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Weighted composition operators via Berezin transform and Carleson measure

Abstract. In this paper, we study the boundedness and the compactness of weighted composition operators on Hardy spaces and weighted Bergman spaces of the unit polydisc in $\mathbf{C}^n$.

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1. Introduction. Let $\mathbf{D}^n$ be the unit polydisc in $\mathbf{C}^n$ and let φ, ψ be analytic functions defined on $\mathbf{D}^n$ such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$. Then a weighted composition operator $W_{\varphi,\psi}$ is defined as $W_{\varphi,\psi}(f)(z) = \psi(z)f(\varphi(z)$, for all $f \in H(\mathbf{D}^n)$, the space of holomorphic functions on the unit polydisc $\mathbf{D}^n$. For composition operators on spaces of analytic functions one can refer to [1], [6] and [20], whereas for weighted composition operator we refer to [4],[5],[7],[14],[17],[21] and references there in. The weighted composition operators appears naturally in the study of isometries on most of the function spaces, see [9] and [19]. Operators of this kind also appears in many branches of analysis; the theory of dynamical systems, semigroups, the theory of operator algebras and so on.

Let T^n denotes the distinguished boundary of $\mathbf{D}^n$, and we use m_n to denote the n–dimensional Lebesgue area measure on T^n, normalised so that $m_n(T^n) = 1$. Also we use the notation f_r for the function $f_r(z) = f(rz)$, where $z = (z_1, z_2, \dots, z_n) \in T^n$. Take $p > 0$. Then $f \in H(\mathbf{D}^n)$ is said to be a member of the Hardy space $H^p(\mathbf{D}^n)$ if and only if

$$\|f\|_{H^p(\mathbf{D}^n)} = (\sup_{0<r<1} \int_{T^n} | f_r |^p \, dm_n)^{\frac{1}{p}} < \infty.$$

For $p=\infty, H^\infty(\mathbf{D}^n)$ is the set of bounded analytic functions on $\mathbf{D}^n$ with the supremum norm.
If $f\in H^p(\mathbf{D}^n)$, then the limit function f^* defined as

$$f^*(z)=\lim_{r\to 1} f(rz) \text{ exists a.e.}$$

on T^n and $f^*\in L^p(m_n)$, the space of measurable functions f on T^n for which

$$\int_{T^n} \mid f(z)\mid^p dm_n<\infty.$$

Let σ_n denote the volume measure on $\overline{\mathbf{D}^n}$, so that $\sigma_n(\overline{\mathbf{D}^n})=1$ and $\sigma_{n,\alpha}$ given by $\prod_{i=1}^n(1-\mid z_i\mid^2)^\alpha\sigma_n$ denotes the weighted measure on $\overline{\mathbf{D}^n}$, where $z=(z_1,z_2,\dots,z_n)$ $\in\overline{\mathbf{D}^n}$. Then $f\in H(\mathbf{D}^n)$ is said to be in the weighted Bergman space $A^p_\alpha(\mathbf{D}^n)$ if and only if

$$\|f\|_{A^p_\alpha(\mathbf{D}^n)}=\left(\int_{\mathbf{D}^n}\mid f(z)\mid^p d\sigma_{n,\alpha}\right)^{\frac{1}{p}}<\infty.$$

It is well known that the spaces $H^p(\mathbf{D}^n)$ and $A^p_\alpha(\mathbf{D}^n)$ are Banach spaces with the above norms,see [18] and [6].
Also, any mapping $\varphi:\mathbf{D}^n\to\mathbf{D}^n$ can be described by n functions $\varphi_1,\varphi_2,\dots,\varphi_n$ as

$$\varphi(z)=(\varphi_1(z),\varphi_2(z),\dots,\varphi_n(z)),$$

where $z=(z_1,z_2,\dots,z_n)\in\mathbf{D}^n$. The mapping φ will be called holomorphic if the n functions $\varphi_1,\varphi_2,\dots,\varphi_n$ are holomorphic functions in $\mathbf{D}^n$.

If R is a rectangle on T^n, then let $S(R)$ denotes the corona associated to R. If $R=I_1\times I_2\times\ldots\times I_n\subset T^n$, where I_j is the interval on T of length δ_j and centre $e^{i(\theta_j^\circ+\delta_j/2)}$ for $j=1,2,\dots,n$, then

$$S(R)=S(I_1)\times S(I_2)\times\ldots S(I_n),$$

where

$$S(I_j)=\{re^{i\theta}\in T:1-\delta_j<r<1 \text{ and } \theta_j^\circ<\theta<\theta_j^\circ+\delta_j\}$$

If V is any open subset in T^n, then $S(V)$ is defined as $S(V)=\cup\{S(R):R$ is a rectangle in $V\}$.

For $a\in\mathbf{D}$, let φ_a be the linear fractional transformation on $\mathbf{D}$ given by

$$\varphi_a(z)=\frac{a-z}{1-\overline{a}\,z}.$$

Then each φ_a is an automorphism on $\mathbf{D}$ and $\varphi_a^{-1}=\varphi_a$. For $\omega=(\omega_1,\omega_2,\dots,\omega_n)\in\mathbf{D}^n$, let $\varphi_\omega(z)=(\varphi_{\omega_1}(z_1),\varphi_{\omega_2}(z_2),\dots,\varphi_{\omega_n}(z_n))$. Then φ_ω is an automorphism on $\mathbf{D}^n$ that exchanges 0 and ω. Since every point evaluation is a bounded linear functional on $A^2_\alpha(\mathbf{D}^n)$. So, for every $a\in\mathbf{D}^n$, there exists a unique function K_a in

$A^2_\alpha(\mathbf{D}^n)$ such that $f(a) =< f, K_a >, f A^2_\alpha(\mathbf{D}^n)$, where $\langle\rangle$ denotes the inner product in $A^2_\alpha(\mathbf{D}^n)$.. The explict formula for the reproducing kernel in $A^2_\alpha(\mathbf{D}^n)$ is given by

$$K^\alpha_a(z) = \prod_{i=1}^{n} \frac{1}{(1-\overline{a_i}\, z_i)^{\alpha+2}}, \quad z, a \in \mathbf{D}^n.$$

Also, using the reproducing property of K^α_a, we have

$$\|K^\alpha_a\|_2^2 =< K^\alpha_a, K^\alpha_a > = K_a(a) = \prod_{i=1}^{n} \frac{1}{(1- \mid a_i \mid^2)^{\alpha+2}}.$$

Thus the normalised reproducing kernel k^α_a is given by

$$\prod_{i=1}^{n} \frac{(1- \mid a_i \mid^2)^{1+\alpha/2}}{(1-\overline{a_i}\, z_i)^{\alpha+2}}.$$

Also, we know that $A^2_{-1}(\mathbf{D}^n) = H^2(\mathbf{D}^n)$, see [12]. Then, for $\alpha = -1$, k^α_a is the normalised reproducing kernel for $H^2(\mathbf{D}^n)$. For an holomorphic self map φ on $\mathbf{D}^n$ and a function $f \in L^1(\mathbf{D}^n)$, the weighted $\varphi-$ Berezin transform of f is defined as

$$B_{\varphi,\alpha}(f)(a) = \int_{\mathbf{D}^n} f(z) \prod_{i=1}^{n} \frac{(1- \mid a_i \mid^2)^{\alpha+2}}{\mid 1-\overline{a_i}\, \varphi_i(z) \mid^{4+2\,\alpha}} d\sigma_{n,\alpha},$$

for $\alpha > -1$ and

$$B_{\varphi,-1}(f)(a) = \int_{T^n} f(z) \prod_{i=1}^{n} \frac{(1- \mid a_i \mid^2)}{\mid 1-\overline{a_i}\, \varphi_i(z) \mid^2} dm_n.$$

If $\varphi_i(z) = z_i$, then $B_{\varphi,\alpha}$ is just the usual Berezin transform.

2. Boundedness. In this section, we characterize the boundedness of $W_{\varphi,\psi}$ using the Carleson measure criterion.

Definition 2.1 Let $1 \le p < \infty$. A finite, non-negative, Borel measure μ on $\overline{\mathbf{D}^n}$ is said to be a p-Carleson measure if $\mu(S(V)) \le C\, m_n(V)^p$ for all connected open sets $V \subset T^n$ and μ is said to be a compact p-Carleson measure if

$$\lim_{m_n(V)\to 0} \sup_{V\subset T^n} \frac{\mu(S(V))}{m_n(V)^p} = 0.$$

Definition 2.2 Let $1 \le p < \infty$. A finite, non-negative, Borel measure μ on $\overline{\mathbf{D}^n}$ is said to be a (p, α) - Carleson measure if $\mu(S(R)) \le C \prod_{i=1}^n \delta_i^{p\,(2+\alpha)}$ for all $R \subset T^n$ and μ is said to be a compact (p, α) - Carleson measure if

$$\lim_{\delta_i\to 0} \sup_{\theta\in T^n} \frac{\mu(S(R))}{\prod_{i=1}^n \delta_i^{p(2+\alpha)}} = 0.$$

Take $\psi \in H^q(\mathbf{D}^n)$. Define the measure ν_n on $\overline{\mathbf{D}^n}$ by

$$\nu_n(E) = \int_{\varphi^{-1}(E)\cap T^n} |\, \psi \,|^q \; d\,m_n,$$

where E is a measurable subset of the closed unit polydisc $\overline{\mathbf{D}^n}$.
Again for $\psi \in A^q_\alpha(\mathbf{D}^n)$, let the measure $\nu_{n,\alpha}$ on $\mathbf{D}^n$, be defined by

$$\nu_{n,\alpha}(E) = \int_{\varphi^{-1}(E)} |\, \psi \,|^q \; d\,\sigma_{n,\alpha},$$

where E is a measurable subset of the unit polydisc $\mathbf{D}^n$.

Using Halmos [10], we can easily prove the following Lemmas.

LEMMA 2.3 *Suppose $\varphi, \psi \in H^q(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$. Then*

$$\int_{\overline{\mathbf{D}^n}} g\, d\nu_n = \int_{T^n} |\, \psi \,|^q \; (g \circ \varphi)\, dm_n,$$

where g is an arbitrary measurable positive function in $\overline{\mathbf{D}^n}$.

LEMMA 2.4 *Suppose $\varphi, \psi \in A^q_\alpha(\mathbf{D}^n)$ and let $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$. Then*

$$\int_{\mathbf{D}^n} g\, d\nu_{n,\alpha} = \int_{\mathbf{D}^n} |\, \psi \,|^q \; (g \circ \varphi)\, d\sigma_{n,\alpha},$$

where g is an arbitrary measurable positive function in $\mathbf{D}^n$.

We will need the following results

THEOREM 2.5 ([11], [12]) *Let μ be a nonnegative, Borel measure on $\overline{\mathbf{D}^n}$. Then the following statements are equivalent:*

(i) *The inclusion map $I : H^p(\mathbf{D}^n) \to L^p(\mathbf{D}^n, \mu)$ is bounded.*

(ii) *The measure μ is p - Carleson measure.*

(iii) *There is a constant $M > 0$ such that, for every $a \in \mathbf{D}^n$,*

$$\sup_{a\in\mathbf{D}^n} \int_{\overline{\mathbf{D}^n}} |\, k_a \,|^p \; d\mu < M < \infty,$$

where k_a is the normalised reproducing kernel function for $H^p(\mathbf{D}^n)$

A similar result holds for the weighted Bergman spaces $A^p_\alpha(\mathbf{D}^n)$.

COROLLARY 2.6 ([12]) *Let μ be a finite, non-negative, Borel measure on $\overline{\mathbf{D}^n}$, and $\beta \geq 1$. Then the following statements are equivalent:*

(*i*) *The identity map* $I : H^p(\mathbf{D}^n) \to L^{p\beta}(\overline{\mathbf{D}^n}, \mu)$ *is bounded.*

(*ii*) *There exists* $C > 0$, *such that the measure* μ *satisfies*

$$\mu(S(V)) \leq C\, m_n(V)^{\beta}.$$

THEOREM 2.7 ([11] THEOREM 2.5) *Take* $1 < p \leq q < \infty, \alpha > -1$ *and let* μ *be a finite positive measure on* $\overline{\mathbf{D}^n}$. *Then the identity map* $I_\alpha : A^p_\alpha(\mathbf{D}^n) \to L^q(\mu)$ *is bounded if and only if* μ *is* $(\frac{q}{p}, \alpha)$ *- Carleson measure, that is, there exists* $C > 0$ *such that*

$$\mu(S(R)) \leq \ C \prod_{i=1}^{n} \delta_i^{q(\alpha+2)/p},$$

for all connected open subsets V *in* T^n.

COROLLARY 2.8 *Let* μ *be a finite, non-negative, Borel measure on* $\overline{\mathbf{D}^n}$. *If there exists* $C > 0$ *and* $\beta > 1$ *such that* $\mu(S(V)) \leq C\, m_n(V)^{\beta}$, *then* $\mu(E) = 0$ *for all measurable subsets* E *of* $\mathbf{D}^n$.

In the following theorem we obtain a lower bound for the norm of weighted composition operator by using the reproducing kernel function .

THEOREM 2.9 *If* $W_{\varphi,\psi}$ *is the weighted composition operator on* $H^p(\mathbf{D^n})$, *then*

$$\sup_{z\in\mathbf{D}^n} \prod_{i=1}^{n} \frac{(1- \mid z_i \mid^2)}{(1- \mid \varphi_i(z) \mid^2)} \mid \psi(z) \mid \leq \|W_{\varphi,\psi}\|_p^p$$

PROOF For the sake of convenience, we shall prove the theorem for the case $p = 2$. For $z \in \mathbf{D}^n$, let K_z be the reproducing kernel in $H^2(\mathbf{D}^n)$. Also, we have

$$\|K_z\|_2^2 = < K_z, K_z > = \ K_z(z) = \prod_{i=1}^{n} \frac{1}{1- \mid z_i \mid^2}.$$

Again, we have

$$\|(W_{\varphi,\psi})^*(K_z)\|_2^2 = \|\overline{\psi(z)}\ K_{\varphi(z)}\|_2^2 = \prod_{i=1}^{n} \frac{\mid \psi(z) \mid^2}{1- \mid \varphi_i(z) \mid^2}.$$

Further,

$$\begin{aligned} \|(W_{\varphi,\psi})^*(K_z)\|_2^2 &= \mid \psi(z) \mid \|K_{\varphi(z)}\|^2 \\ &= \mid \psi(z) \mid K_{\varphi(z)}(\varphi(z)) \\ &= \mid \psi(z) \mid \ (K_{\varphi(z)} \circ \varphi)(z) \\ &\leq \mid (\psi C_\varphi K_{\varphi(z)})(z) \mid \\ &\leq \|W_{\varphi,\psi}(K_{\varphi(z)})\|_2 \ (\prod_{i=1}^{n} \frac{1}{1- \mid z_i \mid^2})^{1/2}. \end{aligned}$$

In the above expression the last inequality holds, since $\mid f(z) \mid \leq \|f\|_2 \, (\prod_{i=1}^n \frac{1}{1-|z_i|^2})^{1/2}$, for every $z \in \mathbf{D}^n$

Hence for every $z \in \mathbf{D}^n$, we have

$$\begin{aligned} \|W_{\varphi,\psi}\|_2^2 &\geq \|(W_{\varphi,\psi})^*(K_z)\|_2^2 \prod_{i=1}^n (1- \mid z_i \mid^2) \\ &= \prod_{i=1}^n \frac{1- \mid z_i \mid^2}{1- \mid \varphi_i(z) \mid^2} \mid \psi(z) \mid . \end{aligned}$$

Taking supremum over all $z \in \mathbf{D}^n$, we have

$$\|W_{\varphi,\psi}\|_2^2 \geq \sup_{z \in \mathbf{D}^n} \prod_{i=1}^n \frac{(1- \mid z_i \mid^2)}{(1- \mid \varphi_i(z) \mid^2)} \mid \psi(z) \mid .$$

■

COROLLARY 2.10 *For* $1 < p < \infty, W_{\varphi,\psi}$ *induces an unbounded weighted composition operator on* $H^p(\mathbf{D}^n)$ *if*

$$\sup_{z \in \mathbf{D}^n} \prod_{i=1}^n \frac{(1- \mid z_i \mid^2)}{(1- \mid \varphi_i(z) \mid^2)} \mid \psi(z) \mid = \infty.$$

THEOREM 2.11 *Let* $1 < p \leq q < \infty$. *If* $\varphi \in H^p(\mathbf{D}^n)$ *is such that* $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$ *and* $\psi \in H^q(\mathbf{D}^n)$, *then* $W_{\varphi,\psi} : H^p(\mathbf{D}^n) \to H^q(\mathbf{D}^n)$ *is bounded if and only if* ν_n *is* $\frac{q}{p}$ *- Carleson measure, that is there exists* $C > 0$ *such that*

$$\nu_n(S(V)) \leq C m_n(V)^{q/p}.$$

PROOF By Corollary 2.4, the measure ν_n is $\frac{q}{p}$ - Carleson if and only if there is a constant $M > 0$ such that

$$\int_{\overline{\mathbf{D}^n}} \mid f \mid^q \, d\nu_n \leq M\|f\|_p^q \text{ for all } f \in H^p(\mathbf{D}^n)$$

Again by using Lemma 2.3 with $g = \mid f \mid^q$, we have

$$\int_{\overline{\mathbf{D}^n}} \mid f \mid^q \, d\nu_n = \int_{T^n} \mid \psi \mid^q \mid f \circ \varphi \mid^q \, dm_n = \|W_{\varphi,\psi}(f)\|_p^q.$$

Hence ν_n is $\frac{q}{p}$ - Carleson measure if and only if there is a constant $M > 0$ such that

$$\|W_{\varphi,\psi}(f)\|_q \leq M^{1/q}\|f\|_q, \text{ for all } f \in H^p(\mathbf{D}^n).$$

■

Now, we give a characterization for the boundedness of $W_{\varphi,\psi}$ on $H^p(\mathbf{D}^n)$ by using the weighted φ - Berezin transform of the function $|\psi|^p$.

THEOREM 2.12 *Suppose $\varphi, \psi \in H^p(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$. Then the weighted composition operator $W_{\varphi,\psi}$ is bounded on $H^p(\mathbf{D}^n)$ if and only if the weighted φ - Berezin transform of the function $|\psi|^p$ is bounded, that is, $B_{\varphi,-1}(|\psi|) \in L^\infty(\mathbf{D}^n)$.*

PROOF The proof follows by using Theorem 2.3 and Theorem 2.9. ■

THEOREM 2.13 *Let $1 < p \leq q < \infty$, and $\varphi \in A^p_\alpha(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$. If $\psi \in A^q_\alpha(\mathbf{D}^n)$, then $W_{\varphi,\psi} : A^p_\alpha(\mathbf{D}^n) \to A^q_\alpha(\mathbf{D}^n)$ is bounded if and only if $\nu_{n,\alpha}$ is $(\frac{q}{p}, \alpha))$ - Carleson measure,that is, there exists $C > 0$ such that*

$$\nu_{n,\alpha}(S(R)) \leq C \prod_{i=1}^{n} \delta_i^{q(\alpha+2)/p}.$$

PROOF Using Theorem 2.5, $\nu_{n,\alpha}$ is $(\frac{q}{p}, \alpha)$ - Carleson measure if and only if there is a constant $C > 0$ such that

$$\int_{\overline{\mathbf{D}^n}} |f|^q \, d\nu_n \leq C\|f\|_p^q \text{ for all } f \in A^p_\alpha(\mathbf{D}^n).$$

Again by Lemma 2.4 with $g = |f|^q$, we have

$$\int_{\overline{\mathbf{D}^n}} |f|^q \, d\nu_n = \int_{\mathbf{D}^n} |\psi|^q |f \circ \varphi|^q \, d\sigma_{n,\alpha} = \|W_{\varphi,\psi}(f)\|_p^q.$$

Hence $\nu_{n,\alpha}$ is $(\frac{q}{p}, \alpha)$ - Carleson measure if and only if there is a constant $C > 0$ such that

$$\|W_{\varphi,\psi}(f)\|_q \leq C^{1/q}\|f\|_p, \text{ for all } f \in A^p_\alpha(\mathbf{D}^n).$$

■

COROLLARY 2.14 *Let $1 < p < q < \infty, \varphi \in H^p(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$ and $\psi \in H^q(\mathbf{D}^n)$. If the operator $W_{\varphi,\psi} : H^p(\mathbf{D}^n) \to H^q(\mathbf{D}^n)$ is bounded, then*

$$m_n(\{z = (z_1, z_2, \ldots, z_n) \in T^n : |\varphi_j(z)|_{j=1,\ldots,n} = 1\}) = 0.$$

PROOF By Theorem 2.9, the measure ν_n satisfies $\nu_n(S(V)) \leq Cm_n(V)^{q/p}$ in $\overline{\mathbf{D}^n}$. Since $\frac{q}{p} > 1$, by Corollary 2.6, the measure $\nu_n|T^n = 0$. Take

$$E = \{z = (z_1, z_2, \ldots, z_n) \in T^n : |\varphi_j(z)|_{j=1,\ldots,n} = 1\}.$$

Then

$$0 = \nu_n(\varphi(E)) = \int_{\varphi^{-1}(\varphi(E)) \cap T^n} |\psi|^q \, dm_n \geq \int_{E \cap T^n} |\psi|^q \, dm_n = \int_E |\psi|^q \, dm_n.$$

Thus $|\psi| = 0$ almost everywhere on E. Since $\psi \in H_1$,we have $m_n(E) = 0$. ■

THEOREM 2.15 *Let $\varphi \in H^p(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$. Then*

(*i*) *Suppose $1 \le q \le \infty$ and $\psi \in H(\mathbf{D}^n)$. Then $W_{\varphi,\psi} : H^\infty(\mathbf{D}^n) \to H^q(\mathbf{D}^n)$ is bounded if and only if $\psi \in H^q(\mathbf{D}^n)$.*

(*ii*) *For $1 \le p < \infty$ and $\psi \in H^p(\mathbf{D}^n)$, we have $W_{\varphi,\psi} : H^p(\mathbf{D}^n) \to H^\infty(\mathbf{D}^n)$ is bounded if and only if*

$$\sup_{z\in\mathbf{D}^n} \prod_{i=1}^{n} \frac{|\psi(z)|}{(1-|\varphi_i(z)|^2)^{1-1/p}} < \infty.$$

PROOF Proof (*i*) is obvious .
(*ii*) For any $z \in \mathbf{D}^n$, the reproducing kernel K_z can be considered as an element of the dual of $H^p(\mathbf{D}^n)$, given by $< K_z, f > = f(z)$. If $W_{\varphi,\psi} : H^p(\mathbf{D}^n) \to H^\infty(\mathbf{D}^n)$ is bounded, then there exists a constant $M > 0$ such that $\|(W_{\varphi,\psi})^* K_z\|_{(H^p)^*} \le M \|K_z\|_{(H^\infty)^*} = M$, for all $z \in \mathbf{D}^n$. Again, $W^*_{\varphi,\psi}(K_z) = \overline{\psi(z)} K_{\varphi(z)}$, so we have

$$\|(W_{\varphi,\psi})^* K_z\|_{(H^p)^*} = \|\overline{\psi(z)} K_{\varphi(z)}\|_{(H^p)^*} = \prod_{i=1}^{n} \frac{|\psi(z)|}{(1-|\varphi_i(z)|^2)^{1-1/p}} \le M, \text{for all } z \in \mathbf{D}^n.$$

Conversely, suppose $M = \sup_{z\in\mathbf{D}^n} \prod_{i=1}^{n} \frac{|\psi(z)|^p}{(1-|\varphi_i(z)|^2)} < \infty$. If $f \in H^p(\mathbf{D}^n)$, then

$$\begin{aligned} |W_{\varphi,\psi}(f)(z)| &= |\psi(z) f(\varphi(z))| \\ &\le |\psi(z)| \, \|K_{\varphi(z)}\|_{(H^p)^*} \|f\|_p \\ &= \prod_{i=1}^{n} \frac{|\psi(z)|}{(1-|\varphi_i(z)|^2)^{1-1/p}} \|f\|_p \\ &\le M^{1/p} \|f\|_p. \end{aligned}$$

■

In the following theorem we give a necessary condition for the boundedness of $W_{\varphi,\psi}$ on $H^p(\mathbf{D}^n)$.

THEOREM 2.16 *Suppose $\varphi, \psi \in H^p(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$. If the weighted composition operator $W_{\varphi,\psi}$ is bounded on $H^p(\mathbf{D}^n)$, then*

$$\sup_{z\in\mathbf{D}^n} \prod_{i=1}^{n} \frac{(1-|z_i|^2)^{1/p}}{(1-|\varphi_i(z)|^2)^{1/p}} |\psi(z)| < \infty.$$

We can easily prove the following characterization for the closed range of $W_{\varphi,\psi}$.

THEOREM 2.17 *Take $1 < p \le q < \infty$ Let $\varphi \in H^p(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$ and $\psi \in H^q(\mathbf{D}^n)$. Suppose $W_{\varphi,\psi} : H^p(\mathbf{D}^n) \to H^q(\mathbf{D}^n)$ is bounded. Then $W_{\varphi,\psi}$ has closed range if and only if there exists a constant $M > 0$ such that*

$$\int_{\overline{\mathbf{D}^n}} |f|^q \, d\nu_n \ge M \|f\|_p^q \text{ for all } f \in H^p(\mathbf{D}^n)$$

The following theorem shows that the question of studying the composition operators between $H^2(\mathbf{C}_+ \times \mathbf{C}_+)$ can be reduced to study the weighted composition operators between $H^2(\mathbf{D}^2)$.

Take $W = H^2(\mathbf{C}_+ \times \mathbf{C}_+) \to H^2(\mathbf{D} \times \mathbf{D})$ given by

$$(Wf)(z,w) = \pi(\frac{2i}{1-z})(\frac{2i}{1-w})f(\tau_1(z), \tau_1(w)),$$

where $\tau_1(z) = i\frac{1+z}{1-z}$ and

$$(W^{-1}g)(z,w) = \pi^{-1}(\frac{1}{i+z})(\frac{1}{i+w})f(\tau_2(z), \tau_2(w)),$$

where $\tau_2(z) = \frac{z-i}{z+i}$. Then clearly W and W^{-1} are isometries [8].

THEOREM 2.18 *Let $\Psi : \mathbf{C}_+ \times \mathbf{C}_+ \to \mathbf{C}_+ \times \mathbf{C}_+$ be holomorphic mapping. Then $C_\Psi : H^2(\mathbf{C}_+ \times \mathbf{C}_+) \to H^2(\mathbf{C}_+ \times \mathbf{C}_+)$ is unitarily equivalent to the operator L_Φ defined by*

$$L_\Phi = (\frac{i+\Phi_1(z)}{i+z})(\frac{i+\Phi_2(w)}{i+w})f \circ \Phi(z,w).$$

where $\Phi(z,w) = (\Phi_1(z), \Phi_2(w)), \Phi_1(z) = \tau_1 \circ \Psi_1 \circ \tau_2(z)$ and $\Phi_2(w) = \tau_1 \circ \Psi_2 \circ \tau_2(w)$.

PROOF We have

$$\begin{aligned}
&(W \circ C_\Psi \circ W^{-1}f)(z,w) \\
&= \pi^{-1}(\frac{1}{i+z})(\frac{1}{i+w})(C_\Psi \circ Wf)(\tau_2(z), \tau_2(w)) \\
&= \pi^{-1}(\frac{1}{i+z})(\frac{1}{i+w})(Wf)(\Psi(\tau_2(z), \tau_2(w))) \\
&= \pi^{-1}(\frac{1}{i+z})(\frac{1}{i+w})(Wf)(\Psi_1 \circ \tau_2(z), \Psi_2 \circ \tau_2(w)) \\
&= \pi^{-1}(\frac{1}{i+z})(\frac{1}{i+w})(\frac{2i}{1-\Psi_1 \circ \tau_2(w)})(\frac{2i}{1-\Psi_2 \circ \tau_2(w)}) \\
&\quad f(\tau_1 \circ \Psi_1 \circ \tau_2(z), \tau_1 \circ \Psi_2 \circ \tau_2(w)).
\end{aligned}$$

Hence

$$L_\Phi = (\frac{i+\Phi_1(z)}{i+z})(\frac{i+\Phi_2(w)}{i+w})f \circ \Phi(z,w).$$

■

For more examples of weighted composition operators, see [14] and [21].

3. Compactness. The following lemma is easy to establish.

LEMMA 3.1 *Let $1 \leq p, q \leq \infty$ and $W_{\varphi,\psi} : H^p(\mathbf{D}^n) \to H^q(\mathbf{D}^n)$ be bounded. Then $W_{\varphi,\psi}$ is compact if and only if whenever $\{f_n\}$ is a bounded sequence in $H^p(\mathbf{D}^n)$ converging to zero uniformly on compact subsets of $\mathbf{D}^n$, we have $\|W_{\varphi,\psi}(f_n)\|_q \to 0$.*

The above lemma is also true for the weighted Bergman spaces

THEOREM 3.2 *[11], [12] Take $1 < p < \infty$. Let μ be a nonnegative, Borel measure on $\overline{\mathbf{D}^n}$. Then the following statements are equivalent:*

(i) *The inclusion map $I : H^p(\mathbf{D}^n) \to L^p(\mathbf{D}^n, \mu)$ is compact.*

(ii) *The measure μ is a compact p - Carleson measure.*

(iii) *For every $a \in \mathbf{D}^n$, we have*

$$\lim_{\|a\|\to 1} \int_{\overline{\mathbf{D}^n}} | k_a |^p \, d\mu = 0.$$

A similar result holds for the weighted Bergman spaces $A^p_\alpha(\mathbf{D}^n)$.

THEOREM 3.3 *Let $\varphi \in H^p(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$, and $\psi \in H^\infty(\mathbf{D}^n)$, and $1 \leq p < \infty$. Then $W_{\varphi,\psi} : H^p(\mathbf{D}^n) \to H^\infty(\mathbf{D}^n)$ is compact if and only if either $\|\varphi\|_\infty < 1$ or*

$$\lim_{|\varphi_i(z)|\to 1} \prod_{i=1}^{n} \frac{| \psi(z) |^p}{(1- | \varphi_i(z) |^2)} = 0.$$

PROOF Suppose the operator $W_{\varphi,\psi}$ is compact, $\|\varphi\|_\infty < 1$ and

$$\lim_{|\varphi_i(z)|\to 1} \prod_{i=1}^{n} \frac{| \psi(z) |^p}{(1- | \varphi_i(z) |^2)} \neq 0.$$

So we can find a sequence $\{z_n\}$ in $\mathbf{D}^n$ and $\epsilon > 0$ such that $| \varphi_i(z_n) | \to 1$ and $\epsilon \prod_{i=1}^{n}(1- | \varphi_i(z_n) |^2) \leq | \psi(z_n) |^p$ for all natural number n. Let us define a function f_n in $H^p(\mathbf{D}^n)$ by

$$f_n(z) = \prod_{i=1}^{n} \frac{(1- | \varphi_i(z_n) |^2)^{1/p}}{(1 - \overline{\varphi_i(z_n)} z)^{2/p}}.$$

Then f_n is a bounded sequence in $H^p(\mathbf{D}^n)$ and tends to 0 uniformly on compact subsets of $\mathbf{D}^n$. So, by Lemma 3.1 ,$\|W_{\varphi,\psi}(f_n)\|_\infty \to 0$. Also, we have that

$$\|W_{\varphi,\psi}(f_n)\|_\infty \geq | \psi(z_n) || f_n(\varphi_i(z_n)) | = \prod_{i=1}^{n} | \psi(z_n) | (1- | \varphi_i(z_n) |^2)^{-1/p} \geq \epsilon^{1/p},$$

for all n, a contraciction.
Conversely, suppose that

$$\lim_{|\varphi_i(z)|\to 1} \prod_{i=1}^{n} \frac{| \psi(z) |^p}{(1- | \varphi_i(z) |^2)} = 0.$$

Let $\{f_n\}$ be a bounded sequence in $H^p(\mathbf{D}^n)$ that tends to 0 uniformly on compact subsets of $\mathbf{D}^n$. Also , take $C = \sup_n \|f_n\|_p$ and $\epsilon > 0$. Again by hypothesis, we can find $r_0 < 1$ such that if $\mid \varphi_i(z) \mid > r_0$, then $\mid \varphi_i(z) \mid^p \leq (\frac{\epsilon}{2C})^P \prod_{i=1}^n (1- \mid \varphi_i(z) \mid^2)$. Also, there is a natural number $n_\circ$ such that if $n \geq n_\circ$, then

$$\sup_{|z_i| \leq r_\circ} \mid f_n(z) \mid \leq \frac{\epsilon}{2\|\psi\|_\infty}.$$

Then for $n \geq n_\circ$, we have

$$\begin{aligned} \|W_{\varphi,\psi}(f_n)\|_\infty &\leq \sup_{|\varphi_i z)| \leq r_\circ} \mid \psi(z) f_n(\varphi_i(z)) \mid + \sup_{|\varphi_i(z)| > r_\circ} \mid \psi(z) f_n(\varphi_i(z)) \mid \\ &\leq \|\psi\|_\infty \sup_{|w_i| \leq r_\circ} \mid f_n(w) \mid + \frac{\epsilon}{2C} \sup_{|\varphi_i(z)| > r_\circ} \prod_{i=1}^n \mid f_n(\varphi_i(z)) \mid \\ &\quad (1- \mid \varphi_i(z) \mid^2)^{1/p} \\ &\leq \|\psi\|_\infty \frac{\epsilon}{2\|\psi\|_\infty} + \frac{\epsilon}{2C} \|f_n\|_p \leq \epsilon. \end{aligned}$$

If we assume that $\|\varphi\|_\infty < 1$ the compactness of $W_{\varphi,\psi}$ follows from the fact that $\overline{\varphi(\mathbf{D}^n)}$ is a compact subset of $\overline{\mathbf{D}^n}$ and $\psi \in H^\infty(\mathbf{D}^n)$. ∎

In the following theorem we give a characterization for the compactness of $W_{\varphi,\psi}$ on $A^p_\alpha(\mathbf{D}^n)$

Theorem 3.4 *Suppose $\varphi : \mathbf{D}^n \to \mathbf{D}^n$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$ and ψ is bounded. Then the weighted composition operator $W_{\varphi,\psi}$ is compact on $A^2_\alpha(\mathbf{D}^n)$, if and only if*

$$\lim_{\|z\| \to 1} \prod_{i=1}^n \frac{(1- \mid z_i \mid^2)}{(1- \mid \varphi_i(z) \mid^2)} \mid \psi(z) \mid^2 = 0. \tag{1}$$

Proof The proof follows on the similar lines as in [16, Corollary 1, page-75]. ∎

Theorem 3.5 *Let $1 \leq q < \infty$, $\varphi \in H^p(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$ and $\psi \in H^q(\mathbf{D}^n)$. Then the operator $W_{\varphi,\psi} : H^\infty(\mathbf{D}^n) \to H^q(\mathbf{D}^n)$ is compact if and only if $m_n(\{z \in T^n :\mid \varphi_i(z) \mid = 1,$ for $i = 1, 2, \ldots, n\}) = 0$.*

Proof Suppose that $m_n(\{z \in T^n :\mid \varphi_i(z) \mid = 1,$ for $i = 1, 2, \ldots, n\}) = 0$. and take a bounded sequence f_n in H^∞ that converges to zero uniformly on compact subsets of $\mathbf{D}^n$. Fix $z \in T^n$ such that $\mid \varphi_i(z) \mid < 1$. Then $f_n(\varphi_i(z)) \to 0$ and so $\psi(z) f_n(\varphi_i(z)) \to 0$ almost everywhere in T^n. Moreover,

$$\mid \psi(z) f_n(\varphi_i(z)) \mid^q \leq \mid \psi(z) \mid^q \|f_n\|^q_\infty \leq C^q \mid \psi(z) \mid^q,$$

where $C = \sup \|f_n\|_\infty$. Since $\mid \psi \mid^q \in L_1(\mathbf{D}^n, m_n)$, we have that

$$\lim_{n \to \infty} \int_{T^n} \mid \psi f_n \circ \varphi \mid^q \, dm_n = 0.$$

∎

Lemma 3.6 *Let $0 < r < 1$ and let $\mathbf{D}_r^n = \{z = (z_1, z_2, \ldots, z_n) \in \mathbf{D}^n : \mid z_i \mid < r, i = 1, 2, \ldots, n\}$. Supppose μ is a positive Borel measure on $\mathbf{D}^n$ and let μ_r denotes the restriction of measure μ to the set $\mathbf{D}_r^n$. Take*

$$\|\mu\| = \sup_{R \subset T^n} \frac{\mu(S(R))}{\prod_{i=1}^n \delta_i^{p(2+\alpha)}} \text{ and } \|\mu\|_r = \sup_{\prod_{i=1}^n \delta_i \leq 1-r} \frac{\mu(S(R))}{\prod_{i=1}^n \delta_i^{p(2+\alpha)}}.$$

Then if μ is Carleson measure for the weighted Bergman space $A_\alpha^p(\mathbf{D}^n)$, so is μ_r. Moreover $\|\mu_r\| \leq K \|\mu\|_r$, where $K > 0$ is a constant.

Lemma 3.7 *Let $0 < r < 1$. Take*

$$\|\mu\|_r^* = \sup_{\|a\| \geq r} \int_{\mathbf{D}^n} \mid k_a^\alpha(z) \mid^p d\mu(z).$$

If μ is a Carleson measure for the weighted Bergman space $A_\alpha^p(\mathbf{D}^n)$, then $\|\mu_r\| \leq M \|\mu\|_r^*$, where M is an absolute constant.

Any holomorphic function f on $\mathbf{D}^n$ has a power series expansion $f(z) = \sum C(\alpha) z^\alpha$, where the sum is over all multi-indices $\alpha = (\alpha_1, \alpha_2 \ldots, \alpha_n)$ of non-negative integers and z^α denotes the monomial $z_1^{\alpha_1}, z_2^{\alpha_2}, \ldots, z_n^{\alpha_n}$. The series converges absolutely and uniformly in every compact subset of the plydisk $\mathbf{D}^n$.

For $s = 0, 1, \ldots,$ let $F_s(z)$ be the sum of those terms $C(\alpha) z^\alpha$ for which $\mid \alpha \mid = \alpha_1 + \alpha_2 + \ldots \alpha_n = s$. Then F_s is a polynomial which is homogeneous of degree s.Thus we can write $f(z) = \sum_{s=0}^\infty F_s(z)$, where F_s is the homogeneous polynomial $\sum_{|\alpha|=s} C(\alpha) z^\alpha$.

For a positive integer n, define the operators $R_n(f) = R_n(\sum_{s=0}^\infty F_s) = \sum_{s=n+1}^\infty F_s$ and $Q_n = I - R_n$ acting from $A_\alpha^2(\mathbf{D}^n)$ to $A_\alpha^2(\mathbf{D}^n)$, where I is the identity map.

Recall that the essential norm of an operator T is defined as :

$$\|T\|_e = \inf\{\|T - K\|, \text{ where K is a compact operator}\}.$$

Now we have the following lemma.

Lemma 3.8 *Suppose $W_{\varphi,\psi}$ is bounded on $A_\alpha^2(\mathbf{D}^n)$. Then*

$$\|W_{\varphi,\psi}\|_e = \lim_{n \to \infty} \|W_{\varphi,\psi} R_n\|_2.$$

The proof is similar to the proof of Lemma given in [6, page–134].

In the following theorem we give the upper and lower estimates for the essential norm of a weighted composition operator.

Theorem 3.9 *Let φ be a holomorphic self–map of $\mathbf{D}^n$ and $\psi \in A^2_\alpha(\mathbf{D}^n)$. Suppose $W_{\varphi,\psi}$ is bounded on $A^2_\alpha(\mathbf{D}^n)$. Then there is an absolute constant $M \geq 1$ such that*

$$\limsup_{\|a\|\to 1} B_{\varphi,\alpha}(\mid \psi \mid^2)(a) \leq \|W_{\varphi,\psi}\|_e^2 \leq \; M \; \limsup_{\|a\|\to 1} B_{\varphi,\alpha}(\mid \psi \mid^2)(a)$$

Proof First we find the upper estimate.
Upper estimate: By Lemma 3.8, we have

$$\|W_{\varphi,\psi}\|_e^2 = \lim_{n\to\infty} \|W_{\varphi,\psi} R_n\|_2 = \lim_{n\to\infty} \sup_{\|f\|_2 \leq 1} \|(W_{\varphi,\psi} R_n) f\|_2.$$

Also, by using Lemma 2.1, we have

$$\begin{aligned}
\|(W_{\varphi,\psi} R_n) f\|_2^2 &= \int_{\mathbf{D}^n} \mid \psi(z) \mid^2 \mid (R_n f)(\varphi(z)) \mid^2 d\sigma_{n,\alpha}. \\
&= \int_{\mathbf{D}^n} \mid (R_n f)(\omega) \mid^2 \; d\nu_{n,\alpha}(\omega). \\
&= \int_{\mathbf{D}^n \setminus \mathbf{D}^n_r} \mid (R_n f)(\omega) \mid^2 d\nu_{n,\alpha}(\omega) + \int_{\mathbf{D}^n_r} \mid (R_n f)(\omega) \mid^2 d\nu_{n,\alpha}(\omega). \\
&= I_1 + I_2. \qquad (2)
\end{aligned}$$

Also, the measure $\nu_{n,\alpha}$ is a Carleson measure, because the operator $W_{\varphi,\psi}$ is bounded on $A^p_\alpha(\mathbf{D}^n)$. Again by using [6, page–133], we can show that, for a fixed r,

$$\sup_{\|f\|_2 \leq 1} \int_{\mathbf{D}^n_r} \mid (R_n f)(\omega) \mid^2 d\nu_{n,\alpha}(\omega) \to 0 \text{ as } n \to \infty.$$

Let $\nu_{n,\alpha,r}$ denotes the restriction of measure $\nu_{n,\alpha}$ to the set $\mathbf{D}^n \setminus \mathbf{D}^n_r$. So by using Lemma 3.7 and Theorem 3.2, we have

$$\begin{aligned}
\int_{\mathbf{D}^n \setminus \mathbf{D}^n_r} \mid (R_n f)(\omega) \mid^2 d\nu_{n,\alpha,r}(\omega) &\leq K \; \|\nu_{n,\alpha,r}\| \|(R_n f)\|_2^2 \leq K \; M \|\nu_{n,\alpha,r}\|_r^* \|f\|_2^2 \\
&\leq K \; M \; \|\nu_{n,\alpha,r}\|_r^*,
\end{aligned}$$

where K and M are absolute constants and $\|\nu_{n,\alpha,r}\|_r^*$ is defined as in Lemma 3.7

Therefore,

$$\lim_{n\to\infty} \sup_{\|f\|_2 \leq 1} \|(W_{\varphi,\psi} R_n) f\|_2 \leq \lim_{n\to\infty} K \; M \; \|\nu_{n,\alpha,r}\|_r^* = K \; M \; \|\nu_{n,\alpha,r}\|_r^*$$

Thus, $\|W_{\varphi,\psi}\|_e^2 \leq K \; M \; \|\nu_{n,\alpha,r}\|_r^*$. Taking $r \to 1$, we have

$$\begin{aligned}
\|W_{\varphi,\psi}\|_e^2 &\leq K \; M \; \lim_{r\to 1} \|\nu_{n,\alpha,r}\|_r^*. \\
&= K \; M \; \limsup_{\|a\|\to 1} \int_{\mathbf{D}^n} \mid k_a^\alpha(\omega) \mid^2 \; d\nu_{n,\alpha}(\omega) \\
&= K \; M \; \limsup_{\|a\|\to 1} B_{\varphi,\alpha}(\mid \psi \mid^2)(a)
\end{aligned}$$

which is the desired upper bound.

Lower estimate : We know that $\|k_a^\alpha\| = 1$ and $k_a^\alpha \to 0$ uniformly on compact subsets of $\mathbf{D}^n$ as $\|a\| \to 1$. Also, for a compact operator K on $A^2_\alpha(\mathbf{D}^n)$, $\|Kk_a^\alpha\|_2 \to 0$ as $\|a\| \to 1$. Therefore,

$$\begin{aligned}\|W_{\varphi,\psi} - K\|_2 &\geq \limsup_{\|a\|\to 1} (\|(W_{\varphi,\psi} - K)k_a^\alpha\|_2) \\ &\geq \limsup_{\|a\|\to 1} (\|(W_{\varphi,\psi})k_a^\alpha\|_2 - \|K\, k_a^\alpha\|_2) \\ &= \limsup_{\|a\|\to 1} \|(W_{\varphi,\psi})k_a^\alpha\|_2.\end{aligned}$$

Thus,

$$\|W_{\varphi,\psi}\|_e^2 \geq \|W_{\varphi,\psi} - K\|_2^2 \geq \limsup_{\|a\|\to 1} \|(W_{\varphi,\psi})k_a^\alpha\|_2^2 = \limsup_{\|a\|\to 1} B_{\varphi,\alpha}(|\,\psi\,|^2)(a).$$

and hence the proof. ∎

Corollary 3.10 *Suppose $\varphi, \psi \in A^2_\alpha(\mathbf{D}^n)$ be such that $\varphi(\mathbf{D}^n) \subseteq \mathbf{D}^n$. Then the weighted composition operator $W_{\varphi,\psi}$ is compact on $A^2_\alpha(\mathbf{D}^n)$ if and only if the the weighted φ - Berezin transform of the function $|\,\psi\,|^2$ tends to zero as $\|a\| \to 1$, that is,*

$$\lim_{\|a\|\to 1} B_{\varphi,\alpha}(|\,\psi\,|^2)(a) = 0.$$

Remark 3.11 All the results in the paper which we have proved for the weighted Bergman spaces are also true for Hardy spaces and vice-versa. Also all the results in this paper after slight modifications can also be proved for Hardy spaces and some weighted Bergman spaces of the unit ball in $\mathbf{C}^n$.

References

[1] P. S. Bourdon and J. H. Shapiro, *Cyclic phenomena for composition operators,* Memoirs of the A.M.S. No.516, Amer. Math. Soc. Providence. RI 1997.

[2] I. Chalender and J. R. Partington,*On structure of invariant subspaces for isometric composition operators on* $H^2(D)$ and $H^2(C_+)$, Archiv der Mathematik **81** (2003), 193–207.

[3] B. R. Chao, Y. J. Lee, K. Nam and D. Zheng, *Products of Bergman space Toeplitz operators on the polydisk,* (preprint).

[4] M. D. Contreras and A. G. Hernandez-Diaz,*Weighted composition operator on Hardy spaces,* J. Math. Anal. Appl., **263** (2001), 224–233.

[5] M. D. Contreras and A. G. Hernandez-Diaz, *Weighted composition operator between different Hardy spaces*, Integral Equation and Operator Theory, **46** (2003), 105–188.

[6] C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions,* CRC Press, Boca Raton FL, 1995.

[7] Z. Cuckovic and R. Zhao, *Weighted composition operators on the Bergman spaces,* J. London Math. Soc., (2) **70** (2004), 499–511.

[8] S. H. Ferguson and M. T. Lacey, *A characterization of product BMO by commutantors,* Acta Math., **189** (2002), 143–160.

[9] R. J. Fleming and J. E. Jamison, *Isometries on Banach Spaces*, CRC, Boca Raton, New York, 2003.

[10] P. R. Halmos, *Measure Theory,* Graduate Texts in Mathematics, **18** Springer–Verlag, New York, 1974.

[11] F. Jafari, *On bounded and compact composition operators in polydiscs,* Can. J. Math., **XLII** No.5 (1990), 869–889.

[12] F. Jafari, *Carleson measure in Hardy and weighted Bergman spaces of polydiscs,* Proc. Amer. Math. Soc., **112** No. 3 (1991), 771–781.

[13] H. Kamowitz, *Compact operators of the form* uC_φ, Pacific J. Math., **80** (1979), 205–211.

[14] R. Kumar and J. R. Partington, *Weighted composition operartors on Hardy and Bergman spaces,* J. Operator Theory: Advances and Applications, Birkhauser Verlag Basel, **153**, (2004), 157–167.

[15] V. Matache , *Composition operators on Hardy spaces of a half plane* , Proc. Amer Math. Soc., **127** (1999),1483 -1491.

[16] J. Moorhouse, *Compact differences of composition operators,* J. Funct. Anal. **219** (2005), 70–92.

[17] S. Ohno and H. Takagi , *Some properties of weighted composition operators on algebra of analytic functions* , J. Nonlinear Convex Anal., **2** (2001), 369 -380.

[18] W. Rudin, *Function Theory in The Unit Ball of* $\mathbf{C}^n$, Springer-Verlag, New-York, 1980.

[19] R. B. Schneider, *Isometries of* $H^p(\mathbf{U}^n)$, Can. J. Math., **XXV** No.1 (1973), 92–95.

[20] J. H. Shapiro, *Composition operators and classical function theory* , Springer-Verlag, New York 1993.

[21] J. H. Shapiro and W. Smith, *Hardy spaces that support no compact composition operators,* J. Funct. Anal. **205**, No.1, (2003), 62–89.

[22] W. Smith, *Composition operators between Bergman and Hardy Spaces*,Trans. Amer. Math. Soc., **348**, No.6 (1996).

[23] R. K. Singh and S. D. Sharma *Composition operators in several complex variables,* Bull. Austral. Math. Soc., **23** (1981), 237-247.

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