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Local uniform rotundity in Musielak-Orlicz sequence space equipped with the Luxemburg norm

Abstract. In this paper, we present criteria for local uniform rotundity and weak local uniform rotundity in Musielak-Orlicz sequence spaces equipped with the Luxemburg norm.

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Key words and phrases: local uniform rotundity, weak local uniform rotundity, Musielak-Orlicz sequence space, Luxemburg norm.

1. Introduction. It is well known that among the many kinds rotundities of Banach space, local uniform rotundity is the most important one. One reason is that this kind of rotundity ensures the fixed point property. Criteria for locally uniform rotundity of Orlicz space have been obtained in [2] and [15], locally uniform rotundity of Musielak-Orlicz function space was discussed also and the result and the proof are similar to those of Orlicz space (see [16]). Because of the complicated structure of Musielak-Orlicz sequence spaces, although the criteria for rotundity and uniform rotundity were obtained by A. Kamińska in [14] and [17], criterion for locally uniform rotundity has not been found. In this paper, we will give criteria for local uniform rotundity and weak local uniform rotundity of Musielak-Orlicz sequence spaces equipped with the Luxemburg norm.

A Banach space $(X, \|\cdot\|)$ is called rotund ($X \in \mathbf{R}$), if $x, y \in X$, $\|x\| = \|y\| = 1$ and $\|x+y\| = 2$ imply $x = y$.

A Banach space X is called an uniformly rotund ($X \in \mathbf{UR}$), if for any two sequences x_n and y_n in X, the conditions $\|x_n\| = 1$ and $\|y_n\| = 1$ for any $n \in N$ and $\|x_n + y_n\| \to 2$ imply $\|x_n - y_n\| \to 0$.

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A Banach space X is called (weakly) locally uniformly rotund ($X \in$ **LUR**, ($X \in$ **WLUR**)), if $\|x_n\| = 1$ for any $n \in N$, $\|x\| = 1$ and $\|x_n + x\| \to 2$ imply that $\|x_n - x\| \to 0$ ($(x_n - x) \xrightarrow{w} 0$).

Obviously,

$$\mathbf{UR} \Rightarrow \mathbf{LUR} \Rightarrow \mathbf{LWUR} \Rightarrow \mathbf{R}.$$

A map $M = (M_i)_{n=1}^{\infty}$, where $M_i : (-\infty, \infty) \to [0, \infty]$ for all $i \in N$, is said to be a Musielak-Orlicz function, if M_i is a nonzero function that is convex, even, vanishing and continuous at zero and left continuous on $(0, \infty)$ for all $i \in N$ (see [9]). By $N = (N_i)_{i=1}^{\infty}$ we denote complementary function of M, that is, $N_i(v) = \sup_{u>0}\{u|v| - M_i(u)\}$ for any $i \in N$. It is easy to show that N is also Musielak-Orlicz function.

$M_i(u)$ is said to be strictly convex on the interval $[a, b]$, if

$$M_i\left(\frac{u+v}{2}\right) < \frac{M_i(u) + M_i(v)}{2}$$

for all $u, v \in [a, b]$, $u \neq v$.

It is said that M satisfies the condition δ_2 (we write $M \in \delta_2$), if there exist constants $i_0 \in N$, $u_0 > 0$, $K > 1$ and a sequence $c = (c_i)_{i=i_0+1}^{\infty} \in l_+^1$ such that

$$M_i(2u) \leq K M_i(u) + c_i$$

for all $i > i_0$ and $u \in R$, satisfying $M_i(u) \leq u_0$.

Given any Musielak-Orlicz function M, we define on l^0 a convex modular ρ_M by

$$\rho_M(x) = \sum_{n=1}^{\infty} M_i(x(i)).$$

The linear space $l_M = \{x : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$ equipped with the Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \rho_M(x/\lambda) \leq 1\}$$

or the Orlicz norm

$$\|x\|^o = \sup\left\{\sum_{i=1}^{\infty} x(i)y(i) : \rho_N(y) \leq 1\right\} = \inf_{k>0} \frac{1}{k}(1 + \rho_M(kx))$$

is a Banach space, denoted by $(l_M, \|\cdot\|)$ or $(l_M, \|\cdot\|^o)$ respectively, and it is called a Musielak-Orlicz sequence space.

2. Results.

Lemma 2.1 *$N \in \delta_2$ if and only if there exist constants $\theta \in (0,1)$, $\delta \in (0,1)$, $i_0 \in N$, $u_0 > 0$ and a sequence $c = (c_i)_{i=i_0+1}^{\infty} \in l_+^1$ such that*

$$M_i(\theta u) \leq (1 - \delta)\theta M_i(u) + c_i$$

for all $i > i_0$ and $u \in R$, satisfying $M_i(u) \leq u_0$.

PROOF This lemma was proved in [3], but we present here a simpler proof. Sufficiency. Let $q_i(v)$ be the right derivative of $N_i(v)$, then

$$M_i\left(q_i\left(v\right)\right)+N_i\left(v\right)=vq_i\left(v\right)=\frac{1}{\theta\left(1-\delta\right)}\theta q_i\left(v\right)\left(1-\delta\right)v$$

$$\leq\frac{1}{\theta\left(1-\delta\right)}\left(M_i\left(\theta q_i\left(v\right)\right)+N_i\left(\left(1-\delta\right)v\right)\right)$$

$$\leq M_i\left(q_i\left(v\right)\right)+\frac{1}{\theta\left(1-\delta\right)}c_i+\frac{1}{\theta\left(1-\delta\right)}N_i\left(\left(1-\delta\right)v\right),$$

we have

$$N_i(v)\leq\frac{1}{\theta(1-\delta)}N_i((1-\delta)v)+\frac{c_i}{\theta(1-\delta)}\ \text{ for }\ i>i_0\ \text{ and }\ M_i(q_i(v))\leq u_0.$$

and thus, $N\in\delta_2$

The proof of the necessity is similar to the proof of the sufficiency, we omit it.■

LEMMA 2.2 (SEE [1] AND [10]) *If $M\in\delta_2$ and M_i vanishes only at zero for any $i\in N$, then $\|x_n\|\to 0$ if $\rho_M(x_n)\to 0$.*

LEMMA 2.3 *If there exists a sequence of positive numbers $(u(i))$ such that $M_i(u(i)) = 1$, $M\in\delta_2$, M_i vanishes only at zero for any $i\in N$, $\|x_n\|\leq 1$, $\|x\|=1$ and $\|x_n+x\|\to 2$, then $\rho_M(x_n)\to 1$.*

PROOF Assume the result is not true. Then we can assume without loss of generality that there exists $\varepsilon>0$ such that $\rho_M(x_n)\leq 1-\varepsilon$ for $n\in N$. In the sequel we shall consider two cases.

1. $|x(i)|<\sup\{u\geq 0: M_i(u)<\infty\}$ for any $i\in N$. Since $M\in\delta_2$, there exist constants $i_0\in N$, $u_0>0$, $K>1$ and a sequence $c=(c_i)_{i=i_0+1}^{\infty}\in l^1_+$ such that

$$M_i(2u)\leq KM_i(u)+c_i$$

for all $i>i_0$ and $u\in R$, satisfying $M_i(u)\leq u_0$. Let $A=\{i: i\leq i_0,\ \text{ or }\ i> i_0\ \text{ and }\ M_i(x(i))>u_0\}$. Since $\rho_M(x)=1$, we deduce that A is a finite set and there exists $\delta\in(0,1)$ such that $(1+\delta)|x(i)|<\sup\{u\geq 0: M_i(u)<\infty\}$ for $i\in A$. Hence

$$\begin{aligned}\rho_M((1+\delta)x) &\leq \sum_{i\in A}M_i((1+\delta)x(i))+\sum_{i\notin A}M_i(2x(i))\\ &\leq \sum_{i\in A}M_i((1+\delta)x(i))+\sum_{i\notin A}(KM_i(x(i))+c_i)<\infty.\end{aligned}$$

Now, we can take $\theta\in(0,\varepsilon)$ satisfying $\rho_M\left(\frac{1+\theta}{1-\theta}x\right)<1+\varepsilon^2$. Then

$$\rho_M\left((1+\theta)\frac{x_n+x}{2}\right)=\rho_M\left(\frac{1+\theta}{2}x_n+\frac{1-\theta}{2}\frac{1+\theta}{1-\theta}x\right)$$

$$\leq\frac{1+\theta}{2}\rho_M(x_n)+\frac{1-\theta}{2}\rho_M\left(\frac{1+\theta}{1-\theta}x\right)\leq\frac{1}{2}((1+\varepsilon)(1-\varepsilon)+1+\varepsilon^2)=1,$$

whence we have $\|\frac{x+x_n}{2}\| \leq \frac{1}{1+\theta}$, which contradicts the condition $\|x+x_n\| \to 2$.

2. Let now $|x(i_1)| = \sup\{u \geq 0 : M_{i_1}(u) < \infty\}$ for some i_1. Without loss of generality, we can assume that $i_1 = 1$. We have $|x(1)| = u(1)$, that is $M_1(x(1)) = 1$, whence $x(i) = 0$ for all $i \geq 2$.

As in case 1 it is easy to get $\delta > 0$ such that $\sum_{i=2}^{\infty} M_i((1+\delta)x(i)) < \infty$. Since $M_1(x_n(1)) \leq \rho_M(x_n) \leq 1-\varepsilon$, so $|x_n(1)| \leq M_1^{-1}(1-\varepsilon)$. Hence

$$\left|\frac{x(1)+x_n(1)}{2}\right| \in \left[\frac{u(1)-M_1^{-1}(1-\varepsilon)}{2}, \frac{u(1)+M_1^{-1}(1-\varepsilon)}{2}\right].$$

Since $\lim_{\theta\to 0} \frac{M_1((1+\theta)u)}{M_1(u)} = 1$ hold uniformly on the interval $[(u(1)-M_1^{-1}(1-\varepsilon))/2, (u(1)+M_1^{-1}(1-\varepsilon))/2]$, we can take $\theta \in (0,\varepsilon)$ small enough such that

$$M_1\left((1+\theta)\frac{x_n(1)+x(1)}{2}\right) \leq \left(1+\frac{\varepsilon^2}{2}\right) M_1\left(\frac{x_n(1)+x(1)}{2}\right).$$

Then

$$\begin{aligned}
&\rho_M\left((1+\theta)\frac{x+x_n}{2}\right) \\
= \; & M_1\left((1+\theta)\frac{x(1)+x_n(1)}{2}\right) + \sum_{i=2}^{\infty} M_i\left(\frac{1+\theta}{2}x_n(i) + \frac{1-\theta}{2}\frac{1+\theta}{1-\theta}x(i)\right) \\
\leq \; & \left(1+\frac{\varepsilon^2}{2}\right) M_1\left(\frac{x(1)+x_n(1)}{2}\right) + \frac{1+\theta}{2}\sum_{i=2}^{\infty} M_i(x_n(i)) \\
\leq \; & \left(1+\frac{\varepsilon^2}{2}\right) \frac{M_1(x(1))+M_1(x_n(1))}{2} + \frac{1+\theta}{2}\sum_{i=2}^{\infty} M_i(x_n(i)) \\
\leq \; & \frac{1}{2}((1+\varepsilon^2/2)M_1(x(1)) + (1+\varepsilon)\rho_M(x_n)) \\
\leq \; & \frac{1}{2}(1+\varepsilon^2/2 + (1+\varepsilon)(1-\varepsilon)) \leq 1.
\end{aligned}$$

Therefore $\|(x+x_n)/2\| \leq \frac{1}{1+\theta}$, a contradiction. ∎

The following are the main results of this paper.

THEOREM 2.4 *The following conditions are equivalent:*

1. $(l_M, \|\cdot\|)$ *is locally uniformly rotund.*
2. $(l_M, \|\cdot\|)$ *is weakly locally uniformly rotund.*
3. *The following condition are satisfied:*
   - *(i) there exists a sequence positive numbers* $(u(i))$ *such that* $M_i(u(i)) = 1$ *for any* $i \in N$,
   - *(ii) each function* M_i *vanishes only at zero,*

(iii) the function M satisfies the condition δ_2,

(iv) a) for any $i \in N$ function M_i is strictly convex on $[0, u(i)]$ or

b) the function N satisfies the condition δ_2 and there exists a sequence positive numbers $(a(i))$ such that $M_i(a(i)) + M_j(a(j)) \geq 1$ for $i \neq j$ and function M_i is strictly convex on $[0, a(i)]$ for any $i \in N$.

Proof The implication $1 \Rightarrow 2$ is obvious. We show now the implication $2 \Rightarrow 3$. Since **WLUR** implies **R**, by Theorem 3 and Theorem 5 in [17], we get the necessity of condition $(i) - (iii)$. If condition (iv) does not hold, we can assume that $N \notin \delta_2$ and there exist $i \in N$ such that M_i is affine on the interval $[a, b]$, where $a(i) \leq a < b \leq u(i)$. Without loss of generality, we can assume that $i = 1$.

For any $u_0 > 0$ and $\theta \in (0, 1)$, let us define

$$c_i = \sup\left\{M_i(u) : M_i\left(\frac{u}{2}\right) > (1-\theta)\frac{M_i(u)}{2};\ M_i(u) \leq u_0\right\}.$$

Then $\sum_{i>i_0} c_i = \infty$ for any $i_0 \in N$. In fact, if it does not hold, then there exists i_0, satisfying $\sum_{i>i_0} c_i < \infty$. By the definition of c_i we have

$$M_i\left(\frac{u}{2}\right) \leq (1-\theta)\frac{M_i(u)}{2} + c_i,$$

for all $i > i_0$ and $u \in R$, satisfying $M_i(u) \leq u_0$. By Lemma 2.1, we get $N \in \delta_2$, a contradiction. Since $N \notin \delta_2$, for any $n \in N$ and $i \geq 3$, we find $u_i^n > 0$, such that $M_i(u_i^n) < \frac{1}{n}$, $M_i((u_i^n)/2) > (1 - (1/n))M_i(u_i^n)/2$ and $\sum_{i=3}^{\infty} M_i(u_i^n) = \infty$. Let $c \geq 0$ be such that $M_1(b) + M_2(c) = 1$ and let us define

$$x = be_1 + ce_2, \qquad x_n = ae_1 + ce_2 + \sum_{i=3}^{i_n-1} u_i^n e_i + v_{i_n}^n e_{i_n} \qquad n = 1, 2, \ldots,$$

where i_n is the smallest natural number for which $\sum_{i=3}^{i_n} M_i(u_i^n) \geq M_1(b) - M_1(a)$ and $v_{i_n}^n \in [0, u_{i_n}^n]$ satisfies condition $\sum_{i=3}^{i_n-1} M_i(u_i^n) + M_{i_n}(v_{i_n}^n) = M_1(b) - M_1(a)$. We have $\sum_{i=3}^{i_n-1} M_i(u_i^n) > M_1(b) - M_1(a) - 1/n$ and $\rho_M(x) = \rho_M(x_n) = 1$, so $\|x\| = \|x_n\| = 1$. Simultaneously

$$\begin{aligned}
\rho_M\left(\frac{x+x_n}{2}\right) &= M_1\left(\frac{a+b}{2}\right) + M_2(c) + \sum_{i=3}^{i_n-1} M_i\left(\frac{u_i^n}{2}\right) + M_{i_n}\left(\frac{v_i^n}{2}\right)\\
&\geq \frac{M_1(a)+M_1(b)}{2} + M_2(c) + \left(1-\frac{1}{n}\right)\sum_{i=3}^{i_n-1}\frac{M_i(u_i^n)}{2}\\
&\geq \frac{1}{2}\left(M_1(b) + M_1(a) + 2M_2(c) + \left(1-\frac{1}{n}\right)\left(M_1(b) - M_1(a) - \frac{1}{n}\right)\right) \to 1.
\end{aligned}$$

Hence $\|x + x_n\| \to 2$. But $x(1) - x_n(1) = b - a > 0$, which contradict with that $WLUR$ of l_M.

$3 \Rightarrow 1$. Let $\|x\| = 1$, $\|x_n\| = 1$ for any $n \in N$ and $\|x + x_n\| \to 2$. By $M \in \delta_2$, we have $\rho_M(x_n) = 1$. Moreover, since $\|x + \frac{x+x_n}{2}\| \to 2$, by Lemma 2.3, we have $\rho_M(\frac{x+x_n}{2}) \to 1$. Hence,

$$\begin{aligned} 0 &\leftarrow \frac{\rho_M(x) + \rho_M(x_n)}{2} - \rho_M\left(\frac{x + x_n}{2}\right) \\ &= \sum_{i=1}^{\infty}\left(\frac{M_i(x(i)) + M_i(x_n(i))}{2} - M_i\left(\frac{x(i) + x_n(i)}{2}\right)\right). \end{aligned}$$

By convexity of $M_i(u)$, all terms of the last series are nonnegative, whence we get

$$\frac{M_i(x(i)) + M_i(x_n(i))}{2} - M_i\left(\frac{x(i) + x_n(i)}{2}\right) \to 0. \tag{1}$$

for any $i \in N$. In the sequel, we will consider in two cases.

Case 1. First we assume that all functions M_i is strictly convex on the intervals $[0, u_i]$. Since $|x(i)|, |x_n(i)| \in [0, u(i)]$ for any $i, n \in N$, by (1), we have $\lim_{n\to\infty} x_n(i) = x(i)$ for $i = 1, 2, \ldots$. Hence, for any $i_0 \in N$, we get

$$\sum_{i>i_0} M_i(x_n(i)) = \rho_M(x_n) - \sum_{i=1}^{i_0} M_i(x_n(i)) \to 1 - \sum_{i=1}^{i_0} M_i(x(i)) = \sum_{i>i_0} M_i(x(i)),$$

whence it follows that $\sum_{i>i_0} M_i(x_n(i))$ converge to zero, uniformly with respect to $n \in N$, as $i_0 \to \infty$. Therefore

$$\rho_M\left(\frac{x_n - x}{2}\right) \le \sum_{i=1}^{i_0} M_i\left(\frac{x_n(i) - x(i)}{2}\right) + \frac{1}{2}\sum_{i>i_0}(M_i(x_n(i)) + M_i(x(i))) \to 0,$$

as $n \to \infty$. By Lemma 2.2, we deduce that $\|\frac{x-x_n}{2}\| \to 0$, that is, $\|x - x_n\| \to 0$.

Case 2. Let now the function N satisfies the condition δ_2 and there exists a sequence $(a(i))_{i=1}^{n}$ of positive numbers such that $M_i(a(i)) + M_j(a(j)) \ge 1$ for $i \ne j$ and all functions M_i are strictly convex on the intervals $[0, a(i)]$. Without loss of generality, we can assume that $x(i) \ge 0$ and $x_n(i) \ge 0$ for any $n, i \in N$.

If $x_n(i) \to x(i)$ for any $i \in N$, then in the same way as in case 1, we get $\|x_n - x\| \to 0$. Now suppose that there exists i_1, we may assume that $i_1 = 1$, such that

$$|x_n(1) - x(1)| \ge c > 0 \tag{2}$$

whence

$$|M_1(x_n(1)) - M_1(x(1))| \ge d$$

with some $d > 0$ and for $n = 1, 2, \ldots$. From (1) we know that $x(1) \in [a, b] \subset [a(1), u(1)]$, where $[a, b]$ is an affine interval of M_1. In virtue of the definitions of the numbers $a(i)$ from condition $(iv - b)$, we get that $x(i) \in [0, a(i)]$ for any $i \ge 2$. If $x(i) < a(i)$, then, by (1), we deduce that $x_n(i) \to x(i)$. If $x(i) = a(i)$, then we have

$x(1) = a(1) < a(1) + c \le x_n(1)$ for $n \in N$. Hence $x_n(i) < a(i)$ for $n \in N$, that is, $x_n(i)$ and $x(i)$ are in the strictly convex interval of M_i. By (1), we can also deduce that $x_n(i) \to x(i)$. So, we always have $x_n(i) \to x(i)$ for $i \ne 1$.

Since $\|x + x_n\| \to 2$, for any $n \in N$ we can find $y_n \in (l_N, \|\cdot\|^o)$ such that $\|y_n\|^o = 1$, $y_n(i) \ge 0$ for any $i \in N$ and $\lim_{n\to\infty} \sum_{i=1}^{\infty}(x(i) + x_n(i))y_n(i) = 2$. Then we have $\sum_{i=1}^{\infty} x(i)y_n(i) \to 1$ and $\sum_{i=1}^{\infty} x_n(i)y_n(i) \to 1$. By definition of the Orlicz norm, we can find $k_n \ge 1$ such that

$$\frac{1}{k_n}(1 + \rho_N(k_n y_n)) \le \|y_n\|^o + \frac{1}{n} = 1 + \frac{1}{n}$$

for $n = 1, 2, \ldots$. Hence we have

$$\begin{aligned} 0 &\leftarrow \frac{1}{k_n}(1 + \rho_N(k_n y_n)) - \sum_{i=1}^{\infty} x(i)y_n(i) \\ &= \frac{1}{k_n}(\rho_M(x) + \rho_N(k_n y_n)) - \sum_{i=1}^{\infty} x(i)y_n(i) \\ &= \sum_{i=1}^{\infty}\left(\frac{M_i(x(i))}{k_n} + \frac{N_i(k_n y_n(i))}{k_n} - x(i)y_n(i)\right) \\ &> \sum_{i>i_0}\left(\frac{M_i(x(i))}{k_n} + \frac{N_i(k_n y_n(i))}{k_n} - x(i)y_n(i)\right). \end{aligned}$$

Therefore $\lim_{i_0\to\infty} \sum_{i>i_0} |N_i(k_n y_n(i))/k_n - x(i)y_n(i)| = 0$, uniformly with respect to $n \in N$. Since $M \in \delta_2$, we get $\sum_{i>i_0} x(i)y_n(i) \le \|\sum_{i>i_0} x(i)e_i\|\|y_n\|^o$ for any $n \in N$, hence $\lim_{i_0\to\infty} \sum_{i>i_0} N_i(y_n(i)) = 0$, uniformly with respect to $n \in N$. Moreover, since $N \in \delta_2$ and N_i vanishes only at zero for $i \ge 2$, by Lemma 2.2, we get $\lim_{i_0\to\infty} \|\sum_{i>i_0} y_n(i)e_i\|^o = 0$, uniformly with respect to $n \in N$. We have

$$\begin{aligned} 0 &\leftarrow \sum_{i=1}^{\infty}(x_n(i) - x(i))y_n(i) = (x_n(1) - x(1))y_n(1) \\ &+ \sum_{i=2}^{i_0}(x_n(i) - x(i))y_n(i) + \sum_{i>i_0}(x_n(i) - x(i))y_n(i). \end{aligned}$$

Since $\|\sum_{i=2}^{i_0}(x_n(i) - x(i))e_i\| + \|x + x_n\| \, \|\sum_{i>i_0} y_n(i)e_i\|^o \to 0$ as $i_0, n \to \infty$, we have $\sum_{i=2}^{i_0}(x_n(i) - x(i))y_n(i) + \sum_{i>i_0}(x_n(i) - x(i))y_n(i) \to 0$ as $i_0, n \to \infty$ and in consequently $\lim_{n\to\infty}(x_n(1) - x(1))y_n(1) = 0$.

But $y_n(1)$ do not converge to 0 as $n \to \infty$. In fact, if $y_n(1) \to 0$ and $x(1) \ne 0$, then $\rho_M(\sum_{i=2}^{\infty} x(i)e_i) \le 1 - M_1(x(1)) < 1$, which contradicts the condition $\|\sum_{i=2}^{\infty} x(i)e_i\| \ge \sum_{i=2}^{\infty} x(i)y_n(i) \to 1$. If $y_n(1) \to 0$ and $x(1) = 0$, then $\rho_M(\sum_{i=2}^{\infty} x_n(i)e_i) \le 1 - M_1(x_n(1)) \le 1 - d$ for any $n \in N$, which contradicts the condition $\|\sum_{i=2}^{\infty} \frac{x_n(i)+x(i)}{2}e_i\| \ge \sum_{i=2}^{\infty} \frac{x_n(i)+x(i)}{2}y_n(i) \to 1$ and Lemma 2.3. So, $x_n(1) \to x(1)$, which contradicts the condition (2). ■

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