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On binary non-associative products and their relation to a classical problem of Euler

1. Introduction. Marshall Hall [4], Chapter 3, raises the question: *in how many ways can the sequence $x_1 x_2 \ldots x_n$ be combined in this order by a binary non-associative product?* Letting a_n represent the number of ways for a sequence of length n and adopting the convention $a_1 = 1$, Table 1 can easily be constructed.

Table 1

n	Different binary non-associative products	a_n — number of different products
2	$x_1 x_2$	1
3	$(x_1)(x_2 x_3)$, $(x_1 x_2)(x_3)$	2
4	$(x_1)(x_2 x_3 x_4)$ (2 ways); $(x_1 x_2 x_3)(x_4)$ (2 ways) $(x_1 x_2)(x_3 x_4)$ (1 way)	5
5	$(x_1)(x_2 x_3 x_4 x_5)$ (5 ways); $(x_1 x_2 x_3 x_4)(x_5)$ (5 ways) $(x_1 x_2)(x_3 x_4 x_5)$ (2 ways); $(x_1 x_2 x_3)(x_4 x_5)$ (2 ways)	14
6	$(x_1)(x_2 x_3 x_4 x_5 x_6)$ (14 ways); $(x_1 x_2 x_3 x_4 x_5)(x_6)$ (14 ways) $(x_1 x_2)(x_3 x_4 x_5 x_6)$ (5 ways); $(x_1 x_2 x_3 x_4)(x_5 x_6)$ (5 ways) $(x_1 x_2 x_3)(x_4 x_5 x_6)$ (4 ways)	42

From Table 1 it is clear that the last product is some composite of the first r letters multiplied by some composite of the last $n-r$, of the form $(x_1 \ldots x_r)(x_{r+1} \ldots x_n)$. Noting that the first r letters can be combined a_r ways and the last $n-r$ letters in a_{n-r} ways, it follows that

$$(1) \qquad a_n = a_1 a_{n-1} + a_2 a_{n-2} + \ldots + a_{n-1} a_1 = \sum_{i=1}^{n-1} a_i a_{n-i}, \qquad n \geqslant 2.$$

Hall [4], in order to determine a_n, introduces the generating function $f(x)$, where

$$f(x) = \sum_{i=1}^{\infty} a_i x^i. \tag{2}$$

At this point the question of convergence of (2) is postponed and Hall later states that "to prove the convergence of (2) on the basis of (1) alone is exceedingly difficult". Squaring (2) yields equation (3) from which (4) immediately follows.

$$\big(f(x)\big)^2 = -x + f(x), \tag{3}$$

$$f(x) = \frac{1-\sqrt{1-4x}}{2}. \tag{4}$$

Now expand (4) as a power series and it follows that the series for $f(x)$ converges for $|x| < \frac{1}{4}$, and has for the coefficient of x^n

$$\frac{(\frac{1}{2})(-\frac{1}{2})\ldots\left(\frac{3-2n}{2}\right)(-4)^n(-\frac{1}{2})}{n!} = \frac{(2n-2)!}{n!(n-1)!}. \tag{5}$$

For these values, $f(x)$ and hence the recursion (1) has for a solution

$$a_n = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1}, \quad n \geqslant 1. \tag{6}$$

Silberger [6] provides a brief history of the integer (6) and he points out that most of the proofs of (6) use generating functions. His paper, however, contains an elementary combinatorial proof of (6) which uses neither generating functions nor equation (1). He also proves that a_n is odd if and only if n is a power 2.

In 1751 Leonhard Euler posed the problem, *in how many ways can a convex n-gon be partitioned into triangles by diagonals which do not intersect inside the n-gon,* to the mathematician Christian Goldbach. Letting E_n represent the number of possible divisions for a polygon of n sides, Euler developed the formula

$$E_n = \frac{2\cdot 6\cdot 10\ldots(4n-10)}{(n-1)!}. \tag{7}$$

However, this was not easy for Euler and he said, "The process of induction I employed was quite laborous". This problem, its solution and a short history of it appear as Problem 7 in Dörrie [1]. In fact, Dörrie also mentions and gives the references to the work done on this problem by Segner

(1758), Rodrigues (1838), Catalan (1838), and Urban (1941). This work includes the relation, since both satisfy the same recurrence, that

$$E_{n+1} = a_n. \tag{8}$$

In the next section the solution to the problem of Marshall Hall, proving (6) by using (1) and not (2), is provided by elementary combinatorial methods. Also, in this section, a combinatorial proof, shorter and more elementary than Silberger [6], that a_n is odd if and only if n is a power of 2 is given. In Section 3 relation (8) is explored. A one to one correspondence is established between the division into triangles of the convex $(n+1)$-gon and the non-associative parsing of the n letter word. A geometric proof that there are an odd number of partitions of the convex $(n+1)$-gon into triangles if and only if n is a power of 2 is also given. In Section 4 an asymptotic expression for the a_n, some applications and other interpretations are given.

The authors would like to thank Mr. Mark Villarino for helpful suggestions on various aspects of this problem.

2. Further properties of the integer a_n. Returning to (1) it is easy to see that

$$a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 5,\ a_5 = 14,\ a_6 = 42, a_7 = 132, \\ a_8 = 429, \ldots \tag{9}$$

Observing (9) and computing further values of the a_n, one is lead to conjecturing that

$$a_n \sim 4a_{n-1}, \tag{10}$$

or more precisely that

$$a_n = \frac{4n-a}{n-b}a_{n-1}, \qquad n \geqslant 2. \tag{11}$$

If (11) is valid it is easy to establish, by choosing $n = 2, 3$ respectively, that $a = 6$ and $b = 0$, and (11) becomes

$$a_n = \left(4 - \frac{6}{n}\right)a_{n-1}, \qquad n \geqslant 2. \tag{12}$$

It shall be proven that (12) and hence also (10) are indeed valid. It is interesting to note that (12) is equivalent to (6) since

$$a_n = \frac{4n-6}{n}a_{n-1} = \frac{4n-6}{n}\cdot\frac{4n-10}{n-1}\cdot\frac{4n-14}{n-2}\cdots\frac{4n-(4n-6)}{3}\cdot\frac{2}{2}\cdot 1,$$

it follows that

$$(13)\qquad a_n = \frac{(2n-3)(2n-5)(2n-7)\ldots 5\cdot 3\cdot 1}{n!}\,2^{n-1}.$$

Now multiply the numerator and denominator by $(n-1)!$ and (6) follows. A combinatorial proof of (6), and hence also (12), will now be provided. However, the following well-known combinatorial identity (Knuth [5], p. 58) is first needed. For a proof of this Lemma and for further information about a much larger class of this type of combinatorial identity, the reader is referred to Gould and Kaucky [3].

LEMMA. *For m, r, s, t integers and $r \neq tj$, it follows that*

$$(14)\qquad \sum_{j=0}^{m}\binom{r-tj}{j}\binom{s-t(m-j)}{m-j}\frac{r}{r-jt} = \binom{r+s-tm}{m}.$$

THEOREM 1. *For the sequence a_n defined by* (1), *with $a_1 = 1$, relation* (6) *holds.*

Proof. By induction for $n = 1, 2, 3$ the Theorem clearly holds. Now assume (6) holds for all $n < k$ and consider $n = k$, clearly

$$a_k = \sum_{i=1}^{k-1} a_i a_{k-i} = \sum_{i=1}^{k-1}\frac{1}{k-i}\cdot\frac{1}{i}\binom{2i-2}{i-1}\binom{2k-2i-2}{k-i-1}.$$

Since,

$$\frac{1}{(k-i)i} = \frac{1}{k}\left(\frac{1}{i}+\frac{1}{k-i}\right)$$

it follows that

$$a_k = \frac{1}{k}\sum_{i=1}^{k-1}\left(\frac{1}{i}+\frac{1}{k-i}\right)\binom{2i-2}{i-1}\binom{2k-2i-2}{k-i-1} = \frac{1}{k}(S_1+S_2).$$

But $S_1 = S_2 = S$, thus $a_k = \dfrac{2S}{k}$ and

$$S = \sum_{i=1}^{k-1}\frac{1}{i}\binom{2i-2}{i-1}\binom{2k-2i-2}{k-i-1} = \sum_{j=0}^{k-2}\frac{1}{j+1}\binom{2j}{j}\binom{2k-4-2j}{k-2-j}.$$

Since

$$\frac{1}{j+1}\binom{2j}{j} = \frac{1}{2j+1}\binom{2j+1}{j}$$

it follows that

$$S = \sum_{j=0}^{k-2}\binom{2j+1}{j}\binom{2k-4-2j}{k-2-j}\frac{1}{2j+1}.$$

Now apply (14) with $m = k-2$, $r = 1$, $s = 0$, $t = -2$. This yields that

$$S = \binom{2k-3}{k-2} \quad \text{and} \quad a_k = \frac{2}{k}\binom{2k-3}{k-2} = \frac{1}{k}\binom{2k-2}{k-1}. \qquad \text{Q. E. D.}$$

Now return to the generating function $f(x)$ of equation (2), it follows immediately, by (12) and the ratio test, that the series converges for all $|x| < \frac{1}{4}$. Thus the convergence of (2) has been established without having to first square the infinite series.

THEOREM 2. *The integer a_n of* (1) *is odd if and only if n is a power of* 2.

Proof. From (1) it is clear that

$$a_{2n+1} = \sum_{i=1}^{2n} a_i a_{2n-i+1} \equiv 0 \pmod 2$$

and

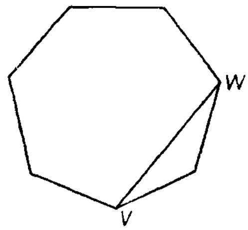
$$a_{2n} = \sum_{i=1}^{2n-1} a_i a_{2n-i} \equiv a_n^2 \equiv a_n \pmod 2$$

from which the Theorem is immediate.

3. Euler's problem. Now the connection between Euler's problem and the binary non-associative product is established. The procedure will be to give a rule for the decomposition of an arbitrary convex polygon and to show inductively that this decomposition parallels a particular method of generating successively the desired products. As an offshoot of this decomposition it is easy to see the geometric significance of the parity of a_n.

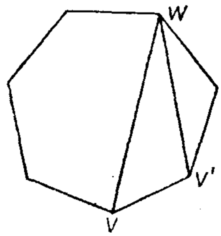
For any convex polygon P choose an arbitrary but fixed vertex, V. In stage one of the decomposition, V is connected sequentially to each non-adjacent vertex, say W (see Figure 1, (a) through (d), in which the process is illustrated for a seven sided polygon, $n = 6$). Each such connection reduces the problem of further dissection of P to a lower order dissection. To ensure that the decomposition avoids duplication of previously considered subpolygons, it is only necessary to join V', a vertex adjacent to V, to W, since $V'W$ will then intersect all previous decompositions. In the general case, this creates a triangle and two convex polygons of lower order having a *total* of $n+2$ sides ($n+3$ in the degenerate cases). After $n-2$ steps, any remaining possibilities must be included in the decomposition obtained by joining V' to V'', the other vertex adjacent to V (Figure 1 (e)). That this is a new decomposition is obvious, since $V'V''$ must intersect all previously constructed rays emanating from V. Further, the possibilities are now exhausted for any other decomposition must include a line joining V to a vertex.

(a)



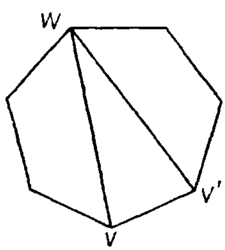
Six and three sides
$14 \times 1 = 14$

(b)



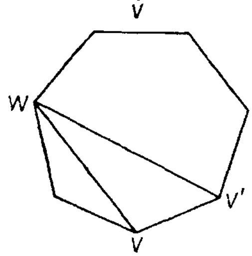
Five and three sides
$5 \times 1 = 5$

(c)



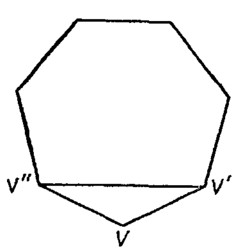
Four and four sides
$2 \times 2 = 4$

(d)



Three and five sides
$5 \times 1 = 5$

(e)



Three and six sides
$1 \times 14 = 14$

Reference to Table 1 shows that the binary non-associative product can be considered as arising from a similiar decomposition The sequence $x_1 \ldots x_n$ is subdivided by enclosing $x_1 \ldots x_i$ and $x_{k+1} \cdot\cdot x_n$ in. parentheses as two factors of the product. Further subdivision and the establishment of the recurrence follow inductively.

To see that the decomposition of the polygon into triangles is in fact in one-to-one correspondence with formation of the binary non-associative product, note first that the identification can be made trivially for $n = 1$ and $n = 2$. Assume that the correspondence is established for $n < N$, and order the 2-factor products in the order of parenthesis insertion, that is the i-th product is characterized by the insertion of a left (opening) and a right (closing) parenthesis between x_i and x_{i+1}. The successive steps of the polygonal decomposition considered in their natural order of counterclockwise succession are assigned as correspondents. The correspondence is clearly one-to-one and the decomposition to lower orders takes place in the identical manner; for consider decomposition of the sequence by insertion of parenthesis following x_i, $1 < i < n-1$. We have $(x_1 \ldots x_i)(x_{i+1} \ldots x_n)$. Under the defined correspondence this product is associated with the polygonal decomposition obtained by designating W as the i-th vertex following V' in the counterclockwise direction. The decomposition of P gives two subpolygons, having $(i+1)$ and $(n-i+1)$ sides respectively and proceeding inductively, it is clear the correspondence is established; a trivial modification is required for the degenerate cases $i = 1$ and $i = n-1$, but involves no additional complexity.

To complete this section a geometric proof of Theorem 2 will now be given. Recalling (8), Table 1, and the decomposition of the $(n+1)$-gon, it follows that if n is odd, a_n and E_{n+1} are both even. Also, if n is even, by the decomposition of the $(n+1)$-gon, there is an isolated decomposition corresponding to $(E_{n+2/2})^2$; all other terms, by symmetry, occur twice. It now follows that E_{n+1} is odd if and only if $E_{n+2/2}$ is odd; but $E_{n+2/2}$ is odd if and only if $E_{n+4/4}$ is odd. Continuing this process, it is clear that E_{n+1} is odd if and only if n is a power of 2.

4. Applications. Applying Stirling's formula (15) to (6) with $(n+1)$ replacing n yields (16), an approximation which enables one to estimate the a_{n+1}, using a logarithm table, for large values of n.

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{15}$$

$$a_{n+1} \sim \frac{2^{2n}}{(n+1)\sqrt{\pi n}}. \tag{16}$$

Sequence (1) has many possible interpretations. For example, let c_n represent the number of ways that $2n$ points on the circumference of a circle can be joined in pairs by n chords which do not intersect within the circle. This of course satisfies a relation similar to (1) and it is easy to show that

$$c_n = \frac{1}{n+1}\binom{2n}{n} = a_{n+1} = E_{n+2}. \tag{17}$$

Feller [2], Chapter 3, points out that the integers a_n arise in problems of arrangements and in fact are a special case of the well-known ballot problem. In this chapter Feller also shows that the a_n are related to problems of coin tossing and random walks.

As a final illustration, the reader is referred to Knuth [5], p. 553. The application given here is the determination of the number of possible rearrangements of a sequence, using a stack (linear list for which all insertions and deletions are made at one end of the list) for temporary storage.

References

[1] H. Dörrie, *100 Great problems of elementary mathematics*, Dover 1965.

[2] W. Feller, *An introduction to probability theory and its applications*, Wiley 1957.

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[6] D. M. Silberger, *Occurrences of the integer* $(2n-2)!/n!(n-1)!$, Prace Mat. 10 (1969), p. 91-96.

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