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The Schur and Steinhaus Theorems for 4-Dimensional Infinite Matrices

Abstract. This paper is a sequel to [2]. Throughout this paper, entries of double sequences, double series and 4-dimensional infinite matrices are real or complex numbers. We prove the Schur and Steinhaus theorems for 4-dimensional infinite matrices.

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1. Introduction and Preliminaries. This paper is a sequel to [2]. Throughout the present paper, entries of double sequences, double series and 4-dimensional infinite matrices are real or complex numbers. If $A = (a_{m,n,k,\ell})$ is a 4-dimensional infinite matrix, and $m, n, k, \ell = 0, 1, 2, \ldots$, by the A-transform of a double sequence $x = \{x_{k,\ell}\}$, $k, \ell = 0, 1, 2, \ldots$, we mean the double sequence $A(x) = \{(Ax)_{m,n}\}$,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell} x_{k,\ell}, \quad m, n = 0, 1, 2, \ldots,$$

where we suppose that the double series on the right converges. The double sequence $x = \{x_{k,\ell}\}$ is said to be summable A or A-summable to s if

$$\lim_{m+n\to\infty} (Ax)_{m,n} = s.$$

Let $c_{ds}, \ell_{ds}^{\infty}$ denote the spaces of convergent double sequences and bounded double sequences, respectively. If $A = (a_{m,n,k,\ell})$ is such that $\{(Ax)_{m,n}\} \in c_{ds}$ whenever $x = \{x_{k,\ell}\} \in c_{ds}$, A is said to be convergence-preserving. If, further, $\lim_{m+n\to\infty} (Ax)_{m,n} = \lim_{k+\ell\to\infty} x_{k,\ell}$, we say that A is regular.

Natarajan proved the following result.

THEOREM 1.1 ([2]) *$A = (a_{m,n,k,\ell})$ is regular if and only if*

(1) $$\lim_{m+n\to\infty} a_{m,n,k,\ell} = 0, \quad k,\ell = 0,1,2,\ldots;$$

(2) $$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell} = 1;$$

(3) $$\lim_{m+n\to\infty} \sum_{k=0}^{\infty} |a_{m,n,k,\ell}| = 0, \quad \ell = 0,1,2,\ldots;$$

(4) $$\lim_{m+n\to\infty} \sum_{\ell=0}^{\infty} |a_{m,n,k,\ell}| = 0, \quad k = 0,1,2,\ldots;$$

and

(5) $$\sup_{m,n} \sum_{k,\ell=0}^{\infty} |a_{m,n,k,\ell}| < \infty.$$

A is called a Schur matrix if $\{(Ax)_{m,n}\} \in c_{ds}$ whenever $x = \{x_{k,\ell}\} \in \ell^{\infty}_{ds}$. The main object of this paper is to get necessary and sufficient conditions for A to be a Schur matrix and then deduce Steinhaus theorem.

DEFINITION 1.2 The double sequence $\{x_{m,n}\}$ is called a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ (the set of all non-negative integers) such that the set

$$\{(m,n),(k,\ell) \in \mathbb{N}^2 : |x_{m,n} - x_{k,\ell}| \geqslant \epsilon, \quad m,n,k,\ell \geqslant N\}$$

if finite.

It is now easy to prove the following result.

THEOREM 1.3 *The double sequence $\{x_{m,n}\}$ is Cauchy if and only if*

(6) $$\lim_{m+n\to\infty} |x_{m+u,n} - x_{m,n}| = 0, \quad u = 0,1,2,\ldots;$$

and

(7) $$\lim_{m+n\to\infty} |x_{m,n+v} - x_{m,n}| = 0, \quad v = 0,1,2,\ldots.$$

DEFINITION 1.4 If every Cauchy double sequence of a normed linear space X converges to an element of X, X is said to be double sequence complete or *ds*-complete.

Note that $\mathbb{R}$ (the set of all real numbers) and $\mathbb{C}$ (the set of all complex numbers) are ds-complete.

For $x = \{x_{m,n}\} \in \ell_{ds}^{\infty}$, define $\|x\| = \sup_{m,n} |x_{m,n}|$. One can easily prove that ℓ_{ds}^{∞} is a normed linear space which is ds-complete. With the same definition of norm for elements of c_{ds}, c_{ds} is a closed subspace of ℓ_{ds}^{∞}.

2. Main Results. Schur's theorem and Steinhaus theorem for 2-dimensional infinite matrices are very well-known (see, for instance, [1]). In this section, we obtain these results for 4-dimensional infinite matrices.

THEOREM 2.1 (SCHUR) *The necessary and sufficient conditions for a 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$ to be a Schur matrix, i.e., $\{(Ax)_{m,n}\} \in c_{ds}$ whenever $x = \{x_{k,\ell}\} \in \ell_{ds}^{\infty}$ are:*

$$\sum_{k,\ell=0}^{\infty} |a_{m,n,k,\ell}| < \infty, \quad m,n = 0,1,2,\ldots; \tag{8}$$

$$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty} |a_{m+u,n,k,\ell} - a_{m,n,k,\ell}| = 0, \quad u = 0,1,2,\ldots; \tag{9}$$

and

$$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty} |a_{m,n+v,k,\ell} - a_{m,n,k,\ell}| = 0, \quad v = 0,1,2,\ldots. \tag{10}$$

PROOF Sufficiency part. Let (8), (9), (10) hold and $x = \{x_{k,\ell}\} \in \ell_{ds}^{\infty}$. First, we note that in view of (8),

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell} x_{k,\ell}, \quad m,n = 0,1,2,\ldots$$

is defined, the double series on the right being convergent. Now, for $u = 0,1,2,\ldots$,

$$\begin{aligned} |(Ax)_{m+u,n} - (Ax)_{m,n}| &= \left| \sum_{k,\ell=0}^{\infty} (a_{m+u,n,k,\ell} - a_{m,n,k,\ell}) x_{k,\ell} \right| \\ &\leqslant M \sum_{k,\ell=0}^{\infty} |a_{m+u,n,k,\ell} - a_{m,n,k,\ell}| \\ &\to 0, \quad m+n \to \infty, \quad \text{using (9)}, \end{aligned}$$

where $|x_{k,\ell}| \leqslant M$, $k,\ell = 0,1,2,\ldots$, $M > 0$. Similarly it follows that

$$|(Ax)_{m,n+v} - (Ax)_{m,n}| \to 0, \quad m+n \to \infty, \quad v = 0,1,2,\ldots,$$

using (10). Thus $\{(Ax)_{m,n}\}$ is a Cauchy double sequence. Since $\mathbb{R}$ (or $\mathbb{C}$) is ds-complete, $\{(Ax)_{m,n}\}$ converges, i.e., $\{(Ax)_{m,n}\} \in c_{ds}$, completing the sufficiency part of the proof.

Necessity part. Let A be a Schur matrix. For $m, n = 0, 1, 2, \dots$, consider the double sequence $\{x_{k,\ell}\}$, where $x_{k,\ell} = sgn\ a_{m,n,k,\ell}$, $k, \ell = 0, 1, 2, \dots$. Then $\{x_{k,\ell}\} \in \ell_{ds}^{\infty}$ so that, by hypothesis,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty} |a_{m,n,k,\ell}|, \quad m, n = 0, 1, 2, \dots$$

is defined. Since the series on the right converges, (8) holds. Suppose (9) does not hold. Then there exist $\ell_0, u_0 \in \mathbb{N}$ such that

$$\lim_{m+n\to\infty} \sum_{k=0}^{\infty} |a_{m+u_0,n,k,\ell_0} - a_{m,n,k,\ell_0}| = 0$$

does not hold. So there exists $\epsilon > 0$ such that the set

$$\left\{(m, n) \in \mathbb{N}^2 : \sum_{k=0}^{\infty} |a_{m+u_0,n,k,\ell_0} - a_{m,n,k,\ell_0}| > 2\epsilon\right\}$$

is infinite. Thus we can choose pairs of integers $m_p, n_p \in \mathbb{N}$ such that $m_1 + n_1 < m_2 + n_2 < \dots < m_p + n_P < \dots$ and

$$\sum_{k=0}^{\infty} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| > 2\epsilon, \quad p = 1, 2, \dots. \tag{11}$$

Using (8), we have,

$$\sum_{k=0}^{\infty} |a_{m_1+u_0,n_1,k,\ell_0} - a_{m_1,n_1,k,\ell_0}| < \infty.$$

Consequently there exists $r_1 \in \mathbb{N}$ such that

$$\sum_{k=r_1}^{\infty} |a_{m_1+u_0,n_1,k,\ell_0} - a_{m_1,n_1,k,\ell_0}| < \frac{\epsilon}{4}. \tag{12}$$

In view of (11) and (12), we have,

$$\sum_{k=0}^{r_1-1} |a_{m_1+u_0,n_1,k,\ell_0} - a_{m_1,n_1,k,\ell_0}| > \frac{7\epsilon}{4} > \epsilon. \tag{13}$$

By hypothesis, (1) holds so that we can suppose that

$$\sum_{k=0}^{r_1-1} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| < \frac{\epsilon}{4}. \tag{14}$$

Using (11), we have,

$$\sum_{k=0}^{\infty} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| > 2\epsilon. \tag{15}$$

Using (8),

$$\sum_{k=0}^{\infty} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| < \infty$$

so that there exists $r_2 \in \mathbb{N}$, $r_2 > r_1$, such that

$$\sum_{k=r_2}^{\infty} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| < \frac{\epsilon}{4}. \tag{16}$$

From (14), (15), (16), we have,

$$\sum_{k=r_1}^{r_2-1} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| > \frac{3\epsilon}{2} > \epsilon.$$

Inductively, we can choose a strictly increasing sequence $\{r_p\}$ such that

$$\sum_{k=0}^{r_{p-1}-1} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| < \frac{\epsilon}{4}; \tag{17}$$

$$\sum_{k=r_p}^{\infty} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| < \frac{\epsilon}{4}; \tag{18}$$

and

$$\sum_{k=r_{p-1}}^{r_p-1} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| > \epsilon. \tag{19}$$

Now, define $\{x_{k,\ell}\} \in \ell_{ds}^{\infty}$, where

$$x_{k,\ell} = \begin{cases} sgn(a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}), \\ \qquad\qquad \text{if } \ell = \ell_0, r_{p-1} \leqslant k < r_p, p = 1, 2, \ldots; \\ 0, \quad \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned}
(Ax)_{m_p+u_0,n_p} - (Ax)_{m_p,n_p} &= \sum_{k,\ell=0}^{\infty} (a_{m_p+u_0,n_p,k,\ell} - a_{m_p,n_p,k,\ell})x_{k,\ell} \\
&= \sum_{k=0}^{\infty}(a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0} \\
&= \sum_{k=0}^{r_{p-1}-1} (a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0} \\
&\quad + \sum_{k=r_{p-1}}^{r_p-1} (a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0} \\
&\quad + \sum_{k=r_p}^{\infty} (a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0} \\
&= \sum_{k=0}^{r_{p-1}-1} (a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0} \\
&\quad + \sum_{k=r_{p-1}}^{r_p-1} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| \\
&\quad + \sum_{k=r_p}^{\infty} (a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0}
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{k=r_{p-1}}^{r_p-1} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| &= \left\{(Ax)_{m_p+u_0,n_p} - (Ax)_{m_p,n_p}\right\} \\
&\quad - \sum_{k=0}^{r_{p-1}-1} (a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0} \\
&\quad - \sum_{k=r_p}^{\infty} (a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0}.
\end{aligned}$$

In view of (17), (18), (19), we have,

$$\begin{aligned}
\epsilon &< \sum_{k=r_{p-1}}^{r_p-1} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| \\
&\leqslant \left|(Ax)_{m_p+u_0,n_p} - (Ax)_{m_p,n_p}\right| + \frac{\epsilon}{4} + \frac{\epsilon}{4},
\end{aligned}$$

from which it follows that

$$\left|(Ax)_{m_p+u_0,n_p} - (Ax)_{m_p,n_p}\right| > \frac{\epsilon}{2}, \quad p = 1, 2, \ldots .$$

Consequently$\{(Ax)_{m,n}\} \notin c_{ds}$, a contradiction. Thus (9) holds. Similarly (10) holds too. This completes the proof of the theorem. ∎

We now deduce the following result.

THEOREM 2.2 (STEINHAUS) *A 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$ cannot be both a regular and a Schur matrix, i.e., given a 4-dimensional regular matrix A, there exists a bounded divergent double sequence which is not A-summable.*

PROOF If A is regular, then (1) and (2) hold. If A is a Schur matrix too, then, $\{a_{m,n,k,\ell}\}_{m,n=0}^{\infty}$ is uniformly Cauchy with respect to $k, \ell = 0, 1, 2, \ldots$. Since $\mathbb{R}$ (or $\mathbb{C}$) is ds-complete, $\{a_{m,n,k,\ell}\}_{m,n=0}^{\infty}$ converges uniformly to 0 with respect to $k, \ell = 0, 1, 2, \ldots$. Consequently, we have,

$$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell} = 0,$$

a contradiction of (2), completing the proof. ∎

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