F}(N_h \times \Omega_h, R^k)$ and $z$ is a bounded function. Then we define

$$\|z\|_i^{(h)} = \sup\{\|z(j, t)\|\colon\ -n_0 \leqslant j \leqslant i,\ t \in \Omega_h\}$$

where $\|\cdot\|$ is a norm in $R^k$. If $(i, t) \in N_h \times \Omega_h$ and $z = (z_1, \ldots, z_k) \in \mathscr{F}(N_h \times \Omega_h, R^k)$, then we write

$$\sup_{(j,\tau) \in V(i,t)} z(j, \tau) = \Big(\sup_{(j,\tau) \in V(i,t)} z_1(j, \tau), \ldots, \sup_{(j,\tau) \in V(i,t)} z_k(j, \tau)\Big).$$

Suppose that $(i, z)$, $(i, \bar{z}) \in N_n^* \times \Gamma_h$ and $F_h\colon N_h^* \times \Gamma_h \to \mathscr{F}(\Omega_h, R^k)$. Then we write $F_n(i, z) \leqslant F_h(i, \bar{z})$ if for each $t \in \Omega_h$ we have $F_h(i, z)(t) \leqslant F_h(i, \bar{z})(t)$. For $p = (p_1, \ldots, p_k) \in R^k$ we write $|p| = (|p_1|, \ldots, |p_k|)$.

Let $\mathscr{F}_0(N_h \times \Omega_h, R_+^k) \subset \mathscr{F}(N_h \times \Omega_h, R_+^k)$ be the class of all functions defined on $N_h \times \Omega_h$ taking values in $R_+^k$ which depend on the first argument only. Suppose that $U_h\colon \mathscr{F}_0(N_h \times \Omega_h, R_+^k) \to \mathscr{F}(N_h, R_+^k)$ is an operator defined by $(U_h z)(i) = z(i, t)$, where $(i, t) \in N_h \times \Omega_h$ and $z \in \mathscr{F}_0(N_h \times \Omega_h, R_+^k)$. It is seen at once that the mapping $U_h$ is bijective. Elements of the sets $\mathscr{F}_0(N_h \times \Omega_h, R_+^k)$ and $\mathscr{F}(N_h, R_+^k)$ will be denoted by the same symbols.

We consider the initial problem for the recurrent equation

$$(1)\qquad \begin{aligned} z(i+1, t) &= F_h(i, z)(t), \quad t \in \Omega_h, \ i \in N_h^*, \\ z(i, t) &= \omega_h(i, t) \quad \text{for } (i, t) \in N_h^{(0)} \times \Omega_h, \end{aligned}$$

where $\omega_h \colon N_h^{(0)} \times \Omega_h \to R^k$ is a given function.

Let $\mathscr{F}_0^*(N_h \times \Omega_h, R_+^k) \subset \mathscr{F}_0(N_h \times \Omega_h, R_+^k)$. In the further parts of the paper we introduce additional assumptions on $\mathscr{F}_0^*(N_h \times \Omega_h, R_+^k)$.

If we assume that $F_h$ satisfies condition (V), then problem (1) has exactly one solution.

Let us consider two problems, the initial problem (1) and the following one:

$$(2)\qquad \begin{aligned} z(i+1, t) &= \tilde{F}_h(i, z)(t), \quad t \in \Omega_h, \ i \in N_h^*, \\ z(i, t) &= \tilde{\omega}_h(i, t) \quad \text{for } (i, t) \in N_h^{(0)} \times \Omega_h, \end{aligned}$$

where $\tilde{F}_h \colon N_h^* \times \Gamma_h \to \mathscr{F}(\Omega_h, R^k)$ and $\tilde{\omega}_h \colon N_h^{(0)} \times \Omega_h \to R^k$. We are interested in finding an estimate of the difference between the solutions of (1) and (2).

LEMMA 1. *Suppose that:*

(a) $F_h, \tilde{F}_h \colon N_h^* \times \Gamma_h \to \mathscr{F}(\Omega_h, R^k)$, $\omega_h, \tilde{\omega}_h \colon N_h^{(0)} \times \Omega_h \to R^k$,

(b) $\Gamma_h \subset \mathscr{F}(N_h \times \Omega_h, R^k)$ *is such a set that if* $z \in \Gamma_h$ *and* $\bar{z} \in \mathscr{F}_0^*(N_h \times \Omega_h, R_+^k)$ *then* $z + \bar{z} \in \Gamma_h$,

(c) *for each* $h \in I_d$ *the function* $F_h$ *is nondecreasing with respect to the functional argument and satisfies condition* (V),

(d) *for each* $h \in I_d$ *there exists a function* $Q_h \colon N_h^* \times R_+^k \to R_+^k$ *such that*

(i) $Q_h$ *is nondecreasing with respect to the second argument and* $Q_h(i, p) \geqslant p$ *for* $(i, p) \in N_h^* \times R_+^k$,

(ii) *for* $z \in \Gamma_h$, $\bar{z} \in \mathscr{F}_0^*(N_h \times \Omega_h, R_+^k)$ *we have*

$$F_h(i, z+\bar{z})(t) - F_h(i, z)(t) \leqslant Q_h\big(i, \sup_{(j,\tau) \in V(i,t)} \bar{z}(j, \tau)\big), \quad (i, t) \in N_h^* \times \Omega_h,$$

(e) $u_h$ *and* $v_h$ *are solutions of* (1) *and* (2), *respectively, and there exists* $\gamma_h \colon N_h^* \to R_+^k$ *such that*

$$|F_h(i, v_h)(t) - \tilde{F}_h(i, v_h)(t)| \leqslant \gamma_h(i) \quad \text{for } (i, t) \in N_h^* \times \Omega_h,$$

(f) $\beta_h, \tilde{\beta}_h \in \mathscr{F}_0^*(N_h \times \Omega_h, R_+^k)$ *are functions such that*

(i) $\beta_h(i) = \tilde{\beta}_h(i)$ *for* $i \in N_h^{(0)}$ *and* $|\omega_h(i, t) - \tilde{\omega}_h(i, t)| \leqslant \beta_h(i)$ *for* $(i, t) \in N_h^{(0)} \times \Omega_h$,

(ii) *for* $i \in N_h^*$ *we have*

$$\beta_h(i+1) = Q_h(i, \tilde{\beta}_h(i)) + \gamma_h(i), \qquad \tilde{\beta}_h(i+1) = Q_h(i, \beta_h(i+1)) + \gamma_h(i).$$

*Under these assumptions we have*

$$(3)\qquad \begin{aligned} &|u_h(i, t)-v_h(i, t)| \leqslant \beta_h(i), \quad i = 0, 1, \ldots, n^*, \ t \in \Omega_h \backslash \Omega_h^{(0)}, \\ &|u_h(i, t)-v_h(i, t)| \leqslant \tilde{\beta}_h(i), \quad i = 0, 1, \ldots, n^*, \ t \in \Omega_h^{(0)}. \end{aligned}$$

Proof. It follows from assumptions (d) and (f) that $\tilde{\beta}_h(i) \geqslant \beta_h(i)$ for $i = 0, 1, \ldots, n^*$ and $\beta_h(i+1) \geqslant \tilde{\beta}_h(i)$ for $i \in N_h^*$. Estimation (3) is equivalent with $\tilde{w}_h(i, t) \leqslant u_h(i, t) \leqslant w_h(i, t)$, $i = 0, 1, \ldots, n^*$, $t \in \Omega_h$, where $w_h(i, t) = v_h(i, t)+\beta_h(i)$ for $(i, t) \in N_h \times (\Omega_h \backslash \Omega_h^{(0)})$, $w_h(i, t) = v_h(i, t)+\tilde{\beta}_h(i)$ for $(i, t) \in N_h \times \Omega_h^{(0)}$ and $\tilde{w}_h(i, t) = v_h(i, t)-\beta_h(i)$ for $(i, t) \in N_h \times (\Omega_h \backslash \Omega_h^{(0)})$, $\tilde{w}_h(i, t) = v_h(i, t)-\tilde{\beta}_h(i)$ for $(i, t) \in N_h \times \Omega_h^{(0)}$.

If $(i, t) \in N_h^* \times (\Omega_h \backslash \Omega_h^{(0)})$ then

$$\begin{aligned} w_h(i+1, t) &= v_h(i+1, t)+\beta_h(i+1) \\ &\geqslant F_h(i, w_h)(t)+\tilde{F}_h(i, v_h)(t)-F_h(i, v_h)(t) \\ &\qquad\qquad -[F_h(i, v_h+\tilde{\beta}_h)(t)-F_h(i, v_h)(t)]+\beta_h(i+1) \\ &\geqslant F_h(i, w_h)(t)-\gamma_h(i)-Q_h(i, \tilde{\beta}_h(i))+\beta_h(i+1) = F_h(i, w_h)(t). \end{aligned}$$

For $(i, t) \in N_h^* \times \Omega_h^{(0)}$ we have

$$\begin{aligned} w_h(i+1, t) &= v_h(i+1, t)+\tilde{\beta}_h(i+1) \\ &\geqslant F_h(i, w_h)(t)+\tilde{F}_h(i, v_h)(t)-F_h(i, v_h)(t) \\ &\qquad\qquad -[F_h(i, v_h+\tilde{\beta}_h)(t)-F_h(i, v_h)(t)]+\tilde{\beta}_h(i+1) \\ &\geqslant F_h(i, w_h)(t)-\gamma_h(i)-Q_h(i, \beta_h(i+1))+\tilde{\beta}_h(i+1) = F_h(i, w_h)(t). \end{aligned}$$

Thus we see that recurrent inequalities

$$(4)\qquad \begin{aligned} &w_h(i+1, t) \geqslant F_h(i, w_h)(t), \quad (i, t) \in N_h^* \times \Omega_h, \\ &u_h(i+1, t) \leqslant F_h(i, u_h)(t), \quad (i, t) \in N_h^* \times \Omega_h, \end{aligned}$$

and the initial inequality

$$(5)\qquad w_h(i, t) \geqslant u_h(i, t), \quad (i, t) \in N_h^{(0)} \times \Omega_h$$

hold. Relations (4), (5) imply $u_h(i, t) \leqslant w_h(i, t)$ for $i = 0, 1, \ldots, n^*$ $t \in \Omega_h$. In a similar way we prove that $\tilde{w}_h(i, t) \leqslant u_h(i, t)$ for $i = 0, 1, \ldots, n^*$, $t \in \Omega_h$. This completes the proof.

EXAMPLE 1. Suppose that assumptions (a)–(c) of Lemma 1 are satisfied and $u_h$, $v_h$ are solutions of (1) and (2), respectively. Assume that

(α) there exists a matrix $\Lambda = [\lambda_{ij}]_{i,j=1,\ldots,k}$ with $\lambda_{ij} \geqslant 0$ such that for $\bar{z} \in \mathscr{F}_0^*(N_h \times \Omega_h, R_+^k)$ we have

$$[F_h(i, z+\bar{z})(t)-F_h(i, z)(t)]^T \leqslant (I+h_0\Lambda)[\sup_{(j,\tau)\in V(i,t)} \bar{z}(j, \tau)]^T$$

(here $I$ denotes the $k\times k$ unit matrix and $T$ means transposition of the vector or matrix),

(β) there exist vectors $C_h = (C_h^{(1)}, \ldots, C_h^{(k)})$, $\sigma_{0,h} = (\sigma_{0,h}^{(1)}, \ldots, \sigma_{0,h}^{(k)})$ such that

$$|F_h(i, v_h)(t)-\tilde{F}_h(i, v_h)(t)| \leqslant h_0 C_h, \quad (i, t)\in N_h^* \times \Omega_h,$$

and

$$|\omega_h(i, t)-\tilde{\omega}_h(i, t)| \leqslant \sigma_{0,h} \quad \text{for } (i, t)\in N_h^{(0)} \times \Omega_h.$$

Then we define $\beta_h,\tilde{\beta}_h$: $N_h \to R_+^k$ by the formulas

$$\begin{aligned}
\beta_h(i) &= \tilde{\beta}_h(i) = \sigma_{0,h} && \text{for } i\in N_h^{(0)},\\
\beta_h(i+1)^T &= (I+h_0\Lambda)\tilde{\beta}_h(i)^T + h_0 C_h^T && \text{for } i\in N_h^*,\\
\tilde{\beta}_h(i+1)^T &= (I+h_0\Lambda)\beta_h(i+1)^T + h_0 C_h^T && \text{for } i\in N_h^*.
\end{aligned}$$

The above relations lead to formulas

$$\begin{aligned}
\beta_h(i)^T &= (I+h_0\Lambda)^{2i-1}\sigma_{0,h}^T + h_0 \sum_{j=0}^{2i-2} (I+h_0\Lambda)^j C_h^T,\\
\tilde{\beta}_h(i)^T &= (I+h_0\Lambda)^{2i}\sigma_{0,h}^T + h_0 \sum_{j=0}^{2i-1} (I+h_0\Lambda)^j C_h^T,
\end{aligned} \tag{6}$$

where $i = 1, 2, \ldots, n$. Assertion (3) has now the form

$$\begin{aligned}
|u_h(i, t)-v_h(i, t)| &\leqslant \beta_h(i), \quad i = 1, \ldots, n^*,\ t\in\Omega_h\backslash\Omega_h^{(0)},\\
|u_h(i, t)-v_h(i, t)| &\leqslant \tilde{\beta}_h(i), \quad i = 1, \ldots, n^*,\ t\in\Omega_h^{(0)}
\end{aligned} \tag{7}$$

with the above given $\beta_h$ and $\tilde{\beta}_h$. Let $\|\cdot\|$ be the norm in $R^k$ defined by $\|p\| = \max_{1\leqslant i\leqslant k} |p_i|$, $p = (p_1, \ldots, p_k)\in R^k$. If $\|\Lambda\| > 0$, where

$$\|\Lambda\| = \max_{1\leqslant i\leqslant k} \sum_{j=1}^{k} |\lambda_{ij}|,$$

then (6), (7) lead to

$$\begin{aligned}
\|u_h(i, t)-v_h(i, t)\| &\leqslant (1+h_0\|\Lambda\|)^{2i-1}\|\sigma_{0,h}\|\\
&\quad +\|C_h\|\,\|\Lambda\|^{-1}[(1+h_0\|\Lambda\|)^{2i-1}-1], \quad t\in\Omega_h\backslash\Omega_h^{(0)},
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\|u_h(i, t)-v_h(i, t)\|\\
&\leqslant (1+h_0\|\Lambda\|)^{2i} + \|C_h\|\,\|\Lambda\|^{-1}[(1+h_0\|\Lambda\|)^{2i}-1], \quad t\in\Omega_h^{(0)},
\end{aligned} \tag{9}$$

where $i = 1, \ldots, n^*$. If $\|A\| = 0$ then (6), (7) lead to

$$\|u_h(i, t) - v_h(i, t)\| \leq \|\sigma_{0,h}\| + h_0(2i-1)\|C_h\|, \qquad t \in \Omega_h \setminus \Omega_h^{(0)},$$

$$\|u_h(i, t) - v_h(i, t)\| \leq \|\sigma_{0,h}\| + h_0 2i\|C_h\|, \qquad t \in \Omega_h^{(0)},$$

where $i = 1, \ldots, n^*$.

**2. The convergence of one-step methods for differential-functional problems.** We will denote by $C(X, Y)$ the set of all continuous functions defined on $X$ and taking values in $Y$; $X$, $Y$ being arbitrary metric spaces. Let $E = (0, a] \times (-b, b)$, where $a > 0$, $b = (b_1, \ldots, b_n)$, $b_i > 0$ for $i = 1, \ldots, n$ and $E^{(0)} = [-\tau_0, 0] \times [-b, b]$, $\tau_0 \geqslant 0$. If $w$: $E^{(0)} \cup E \to R$ is a function of the variables $(x, y)$, $y = (y_1, \ldots, y_n)$, and there exist derivatives $D_{y_i} w$, $D_{y_i y_j} w$ for $i, j = 1, \ldots, n$, then we write $D_y w = (D_{y_1} w, \ldots, D_{y_n} w)$, $D_{yy} w = [D_{y_i y_j} w]_{i,j=1,\ldots,n}$. For a function $z = (z_1, \ldots, z_k)$: $E^{(0)} \cup E \to R$ of the variables $(x, y)$ we write $D_x z = (D_x z_1, \ldots, D_x z_k)$ and $D_{y_i} z = (D_{y_i} z_1, \ldots, D_{y_i} z_k)$, $D_{y_i y_j} z = (D_{y_i y_j} z_1, \ldots, D_{y_i y_j} z_k)$, where $i, j = 1, \ldots, n$. In a similar way we define $D_{y_i y_j y_l} z$ if $z$ has the adequate derivatives of the third order. Let us denote by $\partial E_j^{(-)}$ and $\partial E_j^{(+)}$ the sets

$$\partial E_j^{(-)} = \{(x, y) \in (0, a] \times [-b, b]:\ y_j = -b_j\},$$

$$\partial E_j^{(+)} = \{(x, y) \in (0, a] \times [-b, b]:\ y_j = b_j\}.$$

Let $\Gamma = E \times R^k \times C(E^{(0)} \cup \bar{E}, R^k) \times R^n \times R^{n^2}$ and assume that $f = (f_1, \ldots, f_k)$: $\Gamma \to R^k$, $\varphi = (\varphi_1, \ldots, \varphi_k)$: $E^{(0)} \to R^k$ are given functions. Suppose that

$$g_j = (g_{j,1}, \ldots, g_{j,k}):\ \bar{E} \times C(E^{(0)} \cup \bar{E}, R^k) \to R^k, \qquad j = 1, \ldots, n,$$

and

$$\tilde{g}_j = (\tilde{g}_{j,1}, \ldots, \tilde{g}_{j,k}):\ \bar{E} \times C(E^{(0)} \cup \bar{E}, R^k) \to R^k, \qquad j = 1, \ldots, n$$

($\bar{E}$ is the closure of $E$). For $z = (z_1, \ldots, z_k)$: $E^{(0)} \cup \bar{E} \to R^k$ write

$$f(x, y, z(x, y), z, D_y z(x, y), D_{yy} z(x, y))$$
$$= \big(f_1(x, y, z(x, y), z, D_y z_1(x, y), D_{yy} z_1(x, y)), \ldots$$
$$\ldots, f_k(x, y, z(x, y), z, D_y z_k(x, y), D_{yy} z_k(x, y))\big).$$

We consider the differential-functional problem

$$(10)\quad \begin{aligned} &D_x z(x, y) = f(x, y, z(x, y), z, D_y z(x, y), D_{yy} z(x, y)), && (x, y) \in E,\\ &D_{y_j} z(x, y) = g_j(x, y, z) && \text{for } (x, y) \in \partial E_j^{(-)},\ j = 1, \ldots, n,\\ &D_{y_j} z(x, y) = \tilde{g}_j(x, y, z) && \text{for } (x, y) \in \partial E_j^{(+)},\ j = 1, \ldots, n,\\ &z(x, y) = \varphi(x, y) && \text{for } (x, y) \in E^{(0)}. \end{aligned}$$

We define a mesh in $E^{(0)} \cup \bar{E}$ in the following way. Let $I_0 = \{h = (h_0, h_1, \ldots, h_n)$: there exist natural numbers $\tilde{n}$, $n_j^*$, $j = 1, \ldots, n$, such that $a + \tau_0 = \tilde{n} h_0$, $b_j = h_j n_j^*$ for $j = 1, \ldots, n\}$. For $h \in I_0$ we write $x^{(-n_0+i)} = -\tau_0 + i h_0$, $i = 0, 1, \ldots, n_0 + n^*$, where $n_0$ and $n^*$ are natural numbers such that $a + \tau_0 = (n_0 + n^*) h_0$, $x^{(0)} \leqslant 0 < x^{(1)}$, and $y_j^{(m_j)} = m_j h_j$ for $m_j = 0, \pm 1, \ldots, \pm n_j^*$ where $j = 1, \ldots, n$. For $m = (m_0, m_1, \ldots, m_n)$ we write $m' = (m_1, \ldots, m_n)$ and $y^{(m')} = (y_1^{(m_1)}, \ldots, y_n^{(m_n)})$. Let $M = \{m' \colon m_j = 0, \pm 1, \ldots, \pm(n_j^* - 1)$, $j = 1, \ldots, n\}$ and $M^* = \{m' \colon m_j = 0, \pm 1, \ldots, \pm n_j^*$ for $j = 1, \ldots, n\}$. We define $E_h = \{(x^{(m_0)}, y^{(m')}) \colon m_0 = 1, \ldots, n^*,\ m' \in M\}$ and $E_h^{(0)} = \{(x^{(m_0)}, y^{(m')}) \colon m_0 = -n_0, -n_0 + 1, \ldots, -1, 0,\ m' \in M^*\}$. Let

$$T^* = \{m \colon -n_0 \leqslant m_0 \leqslant n^*,\quad m' \in M^*\}, \qquad \tilde{T} = \{m \colon 1 \leqslant m_0 \leqslant n^*,\ m' \in M^*\}$$

and for $i = 1, \ldots, n$

$$T_i^{(-)} = \{m \in \tilde{T} \colon m_i = -n_i^* \text{ if } 1 \leqslant j < i \text{ then } m_j \neq -n_j^* \text{ and } m_j \neq n_j^*\},$$

$$T_i^{(+)} = \{m \in \tilde{T} \colon m_i = n_i^* \text{ if } 1 \leqslant j < i \text{ then } m_j \neq -n_j^* \text{ and } m_j \neq n_j^*\}.$$

Put $Z_i^{(-)} = \{(x^{(m_0)}, y^{(m')}) \colon m \in T_i^{(-)}\}$ and $Z_i^{(+)} = \{(x^{(m_0)}, y^{(m')}) \colon m \in T_i^{(+)}\}$, where $i = 1, \ldots, n$. If $0 \leqslant j \leqslant n$ then we define $j(m) = (m_0, \ldots, m_{j-1}, m_j + 1, m_{j+1}, \ldots, m_n)$ and $-j(m) = (m_0, \ldots, m_{j-1}, m_j - 1, m_{j+1}, \ldots, m_n)$. For a function $w \colon E_h^{(0)} \cup E_h \to R$ we write $w^{(m)} = w(x^{(m_0)}, y^{(m')})$, $m \in T^*$, and $z^{(m)} = (z_1^{(m)}, \ldots, z_k^{(m)})$ for $z \colon E_h^{(0)} \cup E_h \to R^k$. Let $E_h^* = \{(x^{(m_0)}, y^{(m')}) \colon m_0 = 0, 1, \ldots, n^*,\ m' \in M^*\}$.

Let $j^0(m) = -j^0(m) = m$ and $j^{i+1}(m) = j(j^i(m))$, $-j^{i+1} = -j(-j^i(m))$ for $i = 0, 1, 2, \ldots$ In a similar way we define $j^i(m')$ and $-j^i(m')$, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \ldots$

If $x \in [-\tau_0, a]$ then we write $H_x = \{(\xi, \eta) = (\xi, \eta_1, \ldots, \eta_n) \in E^{(0)} \cup \bar{E} \colon \xi \leqslant x\}$. For $p = (p_1, \ldots, p_k) \in R^k$ we define $\|p\| = \max_{1 \leqslant i \leqslant k} |p_i|$. If $z \in C(E^{(0)} \cup \bar{E}, R^k)$ then $\|z\|_x = \max_{(\xi, \eta) \in H_x} \|z(\xi, \eta)\|$, $x \in [-\tau_0, a]$. Assume that the differential-functional problem (10) is of the Volterra type, i.e., if $x \in (0, a]$, $z, \bar{z} \in C(E^{(0)} \cup \bar{E}, R^k)$ and $z(\xi, \eta) = \bar{z}(\xi, \eta)$ for $(\xi, \eta) \in H_x$, then $f(x, y, p, z, q, r) = f(x, y, p, \bar{z}, q, r)$ for $y \in (-b, b)$, $p \in R^k$, $q = (q_1, \ldots, q_n) \in R^n$, $r = [r_{ij}]_{i,j=1,\ldots,n}$, $r \in R^{n^2}$ and $g_j(x, y, z) = g_j(x, y, \bar{z})$ for $(x, y) \in \bar{E}$, $\tilde{g}_j(x, y, z) = \tilde{g}_j(x, y, \bar{z})$ for $(x, y) \in \bar{E}$, where $j = 1, \ldots, n$.

For $-n_0 \leqslant m_0 \leqslant n^*$ we define $T_{m_0} = \{(j, m') \in T^* \colon j \leqslant m_0\}$ and $E_{h,m_0}$

$= \{(x^{(j)}, y^{(m')}) \in E_h^{(0)} \cup E_h^*: j \leqslant m_0\}$. If $z: E_h^{(0)} \cup E_h^* \to R^k$ then we define the norm $\|z\|_{m_0}^{(h)} = \max\{\|z^{(j,m')}\|: (j, m') \in T_{m_0}\}$. For $-n_0 \leqslant m_0 \leqslant n^*$ and $z = (z_1, \ldots, z_k): E_h^{(0)} \cup E_h^* \to R^k$ we write

$$\max_{(j,m') \in T_{m_0}} z(x^{(j)}, y^{(m')}) = \Big(\max_{(j,m') \in T_{m_0}} z_1(x^{(j)}, y^{(m')}), \ldots, \max_{(j,m') \in T_{m_0}} z_k(x^{(j)}, y^{(m')})\Big).$$

Let $S = \{s = (s_1, \ldots, s_n): s_j \in \{-1, 0, 1\}$ for $j = 1, \ldots, n\}$. If $s \in S$ and $1 \leqslant j \leqslant n$ then we write $j(s) = (s_1, \ldots, s_{j-1}, s_j + 1, s_{j+1}, \ldots, s_n)$ and $-j(s) = (s_1, \ldots, s_{j-1}, s_j - 1, s_{j+1}, \ldots, s_n)$. We define the following operators $A = (A_1, \ldots, A_k)$, $B^{(i)} = (B_1^{(i)}, \ldots, B_k^{(i)})$, $C^{(i)} = (C_1^{(i)}, \ldots, C_n^{(i)})$, $D^{(i)} = [D_{j,l}^{(i)}]_{j,l=1,\ldots,n}$ where $i = 1, \ldots, k$. If $w \in \mathscr{F}(E_h^{(0)} \cup E_h^*, R)$, $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$ then

$$\begin{aligned} A_i w^{(m)} &= \sum_{s \in S} a_s^{(i)} w^{(m_0, m'+s)}, \\ B_j^{(i)} w^{(m)} &= \sum_{s \in S} b_{s,j}^{(i)} w^{(m_0, m'+s)}, \qquad i, j = 1, \ldots, k, \end{aligned} \tag{11}$$

where $a_s^{(i)}, b_{s,j}^{(i)} \in R$ and

$$\begin{aligned} C_j^{(i)} w^{(m)} &= \sum_{s \in S} \frac{1}{h_j} c_{s,j}^{(i)} w^{(m_0, m'+s)}, \\ D_{jl}^{(i)} w^{(m)} &= \sum_{s \in S} \frac{1}{h_j h_l} d_{s,jl}^{(i)} w^{(m_0, m'+s)}, \qquad i = 1, \ldots, k,\ j, l = 1, \ldots, n, \end{aligned} \tag{12}$$

where $c_{s,j}^{(i)}, d_{s,jl}^{(i)} \in R$. For $z = (z_1, \ldots, z_k) \in \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k)$ we write

$$Az^{(m)} = (A_1 z_1^{(m)}, \ldots, A_k z_k^{(m)}), \quad B^{(i)} z^{(m)} = (B_1^{(i)} z_1^{(m)}, \ldots, B_k^{(i)} z_k^{(m)}),$$

where $i = 1, \ldots, k$, $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$. If $w \in \mathscr{F}(E_h^{(0)} \cup E_h^*, R)$ then

$$C^{(i)} w^{(m)} = (C_1^{(i)} w^{(m)}, \ldots, C_n^{(i)} w^{(m)}), \qquad i = 1, \ldots, k,$$

and

$$D^{(i)} w^{(m)} = [D_{jl}^{(i)} w^{(m)}]_{j,l=1,\ldots,n}, \qquad i = 1, \ldots, k,$$

where $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$. Let $\tilde{E}_h = \{(x^{(m_0)}, y^{(m')}): m_0 = 0, 1, \ldots, n^*-1, m' \in M\}$, $\Sigma_h = \tilde{E}_h \times R^k \times \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k) \times R^n \times R^{n^2}$. Assume that for each $h \in I_0$ we have $\Phi_h = (\Phi_h^{(1)}, \ldots, \Phi_h^{(k)}): \Sigma_h \to R^k$. If $z = (z_1, \ldots, z_k) \in \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k)$ then we write

$$\begin{aligned} &\Phi_h(x^{(m_0)}, y^{(m')}, Bz^{(m)}, z, Cz^{(m)}, Dz^{(m)}) \\ &\quad = \big(\Phi_h^{(1)}(x^{(m_0)}, y^{(m')}, B^{(1)} z^{(m)}, z, C^{(1)} z_1^{(m)}, D^{(1)} z_1^{(m)}), \ldots \\ &\qquad \ldots, \Phi_h^{(k)}(x^{(m_0)}, y^{(m')}, B^{(k)} z, z, C^{(k)} z_k^{(m)}, D^{(k)} z_k^{(m)})\big), \\ &\qquad\qquad m_0 = 0, 1, \ldots, n^*-1, \quad m' \in M. \end{aligned}$$

In a similar way we define $f(x^{(m_0)}, y^{(m')}, z^{(m)}, z, Cz^{(m)}, Dz^{(m)})$ for $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$, $z \in C(E^{(0)} \cup \bar{E}, R^k)$.

Suppose that $h \in I_0$, $\bar{m}_0 \in \{1, \ldots, n\}$. We define $V_{h,\bar{m}_0}\colon \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k) \to \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k)$ as follows:

$(V_{h,\bar{m}_0} z)^{(m)} = z^{(j(m))}$ if $m_0 = \bar{m}_0$ and there exists $j$ such that $m \in T_j^{(-)}$,

$(V_{h,\bar{m}_0} z)^{(m)} = z^{(-j(m))}$ if $m_0 = \bar{m}_0$ and there exists $j$ such that $m \in T_j^{(+)}$,.

$(V_{h,\bar{m}_0} z)^{(m)} = z^{(m)}$ for all the rest of $m \in T^*$.

Assume that, for each $h \in I_0$, $g_{h,j}\colon E_h^* \times \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k) \to R^k$, $\tilde{g}_{h,j}\colon E_h^* \times \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k) \to R^k$, $j = 1, \ldots, n$, are given. Suppose that $z \in \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k)$. We define

$$(L_j z)^{(m)} = \sum_{i=1}^{\bar{n}} \alpha_{ji} z^{(j^i(m))} - h_j \sum_{i=0}^{\bar{n}} \beta_{ji} g_{h,j}(x^{(m_0)}, y^{(j^i(m'))}, V_{h,m_0} z),$$

$$m \in T_j^{(-)},\ j = 1, \ldots, n,$$

and

$$(\tilde{L}_j z)^{(m)} = \sum_{i=1}^{\bar{n}} \tilde{\alpha}_{ji} z^{(-j^i(m))} + h_j \sum_{i=0}^{\bar{n}} \tilde{\beta}_{ji} \tilde{g}_{h,j}(x^{(m_0)}, y^{(-j^i(m))}, V_{h,m_0} z),$$

$$m \in T_j^{(+)},\ j = 1, \ldots, n,$$

where $\alpha_{ji}, \tilde{\alpha}_{ji}, \beta_{jl}, \tilde{\beta}_{jl} \in R$, $j = 1, \ldots, n$, $i = 1, \ldots, \bar{n}$, $l = 0, 1, \ldots, \bar{n}$, $1 \leqslant \bar{n} \leqslant \min_{1 \leqslant j \leqslant n} n_j^*$.

We consider the following difference method for the problem (10)

$$\begin{aligned}
& z^{(m_0+1,m')} = Az^{(m)} + h_0 \Phi_h(x^{(m_0)}, y^{(m')}, Bz^{(m)}, z, Cz^{(m)}, Dz^{(m)}), \\
& \qquad\qquad m' \in M,\ m_0 = 0, 1, \ldots, n^*-1, \\
& z^{(m)} = (L_j z)^{(m)}, \quad m \in T_j^{(-)},\ j = 1, \ldots, n, \\
& z^{(m)} = (\tilde{L}_j z)^{(m)}, \quad m \in T_j^{(+)},\ j = 1, \ldots, n, \\
& z^{(m)} = \varphi^{(m)} + \delta^{(m)} \quad \text{on } E_h^{(0)},
\end{aligned} \tag{13}$$

where $\delta = (\delta_1, \ldots, \delta_k)\colon E_h^{(0)} \to R^k$.

Assume that $u_h$ is a solution of the difference-functional problem (13) and $v$ is a solution of (10). We give sufficient conditions for the convergence $\lim_{|h| \to 0} \|u_h^{(m)} - v^{(m)}\| = 0$, $m \in \tilde{T}$.

The function $\Phi_h\colon \Sigma_h \to R^k$ is said to satisfy the *Volterra condition* if for $m_0 \in \{0, 1, \ldots, n^*-1\}$, $z, \bar{z} \in \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k)$ such that $z|_{E_{h,m_0}} = \bar{z}|_{E_{h,m_0}}$ we have $\Phi_h(x^{(m_0)}, y^{(m')}, p, z, q, r) = \Phi_h(x^{(m_0)}, y^{(m')}, p, \bar{z}, q, r)$ for $m' \in M$,

$(p, q, r) \in R^{k+n+n^2}$. In a similar way we define the Volterra condition for $g_{h,j}$ and $\tilde{g}_{h,j}$, $j = 1, \ldots, n$. Let $p = (p_1, \ldots, p_k)$, $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_k)$, $p, \bar{p} \in R^k$. For the index $i$, $1 \leqslant i \leqslant k$, being fixed, we write $p \overset{i}{\leqslant} \bar{p}$ if $p_j \leqslant \bar{p}_j$ for $j = 1, \ldots, k$ and $p_i = \bar{p}_i$. Let $\mathscr{F}_0(E_h^{(0)} \cup E_h, R_+^k)$ be the class of all functions defined on $E_h^{(0)} \cup E_h$ taking values in $R_+^k$ which depend on the first argument only. For a function $z\colon E^{(0)} \cup E \to R^k$ we also use the symbol $z$ for the restriction $z|_{E_h^{(0)} \cup E_n}$.

Our basic assumptions are the following.

Assumption $\mathrm{H}_1$. Suppose that

(i) the function $\Phi_h\colon \Sigma_h \to R^k$, $h \in I_0$, of the variables $(x, y, p, z, q, r)$ is nondecreasing with respect to the functional argument and satisfies the Volterra condition,

(ii) for each $i$, $1 \leqslant i \leqslant k$, if $p \overset{i}{\leqslant} \bar{p}$ then $\Phi_h^{(i)}(x, y, p, z, q, r) \leqslant \Phi_h^{(i)}(x, y, \bar{p}, z, q, r)$ on $\Sigma_h$,

(iii) for each $h \in I_0$, $(x, y, z) \in \tilde{E}_h \times \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k)$ the function $\Phi_h(x, y, \cdot, z, \cdot, \cdot)\colon R^k \times R^n \times R^n \to R^k$ is continuous,

(iv) the derivatives $D_{p_i}\Phi_h^{(i)}$, $D_{q_j}\Phi_h^{(i)}$, $D_{r_{jl}}\Phi_h^{(i)}$, $i = 1, \ldots, k$, $j, l = 1, \ldots, n$, $h \in I_0$, exist on $\Sigma_h$ and for each

$$(x, y, p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k, z) \in \tilde{E}_h \times R^{k-1} \times \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k),$$

$h \in I_0$, the functions

$$\begin{aligned} &D_{p_i}\Phi_h^{(i)}(x, y, p_1, \ldots, p_{i-1}, \cdot, p_{i+1}, \ldots, p_k, z, \cdot, \cdot), \\ &D_{q_j}\Phi_h^{(i)}(x, y, p_1, \ldots, p_{i-1}, \cdot, p_{i+1}, \ldots, p_k, z, \cdot, \cdot), \\ &D_{r_{jl}}\Phi_h^{(i)}(x, y, p_1, \ldots, p_{i-1}, \cdot, p_{i+1}, \ldots, p_k, z, \cdot, \cdot), \\ &\qquad\qquad i = 1, \ldots, k,\ j, l = 1, \ldots, n, \end{aligned}$$

of the variables $(p_i, q, r)$ are continuous on $R \times R^n \times R^n$,

(v) $D_{r_{jl}}\Phi_h^{(i)} = D_{r_{lj}}\Phi_h^{(i)}$ for $i = 1, \ldots, k$, $j, l = 1, \ldots, n$, $h \in I_0$,

(vi) for $P = (x, y, p, z, q, r) \in \Sigma_h$, $h \in I_0$ and for $s \in S$ we have

$$(14)\qquad a_s^{(i)} + h_0 b_{s,i}^{(i)} D_{p_i}\Phi_h^{(i)}(P) + h_0 \sum_{j=1}^{n} \frac{1}{h_j} c_{s,j}^{(i)} D_{q_j}\Phi_h^{(i)}(P) + h_0 \sum_{j,l=1}^{n} \frac{1}{h_j h_l} d_{s,jl}^{(i)} D_{r_{jl}}\Phi_h^{(i)}(P) \geqslant 0, \quad i = 1, \ldots, k,$$

(vii) there exists a matrix $\Lambda = [\lambda_{ij}]_{i,j=1,\ldots,k}$ with $\lambda_{ij} \geqslant 0$ such that for

$h \in I_0$ we have

(15) $$[\Phi_h(x^{(m_0)}, y^{(m')}, p+\bar{p}, z+\bar{z}, q, r) - \Phi_h(x^{(m_0)}, y^{(m')}, p, z, q, r)]^T \leqslant \Lambda[\bar{p} + \max_{(j,m') \in T_{m_0}} \bar{z}^{(j,m')}]^T,$$

where $(x^{(m_0)}, y^{(m')}, p, z, q, r) \in \Sigma_h$, $\bar{z} \in \mathscr{F}_0(E_h^{(0)} \cup E_h^*, R_+^k)$, $\bar{p} \in R_+^k$.

ASSUMPTION $H_2$. Suppose that

(i) for each $h \in I_0$, $j = 1, \ldots, n$, we have $g_{h,j}\colon E_h^* \times \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k) \to R^k$, $\tilde{g}_{h,j}\colon E_h^* \times \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k) \to R^k$ and the functions $g_{h,j}$, $\tilde{g}_{h,j}$ satisfy the Volterra condition,

(ii) for each $h \in I_0$, $j = 1, \ldots, n$, the function $g_{h,j}$ is nonincreasing with respect to $z$ and the function $\tilde{g}_{h,j}$ is nondecreasing with respect to $z$,

(iii) $\alpha_{ji}, \tilde{\alpha}_{ji}, \beta_{jl}, \tilde{\beta}_{jl} \in R_+$ for $j = 1, \ldots, n$, $i = 1, \ldots, \bar{n}$, $l = 0, 1, \ldots, \bar{n}$ and

$$\sum_{i=1}^{\bar{n}} \alpha_{ji} = \sum_{i=1}^{\bar{n}} \tilde{\alpha}_{ji} = 1, \quad j = 1, \ldots, n,$$

(iv) there exists a matrix $\tilde{\Lambda} = [\tilde{\lambda}_{ij}]_{i,j=1,\ldots,n}$ with $\tilde{\lambda}_{ij} \geqslant 0$ such that for $h \in I_0$, $j = 1, \ldots, n$ and for $z \in \mathscr{F}(E_h^{(0)} \cup E_h^*, R^k)$, $\bar{z} \in \mathscr{F}_0(E_h^{(0)} \cup E_h, R_+^k)$ we have

(16) $$[g_{h,j}(x^{(m_0)}, y^{(m')}, z) - g_{h,j}(x^{(m_0)}, y^{(m')}, z+\bar{z})]^T \leqslant \frac{h_0}{h_j} \tilde{\Lambda}[\max_{(j,m') \in T_{m_0}} \bar{z}^{(j,m')}]^T,$$

where $(x^{(m_0)}, y^{(m')}) \in E_h^*$ and

(17) $$[\tilde{g}_{h,j}(x^{(m_0)}, y^{(m')}, z+\bar{z}) - \tilde{g}_{h,j}(x^{(m_0)}, y^{(m')}, z)]^T \leqslant \frac{h_0}{h_j} \tilde{\Lambda}[\max_{(j,m') T_{m_0}} \bar{z}^{(j,m')}]^T,$$

where $(x^{(m_0)}, y^{(m')}) \in E_h^*$.

ASSUMPTION $H_3$. Suppose that the operators $A, B^{(i)}, C^{(i)}, D^{(i)}$, $i = 1, \ldots, k$, satisfy the conditions

(i) $\sum_{s \in S} a_s^{(i)} = 1$ for $i = 1, \ldots, k$ and $b_{s,j}^{(i)} \geqslant 0$ for $s \in S$, $i, j = 1, \ldots, k$, $\sum_{s \in S} b_{s,j}^{(i)} = 1$ for $i, j = 1, \ldots, k$,

(ii) $\sum_{s \in S} c_{s,j}^{(i)} = 0$ and $\sum_{s \in S} d_{s,jl}^{(i)} = 0$ for $i = 1, \ldots, k$, $j, l = 1, \ldots, n$,

(iii) $c_0 \geqslant \max[\max_{1 \leqslant j \leqslant n} \sum_{i=0}^{\bar{n}} \beta_{ji}, \max_{1 \leqslant j \leqslant n} \sum_{i=0}^{\bar{n}} \beta_{ji}]$.

At first we state a result on the stability of method (13).

THEOREM 1. *Suppose that*

(A) *Assumptions* $H_1$–$H_3$ *are satisfied,*

(B) $v \in C(E^{(0)} \cup \bar{E}, R^k)$, $v|_{E^{(0)}} = \varphi$ *and there exist functions*

$$\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k)\colon I_0 \to R^k_+ \quad \text{and} \quad \gamma^* = (\gamma^*_1, \ldots, \gamma^*_k)\colon I_0 \to R^k_+$$

*such that*

$$|v^{(m_0+1,m')} - Av^{(m)} - h_0 \Phi_h(x^{(m_0)}, y^{(m')}, Bv^{(m)}, v, Cv^{(m)}, Dv^{(m)})| \leqslant h_0 \tilde{\gamma}(h), \quad m_0 = 0, 1, \ldots, n^*-1,\ m' \in M \tag{18}$$

*and*

$$\begin{aligned} &|(L_j v)^{(m)} - v^{(m)}| \leqslant h_0 \gamma^*(h), \quad m \in T_i^{(-)},\ j = 1, \ldots, n,\\ &|v^{(m)} - (L_j v)^{(m)}| \leqslant h_0 \gamma^*(h), \quad m \in T_j^{(+)},\ j = 1, \ldots, n, \end{aligned} \tag{19}$$

(C) $u_h$ *is a solution of* (13).

*Under these assumptions we have*

$$\begin{aligned} &|u_h^{(m)} - v^{(m)}| \leqslant (I + h_0 \Lambda^*)^{2m_0-1} \sigma_{0,h}^T + h_0 \sum_{j=0}^{2m_0-2} (I + h_0 \Lambda^*)^j \gamma(h)^T,\\ &\qquad m_0 = 1, \ldots, n^*,\ m' \in M,\\ &|u_h^{(m)} - v^{(m)}| \leqslant (I + h_0 \Lambda^*)^{2m_0} \sigma_{0,h}^T + h_0 \sum_{j=0}^{2m_0-1} (I + h_0 \Lambda^*)^j \gamma(h)^T,\\ &\qquad m_0 = 1, \ldots, n^*,\ m' \in M^* \setminus M, \end{aligned} \tag{20}$$

*where* $\gamma = (\gamma_1, \ldots, \gamma_k)\colon I_0 \to R^k_+$, $\gamma_i(h) = \max(\tilde{\gamma}_i(h), \gamma^*_i(h))$, $i = 1, \ldots, k$, $\sigma_{0,h} = \max_{-n_0 \leqslant j \leqslant 0, m' \in M^*} |\delta^{(j,m)}|$ *and*

$$\Lambda^* = [\lambda^*_{ij}]_{i,j=1,\ldots,k}, \quad \lambda^*_{ij} = \max[2\lambda_{ij}, c_0 \tilde{\lambda}_{ij}], \quad i, j = 1, \ldots, k. \tag{21}$$

Proof. We apply Lemma 1 for to prove (20). Let $\Omega_h = M^*$ and $N_h$, $N_h^{(0)}$, $N_h^*$ are sets defined in Section 1 with $n_0$ and $n^*$ given in 2. Put

$$\Omega_h^{(0)} = \{m'\colon (1, m) \in [\bigcup_{j=1}^n T_j^{(-)} \cup \bigcup_{j=1}^n T_j^{(+)}]\}.$$

To each function $z\colon E_h^{(0)} \cup E_h^* \to R^k$ there corresponds a function $\tilde{z}\colon N_h \times \Omega_h \to R^k$ given by $\tilde{z}^{(m)} = z(x^{(m_0)}, y^{(m')})$. We will use the same notation for elements of $\mathscr{F}(E_h^{(0)} \cup E_h^*, R^k)$ and $\mathscr{F}(N_h \times \Omega_h, R^k)$. Define $\Gamma_h = \mathscr{F}(N_h \times \Omega_h, R^k)$ and

$$F_h = (F_h^{(1)}, \ldots, F_h^{(k)})\colon N_h^* \times \Gamma_h \to \mathscr{F}(\Omega_h, R^k),$$

where

$$\text{(22)} \qquad F_h(m_0, z)(m') = Az^{(m)} + h_0\, \Phi_h(x^{(m_0)}, y^{(m')}, Bz^{(m)}, z, Cz^{(m)}, Dz^{(m)}), \qquad m_0 \in N_h^*,\ m' \in M,$$

and

$$\text{(23)} \qquad \begin{aligned} F_h(m_0, z)(m') &= (L_j z)^{(m_0+1, m')} &&\text{for } (m_0+1, m') \in T_j^{(-)},\\ F_h(m_0, z)(m') &= (\tilde{L}_j z)^{(m_0+1, m')} &&\text{for } (m_0+1, m') \in T_j^{(+)}, \end{aligned}$$

where $j = 1, \ldots, n$. Then we have

$$u_h^{(m_0+1, m')} = F_h(m_0, u_h)(m'), \qquad m_0 \in N_h^*,\ m' \in \Omega_h.$$

It follows from Assumptions $H_1$ and $H_2$ that $F_h$ satisfies condition (V) with respect to $\Omega_h^{(0)}$. Now we prove that $F_h$ is nondecreasing with respect to the functional argument. Suppose that $z, \bar{z} \in \Gamma_h$ and $z \leqslant \bar{z}$. Then we have for $m' \in M$, $m_0 \in N_h^*$

$$\begin{aligned}
&F_h^{(i)}(m_0, z)(m') - F_h^{(i)}(m_0, \bar{z})(m')\\
&= A_i z_i^{(m)} - A_i \bar{z}_i^{(m)} + h_0 [\Phi_h^{(i)}(x^{(m_0)}, y^{(m')}, B^{(i)} z^{(m)}, z, C^{(i)} z_i^{(m)}, D^{(i)} z_i^{(m)})\\
&\qquad - \Phi_h^{(i)}(x^{(m_0)}, y^{(m')}, B^{(i)} \bar{z}^{(m)}, \bar{z}, C^{(i)} \bar{z}_i^{(m)}, D^{(i)} \bar{z}_i^{(m)})]\\
&\leqslant A_i z_i^{(m)} - A_i \bar{z}_i^{(m)}\\
&\quad + h_0 [\Phi_h^{(i)}(x^{(m_0)}, y^{(m')}, B_1^{(i)} \bar{z}_1^{(m)}, \ldots, B_{i-1}^{(i)} \bar{z}_{i-1}^{(m)}, B_i^{(i)} z_i^{(m)},\\
&\qquad B_{i+1}^{(i)} \bar{z}_{i+1}^{(m)}, \ldots, B_k^{(i)} \bar{z}_k^{(m)}, \bar{z}, C^{(i)} z_i^{(m)}, D^{(i)} z_i^{(m)}) -\\
&\qquad - \Phi_h^{(i)}(x^{(m_0)}, y^{(m')}, B^{(i)} \bar{z}^{(m)}, \bar{z}, C^{(i)} \bar{z}_i^{(m)}, D^{(i)} \bar{z}_i^{(m)})]\\
&= A_i z_i^{(m)} - A_i \bar{z}_i^{(m)} + h_0 D_{p_i} \Phi_h^{(i)}(\tilde{P}_i) [B_i^{(i)} z_i^{(m)} - B_i^{(i)} \bar{z}_i^{(m)}]\\
&\quad + h_0 \sum_{j=1}^{n} D_{q_j} \Phi_h^{(i)}(\tilde{P}_i) [C_j^{(i)} z_i^{(m)} - C_j^{(i)} \bar{z}_i^{(m)}]\\
&\quad + h_0 \sum_{j,l=1}^{n} D_{r_{jl}} \Phi_h^{(i)}(\tilde{P}_i) [D_{jl}^{(i)} z_i^{(m)} - D_{jl}^{(i)} \bar{z}_i^{(m)}]\\
&= \sum_{s \in S} [z_i^{(m_0, m'+s)} - \bar{z}_i^{(m_0, m'+s)}] \Bigg[ a_s^{(i)} + h_0 b_{s,i}^{(i)} D_{p_i} \Phi_h^{(i)}(\tilde{P}_i)\\
&\quad + \sum_{j=1}^{n} \frac{h_0}{h_j} D_{q_j} \Phi_h^{(i)}(\tilde{P}_i) c_{s,j}^{(i)} + h_0 \sum_{j,l=1}^{n} \frac{1}{h_j h_l} D_{r_{jl}} \Phi_h^{(i)}(\tilde{P}_i) d_{s,jl}^{(i)} \Bigg], \qquad i = 1, \ldots, k,
\end{aligned}$$

where $\tilde{P}_i$ are intermediate points. The above estimation and (14) imply

$$\text{(24)} \qquad F_h(m_0, z)(m') \leqslant F_h(m_0, \bar{z})(m'), \qquad m_0 \in N_h^*,$$

where $m' \in M$. It follows from Assumption $H_2$ that (24) holds for $m \in \Omega_h \backslash M$ and for $z, \bar{z} \in \Gamma_h$, $z \leqslant \bar{z}$.

Let $\Psi_h = (\psi_h^{(1)}, \ldots, \psi_h^{(k)})\colon N_h^* \times \Gamma_h \to \mathscr{F}(\dot{\Omega}_h, R^k)$ be the function given by

$$\Psi_h(m_0, z)(m') = z^{(m_0+1,m')} - Az^{(m)} - h_0 \Phi_h(x^{(m_0)}, y^{(m')}, Bz^{(m)}, z, Cz^{(m)}, Dz^{(m)}) \quad \text{for } m_0 \in N_h^*,\ m' \in M, \tag{25}$$

and

$$\begin{aligned} \Psi_h(m_0, z)(m') &= z^{(m_0+1,m')} - (L_j z)^{(m_0+1,m')} && \text{for } m_0 \in N_h^*,\ (m_0, m') \in T_j^{(-)}, \\ \Psi_h(m_o, z)(m') &= z^{(m_0+1,m')} - (\tilde{L}_j z)^{(m_0+1,m')} && \text{for } m_0 \in N_h^*,\ (m_0, m') \in T_j^{(+)}, \end{aligned} \tag{26}$$

where $j = 1, \ldots, n$. Let $\tilde{F}_h = (\tilde{F}_h^{(1)}, \ldots, \tilde{F}_h^{(k)})\colon N_h^* \times \Gamma_h \to \mathscr{F}(\Omega_h, R^k)$, where

$$\tilde{F}_h(m_0, z)(m') = F_h(m_0, z)(m') + \Psi_h(m_0, z)(m'). \tag{27}$$

Then

$$v^{(m_0+1,m')} = \tilde{F}_h(m_0, v)(m'), \qquad m_0 \in N_h^*,\ m' \in \Omega_h, \tag{28}$$

and

$$|F_h(m_0, v)(m') - \tilde{F}_h(m_0, v)(m')| \leqslant h_0 \gamma(h), \qquad m_0 \in N_h^*,\ m \in \Omega_h. \tag{29}$$

For each $m \in N_h^* \times \Omega_h$ we define the set $V(m)$ in the following way. If $m \in N_h^* \times (\Omega_h \setminus \Omega_h^{(0)})$ then $V(m) = T_{m_0}$. For $m \in N_h^* \times \Omega_h^{(0)}$ we set

$$V(m) = T_{m_0} \cup \{(j, \bar{m}') \in N_h \times (\Omega_h \setminus \Omega_h^{(0)})\colon j = m_0 + 1\}.$$

It follows from (15)–(17) and from (22), (23) that

$$[F_h(m_0, z+\bar{z})(m') - F_h(m_0, z)(m')]^T \leqslant (I + h_0 \Lambda^*)(\max_{(j,\bar{m}') \in V(m)} \bar{z}^{(j,\bar{m}')})^T, \tag{30}$$

where $m_0 \in N_h^*$, $m' \in \Omega_h$, $z \in \Gamma_h$, $\bar{z} \in \mathscr{F}_0(E_h^{(0)} \cup E_h, R_+^k)$. Since

$$|u_h^{(j,m)} - v^{(j,m)}| \leqslant \sigma_{0,h} \quad \text{for } (j, m') \in N_h^{(0)} \times \Omega_h,$$

we obtain estimation (20) from Lemma 1 (see Example 1).

Remark 1. If the assumptions of Theorem 1 are satisfied and $L = \|\Lambda^*\| > 0$, then we obtain from (20)

$$\|u_h - v\|_{m_0}^{(h)} \leqslant (1 + h_0 L)^{2m_0} \|\sigma_{0,h}\| + L^{-1}[(1 + h_0 L)^{2m_0} - 1]\,\|\gamma(h)\|, \quad m_0 = 1, 2, \ldots, n^*. \tag{31}$$

If $\|\Lambda^*\|$ then

$$\|u_h - v\|_{m_0}^{(h)} \leqslant \|\sigma_{0,h}\| + 2m_0 h_0 \|C_h\|, \qquad m_0 = 1, \ldots, n^*.$$

ASSUMPTION $H_4$. Suppose that

(i) the operator $A$ defined by (11) satisfies the conditions:

$$\sum_{s\in S} s_j a_s^{(i)} = 0, \quad \sum_{s\in S} s_j s_l a_s^{(i)} = 0, \quad j, l = 1, \ldots, n, \ i = 1, \ldots, k,$$

(ii) the difference operators $C^{(i)}$, $D^{(i)}$, $i = 1, \ldots, k$, defined by (12) satisfy

$$\sum_{s\in S} s_l c_{s,j}^{(i)} = \delta_{jl}, \quad j, l = 1, \ldots, n, \ i = 1, \ldots, k,$$

where $\delta_{jl}$ is the Kronecker symbol and

$$\sum_{s\in S} s_{j'} d_{s,jl}^{(i)} = 0, \quad \sum_{s\in S} s_{j'} s_{l'} d_{s,jl}^{(i)} = \delta_{jj'}\delta_{ll'} \quad \text{for } j \neq l,$$

$$\sum_{s\in S} s_{j'} s_{l'} d_{s,jl}^{(i)} = 2\delta_{jj'}\delta_{ll'} \quad \text{for } j = l,$$

where $i = 1, \ldots, k$, $j, j', l, l' = 1, \ldots, n$,

(iii) there exists $\bar{c} > 0$ such that $h_j h_i^{-1} \leqslant \bar{c}$ for $i, j = 1, \ldots, n$, and $h_i h_j \leqslant \bar{c} h_0$, $i, j = 1, \ldots, n$,

Now we prove a theorem on the convergence of the method (13).

THEOREM 2. *Suppose that*

(A) *Assumptions* $H_1$–$H_4$ *are satisfied,*

(B) $f \in C(\Gamma, R^k)$, $\varphi \in C(E^{(0)}, R^k)$ *and* $g_j \in C(\bar{E} \times C(E^{(0)} \cup \bar{E}, R^k), R^k)$, $\tilde{g}_j \in C(\bar{E} \times C(E^{(0)} \cup \bar{E}, R^k), R^k)$ *where* $j = 1, \ldots, n$,

(C) $v = (v_1, \ldots, v_k) \in C(E^{(0)} \cup E, R^k)$ *is a solution of* (10) *such that* $v|_{\bar{E}}$ *is of class* $C^3$,

(D) *there exist functions* $\beta, \beta_0 \colon I_0 \to R_+^k$ *such that*

$$(32) \quad \begin{aligned} &|\Phi_h(x^{(m_0)}, y^{(m')}, v^{(m)}, v, Cv^{(m)}, Dv^{(m)}) \\ &\qquad - f(x^{(m_0)}, y^{(m')}, v^{(m)}, v, Cv^{(m)}, Dv^{(m)})| \leqslant \beta(h), \\ &\qquad\qquad m_0 = 0, 1, \ldots, n^*-1, \ m' \in M, \ h \in I_0, \end{aligned}$$

*and*

$$(33) \quad \begin{aligned} &|g_{h,j}(x^{(m_0)}, y^{(m')}, V_{h,m_0} v) - g_j(x^{(m_0)}, y^{(m')}, v)| \leqslant \frac{h_0}{h_j}\beta_0(h), \quad m \in T_j^{(-)}, \\ &|\tilde{g}_{h,j}(x^{(m_0)}, y^{(m')}, V_{h,m_0} v) - \tilde{g}_j(x^{(m_0)}, y^{(m')}, v)| \leqslant \frac{h_0}{h_i}\beta_0(h), \quad m \in T_j^{(+)}, \\ &\qquad\qquad j = 1, \ldots, n, \end{aligned}$$

*where*

$$(34) \quad \lim_{|h|\to 0} \beta(h) = \lim_{|h|\to 0} \beta_0(h) = 0,$$

(E) *there exists a function* $\beta_1 \colon I_0 \to R_+^k$ *such that*

$$(35)\qquad \begin{aligned}&\Big|\sum_{i=1}^{\bar{n}} \alpha_{ji}\, v^{(j^i(m))} - h_j \sum_{i=0}^{\bar{n}} \beta_{ji}\, g_j(x^{(m_0)}, y^{(j^i(m'))}, v) - v^{(m)}\Big| \leqslant h_0 \beta_1(h),\\ &\qquad\qquad\qquad\qquad m \in T_j^{(-)},\ j = 1, \ldots, n,\\ &\Big|\sum_{i=1}^{\bar{n}} \tilde{\alpha}_{ji}\, v^{(-j^i(m))} + h_j \sum_{i=0}^{\bar{n}} \tilde{\beta}_{ji}\, \tilde{g}_j(x^{(m_0)}, y^{(-j^i(m'))}, v) - v^{(m)}\Big| \leqslant h_0 \beta_1(h),\\ &\qquad\qquad\qquad\qquad m \in T_j^{(+)},\ j = 1, \ldots, n,\end{aligned}$$

*and*

$$(36)\qquad \lim_{|h| \to 0} \beta_1(h) = 0,$$

(F) $\delta\colon E_h^{(0)} \to R^k$ *and* $\sigma_{0,h} = \max_{-n_0 \leqslant m_0 \leqslant 0,\, m' \in M^*} |\delta(m)|$, $\lim_{|h| \to 0} \sigma_{0,h} = 0$,

(G) $u_h$ *is a solution of* (13).

*Under these assumptions there exists a function* $\gamma\colon I_0 \to R_+^k$ *such that* $\lim_{|h| \to 0} \gamma(h) = 0$ *and estimates* (20) *hold with* $\Lambda^*$ *given by* (21). *In particular the estimates*

$$(37)\qquad \|u_h - v\|_{m_0}^{(h)} \leqslant e^{2aL} \|\sigma_{0,h}\| + L^{-1} [e^{2aL} - 1] \|\gamma(h)\|, \qquad m_0 = 0, 1, \ldots, n^*,$$

*where* $L = \|\Lambda^*\| > 0$ *and for* $L = 0$

$$(38)\qquad \|u_h - v\|_{m_0}^{(h)} \leqslant e^{2aL} \|\sigma_{0,h}\| + 2a \|\gamma(h)\|, \qquad m_0 = 0, 1, \ldots, n^*,$$

*hold.*

Proof. Let $C \geqslant 0$ be such a constant that

$$\|v(x, y)\|,\ \|D_{y_i} v(x, y)\|,\ \|D_{y_i y_j} v(x, y)\|,\ \|D_{y_i y_j y_l} v(x, y)\| \leqslant C,$$
$$i, j, l = 1, \ldots, n,\ (x, y) \in \bar{E}.$$

Using for $v_i$ the Taylor expansion of the third order with respect to $y$ we get for some $\theta_i \in (0, 1)$

$$\frac{1}{h_0} \Big[v_i^{(m_0+1, m')} - \sum_{s \in S} a_s^{(i)} v_i^{(m_0, m'+s)}\Big] = D_x v_i(x^{(m_0)} + \theta_i h_0, y^{(m')}) + R_{0,i}^{(m)}(h),$$
$$m_0 = 0, 1, \ldots, n^* - 1,\ m' \in M,$$

where $i = 1, \ldots, k$ and

$$|R_{0,i}^{(m)}(h)| \leqslant \tilde{\gamma}_0^{(i)}(h), \quad i = 1, \ldots, k,\ m \in N_h^* \times M,\ h \in I_0,$$

where

$$\tilde{\gamma}_0^{(i)}(h) = \tfrac{1}{6} \bar{c} n^2 C |h| \sum_{s \in S} |a_s^{(i)}|.$$

Let $\tilde{\gamma}_0(h) = (\tilde{\gamma}_0^{(1)}(h), \ldots, \tilde{\gamma}_0^{(k)}(h))$, $h \in I_0$, and

$$R_0^{(m)}(h) = (R_{0,1}^{(m)}(h), \ldots, R_{0,k}^{(m)}(h)).$$

We define for $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$, $h \in I_0$,

$$R_1^{(m)}(h) = \Phi_h(x^{(m_0)}, y^{(m')}, Bv^{(m)}, v, Cv^{(m)}, Dv^{(m)}) - \Phi_h(x^{(m_0)}, y^{(m')}, v^{(m)}, v, Cv^{(m)}, Dv^{(m)}),$$

$$R_2^{(m)}(h) = \Phi_h(x^{(m_0)}, y^{(m')}, v^{(m)}, v, Cv^{(m)}, Dv^{(m)}) - f(x^{(m_0)}, y^{(m')}, v^{(m)}, v, Cv^{(m)}, Dv^{(m)}),$$

$$R_3^{(m)}(h) = f(x^{(m_0)}, y^{(m')}, v^{(m)}, v, Cv^{(m)}, Dv^{(m)}) - f(x^{(m_0)}, y^{(m')}, v^{(m)}, v, D_y v^{(m)}, D_{yy} v^{(m)}),$$

and

$$R_4^{(m)}(\eta, h) = \big(R_{4,1}^{(m)}(\eta_1, h), R_{4,2}^{(m)}(\eta_2, h), \ldots, R_{4,k}^{(m)}(\eta_k, h)\big),$$

where

$$\begin{aligned} R_{4,i}^{(m)}(\eta_i, h) = f_i(x^{(m_0)}, y^{(m')}, v^{(m)}, v, D_y v_i^{(m)}, D_{yy} v_i^{(m)}) - \\ - f_i\big(x^{(m_0)} + \eta_i h_0, y^{(m')}, v(x^{(m_0)} + \eta_i h_0, y^{(m')}), \\ v, D_y v_i(x^{(m_0)} + \eta_i h_0, y^{(m')}), D_{yy} v_i(x^{(m_0)} + \eta_i h_0, y^{(m')})\big), \quad i = 1, \ldots, k. \end{aligned}$$

Then we have

$$\begin{aligned} (39) \quad & v^{(m_0+1, m')} - Av^{(m)} - h_0 \Phi_h(x^{(m_0)}, y^{(m')}, Bv^{(m)}, v, Cv^{(m)}, Dv^{(m)}) \\ & = -h_0[-R_0^{(m)}(h) + R_1^{(m)}(h) + R_2^{(m)}(h) + R_3^{(m)}(h) + R_4^{(m)}(\theta_i, h)], \\ & \qquad m_0 = 0, 1, \ldots, n^*-1, \ h \in I_0. \end{aligned}$$

Since $\|B^{(i)} v^{(m)} - v^{(m)}\| \leqslant C|h|$, $i = 1, \ldots, k$, for $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$ it follows that there exists a function $\tilde{\gamma}_1\colon I_0 \to R_+^k$ such that $|R_1^{(m)}(h)| \leqslant \tilde{\gamma}_1(h)$, $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$ and $\lim_{|h| \to 0} \tilde{\gamma}_1(h) = 0$. Let $\tilde{\gamma}_2(h) = \beta(h)$ for $h \in I_0$. Then $|R_2^{(m)}(h)| \leqslant \tilde{\gamma}_2(h)$ for $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$ and $\lim_{|h| \to 0} \tilde{\gamma}_2(h) = 0$.

Put $C_j v^{(m)} = (C_j^{(1)} v_1^{(m)}, \ldots, C_j^{(k)} v_k^{(m)})$, $D_{jl} v^{(m)} = (D_{jl}^{(1)} v_1^{(m)}, \ldots, D_{jl}^{(k)} v_k^{(m)})$. It follows from Assumptions $H_3$, $H_4$ that for $j = 1, \ldots, n$, $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$

$$|C_j v^{(m)} - D_{y_j} v^{(m)}| \leqslant \tfrac{1}{2} \bar{c} n C |h| \max_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n}} \sum_{s \in S} |c_{s,j}^{(i)}|$$

and

$$\|D_{jl} v^{(m)} - D_{y_j y_l} v^{(m)}\| \leqslant \frac{|h|}{6} C (\bar{c} n)^2 \max_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant \tau, \tau' \leqslant n}} \sum_{s \in S} d_{s,\tau\tau'}^{(i)},$$

where $j, l = 1, \ldots, n$, $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$. The above estimations imply the existence of a function $\tilde{\gamma}_3\colon I_0 \to R^k_+$ such that $|R_3^{(m)}(h)| \leqslant \tilde{\gamma}_3(h)$, $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$ and $\lim_{|h|\to 0} \tilde{\gamma}_3(h) = 0$. Let

$$\tilde{\gamma}_{4,i}(h) = \max_{\substack{0 \leqslant m_0 \leqslant n^*-1 \\ m' \in M}} \max_{\eta_i \in [0,1]} |R_{4,i}^{(m)}(\eta_i, h)|, \qquad h \in I_0,$$

and

$$\tilde{\gamma}_4(h) = (\tilde{\gamma}_{4,1}(h), \ldots, \tilde{\gamma}_{4,k}(h)), \qquad h \in I_0.$$

Then $|R_4^{(m)}(\eta, h)| \leqslant \tilde{\gamma}_4(h)$ for $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$, $\eta_i \in [0, 1]$, $i = 1, \ldots, n$, and $\lim_{|h|\to 0} \tilde{\gamma}_4(h) = 0$. Let

$$\tilde{\gamma}(h) = \sum_{i=0}^{4} \tilde{\gamma}_i(h), \qquad h \in I_0.$$

It follows from (39) that estimation (18) holds true with the above given $\tilde{\gamma}$ and $\lim_{|h|\to 0} \tilde{\gamma}(h) = 0$.

With the notation

$$S_{j,1}^{(m)}(h) = \sum_{i=1}^{\bar{n}} \alpha_{ji} v^{(j^i(m))} - h_j \sum_{i=0}^{\bar{n}} \beta_{ji} g_j(x^{(m_0)}, y^{(j^i(m))}, v) - v^{(m)},$$

$$S_{j,2}^{(m)}(h) = h_j \sum_{i=0}^{\bar{n}} \beta_{ji} [g_j(x^{(m_0)}, y^{(j^i(m'))}, v) - g_{h,j}(x^{(m_0)}, y^{(j^i(m'))}, V_{h,m_0} v)],$$

where $m \in T_j^{(-)}$, $j = 1, \ldots, n$, we have

(40) $$|(L_j v)^{(m)} - v^{(m)}| \leqslant |S_{j,1}^{(m)}(h)| + |S_{j,2}^{(m)}(h)| \leqslant h_0(1+c_0)\beta_1(h), \qquad m \in T_j^{(-)},\ j = 1, \ldots, n.$$

In a similar way we prove that

$$|(\tilde{L}_j v)^{(m)} - v^{(m)}| \leqslant h_0(1+c_0)\beta_1(h).$$

Thus we see that estimation (19) holds true with $\gamma^*(h) = (1+c_0)\beta_1(h)$ and $\lim_{|h|\to 0} \gamma^*(h) = 0$.

Now we obtain the estimation (20), (21) and (37), (38) from Theorem 1 (see Remark 1).

Rermark 2. In Theorem 2 we assume that the consistency condition (32)–(34) is satisfied on the solution of (10). Theorem 2 remains to be true if instead of (32)–(34) we assume one of the conditions

(i) the consistency condition (32)–(34) holds for each function $v \in C(E^{(0)} \cup \bar{E}, R^k)$ such that $v|_{\bar{E}}$ is class $C^3$,

(ii) there exist functions $\beta, \beta_0\colon I_0 \to R^k_+$ such that for $h \in I_0$

$$|\Phi_h(x, y, p, z, q, r) - f(x, y, p, z, q, r)| \leqslant \beta(h), \quad (x, y, p, z, q, r) \in \Gamma,$$

$$|g_{h,j}(x, y, z) - g_j(x, y, z)| \leqslant \frac{h_0}{h_j} \beta_0(h), \quad (x, y) \in \bar{E},\ z \in C(E^{(0)} \cup \bar{E}, R^k),$$

$$|\tilde{g}_{h,j}(x, y, z) - \tilde{g}_j(x, y, z)| \leqslant \frac{h_0}{h_j} \beta_0(h), \quad (x, y) \in \bar{E},\ z \in C(E^{(0)} \cup \bar{E}, R^k),$$

where $j = 1, \ldots, n$ and $\lim\limits_{|h| \to 0} \beta(h) = \lim\limits_{|h| \to 0} \beta_0(h) = 0$.

**3. Examples of difference methods.** Let $J = \{(j, l)\colon j, l = 1, \ldots, n,\ j \neq l\}$. Suppose that for each $i$, $1 \leqslant i \leqslant k$, we have defined sets $J_i^{(-)}, J_i^{(+)} \subset J$ such that $J^{(-)} \cup J^{(+)} = J$ and $J_i^{(-)} \cap J_i^{(+)} = \emptyset$, $i = 1, \ldots, k$ (in particular it may be $J_i^{(-)} = \emptyset$ or $J_i^{(+)} = \emptyset$). Assume that if $(j, l) \in J_i^{(-)}$ then $(l, j) \in J_i^{(-)}$. Suppose that $\Delta^{(1)} = (\Delta_1^{(1)}, \ldots, \Delta_n^{(1)})$ and $\Delta^{(2,i)} = [\Delta_{jl}^{(2,i)}]_{j,l=1,\ldots,n}$, $i = 1, \ldots, k$, are difference operators defined by

$$\Delta^{(1)} w^{(m)} = (\Delta_1^{(1)} w^{(m)}, \ldots, \Delta_n^{(1)} w^{(m)}), \quad \Delta^{(2,i)} w^{(m)} = [\Delta_{jl}^{(2,i)} w^{(m)}]_{j,l=1,\ldots,n} \tag{41}$$

and

$$\begin{aligned}
\Delta_j^{(1)} w^{(m)} &= (2h_j)^{-1} [w^{(j(m))} - w^{(-j(m))}], \quad j = 1, \ldots, n,\\
\Delta_{jl}^{(2,i)} w^{(m)} &= (2h_j h_l)^{-1} [w^{(j(m))} + w^{(l(m))} + w^{(-j(m))}\\
&\qquad + w^{(-l(m))} - 2w^{(m)} - w^{(j(-l(m)))} - w^{(-j(l(m)))}]\\
&\qquad \text{for } (j, l) \in J_i^{(-)} \text{ and for } j = l = 1, \ldots, n,\ i = 1, \ldots, k,\\
\Delta_{jl}^{(2,i)} w^{(m)} &= (2h_j h_l)^{-1} [-w^{(j(m))} - w^{(l(m))} - w^{(-j(m))} - w^{(-l(m))} + 2w^{(m)}\\
&\qquad + w^{(j(l(m)))} + w^{(-j(-l(m)))}] \quad \text{for } (j, l) \in J_i^{(+)},\ i = 1, \ldots, k.
\end{aligned} \tag{42}$$

Let $1 \leqslant k_0 \leqslant k$. The difference method

$$\begin{aligned}
z_i^{(m_0+1,m')} &= z_i^{(m)} + h_0 \Phi_h^{(i)}(x^{(m_0)}, y^{(m')}, z^{(m)}, z, \Delta^{(1)} z_i^{(m)}, \Delta^{(2,i)} z_i^{(m)}),\\
&\qquad i = 1, \ldots, k_0,\ m_0 = 0, 1, \ldots, n^* - 1,\ m' \in M,\\
z_i^{(m_0+1,m')} &= \tfrac{1}{2} z_i^{(m)} + \frac{1}{4n} \sum_{j=1}^{n} (z_i^{(j(m))} + z_i^{(-j(m))}) + h_0 \Phi_h^{(i)}\Big(x^{(m_0)}, y^{(m')}, \tfrac{1}{2} z^{(m)}\\
&\qquad + \sum_{j=1}^{n} (z^{(j(m))} + z^{(-j(m))})(4n)^{-1}, z, \Delta^{(1)} z_i^{(m)}, \Delta^{(2,i)} z_i^{(m)}\Big),\\
&\qquad i = k_0 + 1, \ldots, k,\ m_0 = 0, 1, \ldots, n^* - 1,\ m' \in M,\\
z^{(m)} &= (L_j z)^{(m)}, \quad m \in T_j^{(-)},\ j = 1, \ldots, n,\\
z^{(m)} &= (\tilde{L}_j z)^{(m)}, \quad m \in T_j^{(+)},\ j = 1, \ldots, n,\\
z^{(m)} &= \varphi(m) + \delta^{(m)} \quad \text{for } -n_0 \leqslant m_0 \leqslant 0,\ m' \in M^*,
\end{aligned} \tag{43}$$

can be obtained from (13) by specializing the operators $A$, $B^{(i)}$, $C^{(i)}$, $D^{(i)}$.

THEOREM 3. *Suppose that*

(A) *conditions* (i)–(v), (vii) *from Assumption* $H_1$ *and condition* (iii) *of Assumption* $H_3$ *are satisfied,*

(B) *Assumption* $H_2$ *and conditions* (iii), (iv) *of Assumption* $H_4$ *hold,*

(C) *conditions* (B)–(G) *of Theorem* 2 *are satisfied,*

(D) *for each* $P = (x, y, p, z, q, r) \in \Sigma_h$ *and for* $i = 1, \ldots, k_0$ *we have*

$$(44)\qquad \begin{aligned} &1+h_0 D_{p_i}\Phi_h^{(i)}(P)-2h_0\sum_{j=1}^{n}\frac{1}{h_j^2}D_{r_{jj}}\Phi_h^{(i)}(P)+h_0\sum_{\substack{j,l=1\\ j\neq l}}^{n}\frac{1}{h_j h_l}|D_{r_{jl}}\Phi_h^{(i)}(P)| \geqslant 0,\\ &-\tfrac{1}{2}|D_{q_j}\Phi_h^{(i)}(P)|+\frac{1}{h_j}D_{r_{jj}}\Phi_h^{(i)}(P)-\sum_{\substack{l=1\\ l\neq j}}^{n}\frac{1}{h_l}|D_{r_{jl}}\Phi_h^{(i)}(P)| \geqslant 0,\\ &\qquad\qquad j = 1, \ldots, n, \end{aligned}$$

(E) *for each* $P \in \Sigma_h$ *and for* $i = k_0+1, \ldots, k$ *we have*

$$(45)\qquad \begin{aligned} &\tfrac{1}{2}[1+h_0 D_{p_i}\Phi_h^{(i)}(P)]-2h_0\sum_{j=1}^{n}\frac{1}{h_j^2}D_{r_{jj}}\Phi_h^{(i)}(P)+\sum_{\substack{j,l=1\\ j\neq l}}^{n}\frac{h_0}{h_j h_l}|D_{r_{jl}}\Phi_h^{(i)}(P)| \geqslant 0,\\ &\frac{1}{4n}\big(1+h_0 D_{p_i}\Phi_h^{(i)}(P)\big)-\frac{1}{2}\frac{h_0}{h_j}|D_{q_j}\Phi_h^{(i)}(P)|\\ &\qquad +\frac{h_0}{h_j^2}D_{r_{jj}}\Phi_h^{(i)}(P)-\sum_{\substack{l=1\\ l\neq j}}^{n}\frac{h_0}{h_j h_l}|D_{r_{jl}}\Phi_h^{(i)}(P)| \geqslant 0, \quad j = 1, \ldots, n, \end{aligned}$$

(F) *for* $P \in \Sigma_h$ *and for* $i = 1, \ldots, k$ *we have*

$$(46)\qquad \begin{aligned} &D_{r_{jl}}\Phi_h^{(i)}(P) \geqslant 0 \quad \textit{for } (j, l) \in J_i^{(+)},\\ &D_{r_{jl}}\Phi_h^{(i)}(P) \leqslant 0 \quad \textit{for } (j, l) \in J_i^{(-)}. \end{aligned}$$

*Under these assumptions we have*

$$\lim_{|h|\to 0} \|u_h^{(m)} - v^{(m)}\| = 0.$$

Proof. The difference method (41)–(43) satisfies all the assumptions of Theorem 2. In particular, conditions (14) for the method (41)–(43) are equivalent with (44)–(46). This completes the proof.

Remark 3. If we assume in (41)–(43) that $k_0 = k$ and $\Phi_h = f$, $g_{h,j} = g_j$, $\tilde{g}_{h,j} = \tilde{g}_j$ for $j = 1, \ldots, n$, then we obtain the method considered in [13], [16]–[19].

Remark 4. We define $\mathscr{F}_0^*(E_h^{(0)} \cup E_h, R_+^k) \subset \mathscr{F}_0(E_h^{(0)} \cup E_h, R_+^k)$ in the following way: $\bar{z} \in \mathcal{F}_0^*(E_h^{(0)} \cup E_h, R_+^k)$ iff there exist constants $K(\bar{z}), M(\bar{z}) \in R_+$ such that

$$\bar{z}^{(m)} \leqslant K(\bar{z})\sigma_{0,h} + M(\bar{z})\gamma(h), \qquad m_0 = -n_0, \ldots, n^*, \qquad m' \in M,$$

where $\sigma_{0,h}$ and $\gamma(h)$ are defined in Theorem 1. It is easy to see that Theorems 1 and 2 hold if we assume estimations (15)–(19) for $\bar{z} \in \mathcal{F}_0^*(E_h^{(0)} \cup E_h, R_+^k)$.

**4. Difference methods for almost linear problems.** In this section we consider the differential-functional problem (10) with

$$f_i(x, y, p, z, q, r) = \tilde{f}_i(x, y, p, z, q) + \sum_{j,l=1}^{n} a_{jl}^{(i)}(x, y) r_{jl}, \qquad i = 1, \ldots, k, \tag{47}$$

where $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_k)$: $E \times R^k \times C(E^{(0)} \cup \bar{E}, R^k) \times R^n \to R^k$, $A^{(i)} = [a_{jl}^{(i)}]_{j,l=1,\ldots,n}$, $A^{(i)}$: $E \to R^{n^2}$. Let

$$\tilde{\Sigma}_h = \tilde{E}_h \times R^k \times \mathcal{F}(E_h^{(0)} \cup E_h^*, R^k) \times R^n.$$

Suppose that

$$\tilde{\Phi}_h = (\tilde{\Phi}_h^{(1)}, \ldots, \tilde{\Phi}_h^{(k)}): \tilde{\Sigma}_h \to R^k \quad \text{and} \quad \delta: E_h^{(0)} \to R^k$$

where $h \in I_0$. Define difference operators $\Delta = (\Delta_1, \ldots, \Delta_n)$, $\Delta^{(i,m)} = [\Delta_{jl}^{(i,m)}]_{j,l=1,\ldots,n}$ as follows

$$\Delta_j w^{(m)} = (2h_j)^{-1}[w^{(j(m))} - w^{(-j(m))}], \qquad j = 1, \ldots, n,$$

$$\begin{aligned} \Delta_{jl}^{(i,m)} w^{(m)} &= (2h_j h_l)^{-1}[-w^{(j(m))} - w^{(l(m))} - w^{(-j(m))} \\ &\quad - w^{(-l(m))} + 2w^{(m)} + w^{(j(l(m)))} + w^{(-j(-l(m)))}] \\ &\qquad \text{if } j \neq l \text{ and } a_{jl}^{(i)}(x^{(m_0)}, y^{(m')}) \geqslant 0,\ j, l = 1, \ldots, n, \\ \Delta_{jl}^{(i,m)} w^{(m)} &= (2h_j h_l)^{-1}[w^{(j(m))} + w^{(l(m))} + w^{(-j(m))} \\ &\quad + w^{(-l(m))} - 2w^{(m)} - w^{(j(-l(m)))} - w^{(-j(l(m)))}] \\ &\qquad \text{if } j = l \text{ or } a_{jl}^{(i)}(x^{(m_0)}, y^{(m')}) < 0,\ j, l = 1, \ldots, n, \end{aligned} \tag{48}$$

where $w$: $E_h^{(0)} \cup E_h^* \to R$. Consider the difference method

$$\begin{aligned} z_i^{(m_0+1,m')} &= A_i z_i^{(m)} + h_0 \sum_{j,l=1}^{n} a_{jl}^{(i)}(x^{(m_0)}, y^{(m')}) \Delta_{jl}^{(i,m)} z_i^{(m)} \\ &\quad + h_0 \tilde{\Phi}_h^{(i)}(x^{(m_0)}, y^{(m')}, B^{(i)} z^{(m)}, z, \Delta z_i^{(m)}), \\ &\qquad i = 1, \ldots, k,\ m_0 = 0, 1, \ldots, n^*-1,\ m' \in M, \end{aligned}$$

$$
\begin{aligned}
&z^{(m)} = (L_j z)^{(m)}, && m \in T_j^{(-)}, \\
(49)\qquad &z^{(m)} = (\tilde{L}_j z)^{(m)}, && m \in T_j^{(+)},\ j = 1, \ldots, n, \\
&z^{(m)} = \varphi^{(m)} + \delta^{(m)} && \text{for } -n_0 \leqslant m_0 \leqslant 0,\ m' \in M.
\end{aligned}
$$

In Theorem 3 we assume that the derivatives $D_{r_{jl}} \Phi_h^{(i)}$, $i = 1, \ldots, k$, $j \neq l$ satisfy (46). Now we omit this assumption for method (48), (49).

Assumption H$_5$. Suppose that

(i) the function $\Phi_h = (\Phi_h^{(1)}, \ldots, \Phi_h^{(k)})$ given by

$$
(50)\qquad \Phi_h^{(i)}(x, y, p, z, q, r) = \tilde{\Phi}_h^{(i)}(x, y, p, z, q) + \sum_{j,l=1}^{n} a_{jl}^{(i)}(x, y) r_{jl}, \qquad i = 1, \ldots, k,
$$

satisfies conditions (i)–(v) and (vii) of Assumption H$_1$,

(ii) Assumption H$_2$ holds,

(iii) condition (iii) of Assumption H$_3$ and conditions (iii), (iv) of Assumption H$_4$ are satisfied,

(iv) for $Q = (x, y, p, z, q) \in \tilde{\Sigma}_h$, $h \in I_0$, and for $i = 1, \ldots, k$ we have

$$
(51)\qquad
\begin{aligned}
&a_\theta^{(i)} + h_0 b_{\theta,i}^{(i)} D_{p_i} \tilde{\Phi}_h^{(i)}(Q) - 2h_0 \sum_{j=1}^{n} \frac{1}{h_j^2} a_{jj}^{(i)}(x, y) + h_0 \sum_{\substack{j,l=1 \\ j \neq l}}^{n} \frac{1}{h_j h_l} |a_{jl}^{(i)}(x, y)| \geqslant 0, \\
&a_{j(\theta)}^{(i)} + h_0 b_{j(\theta),i}^{(i)} D_{p_i} \tilde{\Phi}_h^{(i)}(Q) + \frac{h_0}{2h_j} D_{q_j} \tilde{\Phi}_h^{(i)}(Q) \\
&\qquad + h_0 \frac{1}{h_j^2} a_{jj}^{(i)}(x, y) - h_0 \sum_{\substack{l=1 \\ l \neq j}}^{n} \frac{1}{h_j h_l} |a_{jl}^{(i)}(x, y)| \geqslant 0, \quad j = 1, \ldots, n,
\end{aligned}
$$

where $\theta = (0, \ldots, 0) \in S$ and

$$
(52)\qquad
\begin{aligned}
&a_{-j(\theta)}^{(i)} + h_0 b_{-j(\theta),i}^{(i)} D_{p_i} \tilde{\Phi}_h^{(i)}(Q) - \frac{h_0}{2h_j} D_{q_j} \tilde{\Phi}_h^{(i)}(Q) \\
&\qquad + h_0 \frac{1}{h_j^2} a_{jj}^{(i)}(x, y) - h_0 \sum_{\substack{l=1 \\ l \neq j}}^{n} \frac{1}{h_j h_l} |a_{jl}^{(i)}(x, y)| \geqslant 0, \quad j = 1, \ldots, n, \\
&a_s^{(i)} + h_0 b_{s,i}^{(i)} D_{p_i} \tilde{\Phi}_h^{(i)}(Q) \geqslant 0 \quad \text{for } s \in S^{(0)},
\end{aligned}
$$

where $S^{(0)} = \{s \in S\colon s \neq \theta,\ s \neq j(\theta),\ s \neq -j(\theta) \text{ for } j = 1, \ldots, n\}$,

(v) the operators $A$ and $B^{(i)}$, $i = 1, \ldots, k$, satisfy condition (i) of Assumption H$_3$ and condition (i) of Assumption H$_4$.

THEOREM 4. *Suppose that*

(A) *Assumption* $H_5$ *is satisfied,*

(B) $v \in C(E^{(0)} \cup \bar{E}, R^k)$ *is a solution of* (10), (47) *such that* $v|_{\bar{E}}$ *is of class* $C^3$,

(C) *there exist functions* $\beta$, $\beta_1$, $\beta_0$: $I_0 \to R^k$ *such that the consistency condition* (32)–(36) *is satisfied with* $f$ *and* $\Phi_h$ *given by* (47) *and* (50), *respectively,*

(D) *the functions* $\tilde{f}$, $g_j$, $\tilde{g}_j$, $j = 1, \ldots, n$, $\varphi$, $A^{(i)}$, $i = 1, \ldots, k$, *are continuous on their domains and* $u_h$ *is a solution of* (48), (49),

(E) *conditions* (E), (G) *of Theorem* 2 *are satisfied.*

*Under these assumptions there exists a function* $\gamma$: $I_0 \to R_+^k$ *such that* $\lim_{|h| \to 0} \gamma(h) = 0$ *and estimates* (20) *hold with* $\Lambda^*$ *given by* (21). *In particular we have estimates* (35), (36).

Proof. There exists a function $\gamma^* = (\gamma_1^*, \ldots, \gamma_k^*)$: $I_0 \to R_+^k$ such that

(a) for $i = 1, \ldots, k$, $m_0 = 0, 1, \ldots, n^*-1$, $m' \in M$ we have

$$|v_i^{(m_0+1,m')} - A_i v_i^{(m)} - h_0 \tilde{\Phi}_h^{(i)}(x^{(m_0)}, y^{(m')}, B^{(i)} v^{(m)}, v, \Delta v^{(m)}) - h_0 \sum_{j,l=1}^{n} a_{jl}^{(i)}(x^{(m_0)}, y^{(m')}) \Delta_{jl}^{(i,m)} v_i^{(m)}| \leqslant h_0 \gamma_i^*(h),$$

(b) estimates (19) hold.

The proof of the existence of $\gamma^*$ is similar to that used in the proof of Theorem 2. We omit the details. Now we apply Lemma 1 for the proof of our assertion. Let $\Omega_h = M^*$, $\Omega_h^{(0)}$ is the set given in the proof of Theorem 1 and $N_h$, $N_h^{(0)}$, $N_h^*$ are the sets defined in Section 1 with $n_0$ and $n^*$ given in Section 2. Let $\Gamma_h = \mathscr{F}(N_h \times \Omega_h, R^k)$ and

$$F_h = (F_h^{(1)}, \ldots, F_h^{(k)}): N_h^* \times \Omega_h \to \mathscr{F}(\Omega_h, R^k)$$

is defined by

$$F_h^{(i)}(m_0, z)(m') = A_i z_i^{(m)} + h_0 \sum_{j,l=1}^{n} a_{jl}^{(i)}(x^{(m_0)}, y^{(m')}) \Delta_{jl}^{(i,m)} z_i^{(m)} + h_0 \tilde{\Phi}_h^{(i)}(x^{(m_0)}, y^{(m')}, B^{(i)} z^{(m)}, z, \Delta z_i^{(m)}), \quad m \in N_h^* \times M. \tag{53}$$

If $m_0 \in N_h^*$, $m' \in \Omega_h \backslash M$ then we define $F_h(m_0, z)(m')$ by (23). Then

$$u_h^{(m_0+1,m')} = F_h(m_0, u_h)(m'), \quad m_0 \in N_h^*, \ m' \in \Omega_h.$$

Let $V(m) = T_{m_0}$ if $m \in N_h^* \times (\Omega_h \backslash \Omega_h^{(0)})$ and

$$V(m) = T_{m_0} \cup \{(j, \bar{m}') \in N_h \times (\Omega_h \backslash \Omega_h^{(0)}): j = m_0 + 1\}$$

for $m \in N_h^* \times \Omega_h^{(0)}$. It follows from Assumption $H_5$ that $F_h$ satisfies condition (V) with respect to $\Omega_h^{(0)}$. Now we prove that $F_h$ is nondecreasing with respect to the functional argument. Suppose that $z,\ \bar{z}\in\Gamma_h$ and $z\leqslant\bar{z}$. Let

$$J^{(+)}_{i,m}=\{(j,l)\colon j,l\in\{1,\ldots,n\},\ j\neq l,\ a^{(i)}_{jl}(x^{(m_0)},y^{(m')})\geqslant 0\},$$
$$J^{(-)}_{i,m}=\{(j,l)\colon j,l\in\{1,\ldots,n\},\ j\neq l,\ a^{(i)}_{jl}(x^{(m_0)},y^{(m')})<0\}.$$

Then, using the mean value theorem we have for $m'\in M$, $m_0\in N^*_h$

$$\begin{aligned}
&F^{(i)}_h(m_0,z)(m')-F^{(i)}_h(m_0,\bar{z})(m')\\
&\leqslant (z^{(m)}_i-\bar{z}^{(m)}_i)\Big[a^{(i)}_\theta+h_0 b^{(i)}_{\theta,i}D_{p_i}\tilde{\Phi}^{(i)}_h(Q_i)-2h_0\sum_{j=1}^{n}\frac{1}{h_j^2}a^{(i)}_{jj}(x^{(m_0)},y^{(m')})\\
&\qquad+h_0\sum_{\substack{j,l=1\\ j\neq l}}^{n}\frac{1}{h_jh_l}|a^{(i)}_{jl}(x^{(m_0)},y^{(m')})|\Big]\\
&+\sum_{j=1}^{n}(z^{(j(m))}_i-\bar{z}^{(j(m))}_i)\Big[a^{(i)}_{j(\theta)}+h_0 b^{(i)}_{j(\theta),i}D_{p_i}\tilde{\Phi}^{(i)}_h(Q_i)\\
&\qquad+h_0\frac{1}{2h_j}D_{q_j}\tilde{\Phi}^{(i)}_h(Q_i)+h_0\frac{1}{h_j^2}a^{(i)}_{jj}(x^{(m_0)},y^{(m')})\\
&\qquad-h_0\sum_{\substack{l=1\\ l\neq j}}^{n}\frac{1}{h_jh_l}|a^{(i)}_{jl}(x^{(m_0)},y^{(m')})|\Big]\\
&+\sum_{j=1}^{n}(z^{(-j(m))}_i-\bar{z}^{(-j(m))}_i)\Big[a^{(i)}_{-j(\theta)}+h_0 b^{(i)}_{-j(\theta),i}D_{p_i}\tilde{\Phi}^{(i)}_h(Q_i)\\
&\qquad-h_0\frac{1}{2h_j}D_{q_j}\tilde{\Phi}^{(i)}_h(Q_i)+h_0\frac{1}{h_j^2}a^{(i)}_{jj}(x^{(m_0)},y^{(m')})\\
&\qquad-h_0\sum_{\substack{l=1\\ l\neq j}}^{n}\frac{1}{h_jh_l}|a^{(i)}_{jl}(x^{(m_0)},y^{(m')})|\Big]+\sum_{s\in S^{(0)}}(z^{(m_0,m'+s)}_i-\bar{z}^{(m_0,m'+s)}_i)\times\\
&\times[a^{(i)}_s+h_0 b^{(i)}_{s,i}D_{p_i}\tilde{\Phi}^{(i)}_h(Q_i)]+h_0\sum_{(j,l)\in J^{(+)}_{i,m}}a^{(i)}_{jl}(x^{(m_0)},y^{(m')})(2h_jh_l)^{-1}\times\\
&\times[z^{(j(l(m)))}_i-\bar{z}^{(j(l(m)))}_i+z^{(-j(-l(m)))}_i-\bar{z}^{(-j(-l(m)))}_i]\\
&-h_0\sum_{(j,l)\in J^{(-)}_{i,m}}a^{(i)}_{jl}(x^{(m_0)},y^{(m')})(2h_jh_l)^{-1}\times\\
&\times[z^{(j(-l(m)))}-\bar{z}^{(j(-l(m)))}_i+z^{(-j(l(m)))}_i-\bar{z}^{(-j(l(m)))}_i],\qquad i=1,\ldots,k.
\end{aligned}$$

These estimates and (51), (52) imply

(54) $$F_h(m_0,z)(m')\leqslant F_h(m_0,\bar{z})(m'),\qquad m_0\in N^*_h,$$

for $m' \in M$. It follows from the definition of $F_h$ and from Assumption $H_2$ that (54) holds true for $m \in \Omega_h \setminus M$, $z, \bar{z} \in \Gamma_h$, $z \leqslant \bar{z}$. Let

$$\Psi_h = (\psi_h^{(1)}, \ldots, \psi_h^{(k)}) \colon N_h^* \times \Gamma_h \to \mathcal{F}(\Omega_h, R^k)$$

be a function given by

$$\psi_h^{(i)}(m_0, z)(m') = z_i^{(m_0+1, m')} - A_i z_i^{(m)} - h_0 \tilde{\Phi}_h^{(i)}(x^{(m_0)}, y^{(m')}, B^{(i)} z^{(m)}, z, \Delta z_i^{(m)})$$
$$- h_0 \sum_{j,l=1}^{n} a_{jl}^{(i)}(x^{(m_0)}, y^{(m')}) \Delta_{jl}^{(i,m)} z_i^{(m)},$$
$$i = 1, \ldots, k, \; m_0 \in N_h^*, \; m' \in M.$$

If $m_0 \in N_h^*$ and $m' \in \Omega_h \setminus M$ then we define $\Psi_h(m_0, z)(m')$ by (26).

For $\tilde{F}_h \colon N_h^* \times \Gamma_h \to \mathcal{F}(\Omega_h, R^k)$ given by (27) we have (28) and (29). Since $F_h$ given by (53), (23) satisfies (30) then using Lemma 1 (see also Example 1) we obtain the assertion of our theorem.

Remark 5. A theorem on difference-functional inequalities related to (10), (47) can be proved without assumption (46) (see [9]).

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