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On distributive lattices

In this paper we generalize Glivenko's theorem and then show how this simplifies the proof that the clusters of a linear lattice form a complete Boolean algebra.

Let S be a linear lattice. A set of S is called a *manifold* if is not empty. A manifold M is said to be a *cluster* if $M = M^{\perp\perp}$, where $M^{\perp} = \{x \mid |x| \wedge |y| = 0 \text{ for every } y \in M\}$. In [1] it is proved that the clusters form a Boolean algebra by the order of inclusion. In [2] we gave two different proofs. In these proofs we used stars: a manifold M of S is called a *star* if $x \in M$ and $0 \leqslant y \leqslant |x|$ imply $y \in M$. The purpose of this paper is to simplify the proof of [1] by stars instead of ideals.

According to [1], a Brouwerian lattice L is a lattice such that for any $a, b \in L$, $\{x \mid a \wedge x \leqslant b\}$ contains a greatest element which is denoted by $b : a$. In a Brouwerian lattice with 0, $0 : a$ is denoted by a^*.

GLIVENKO'S THEOREM. *If L is a Brouwerian lattice with 0, then the correspondence $a \to a^{**}$ is a lattice-epimorphism of L onto the Boolean algebra of elements of L such that $a = a^{**}$.*

In [1], p. 45, it is shown that a Brouwerian lattice is a distributive lattice. Thus, if L is a Brouwerian lattice with 0, then L is a distributive lattice with 0 and $x \wedge y = 0$ if and only if $x \leqslant y^*$. We now generalize Glivenko's theorem by these conditions.

In a previous paper [2], we proved the

LEMMA. *Let A be a lattice with 0. If there is a map $x \to x'$ from A into A such that*

(i) $x'' = x$,

(ii) $x \wedge y = 0$ *if and only if* $x \leqslant y'$, *and*

(iii) $(x \vee y)' = x' \wedge y'$,

then A is a Boolean algebra.

Using this lemma we can prove the

THEOREM. *Let L be a distributive lattice with 0. If there is a map $x \to x'$ from L into L such that*

$$x \wedge y = 0 \text{ if and only if } x \leqslant y', \tag{1}$$

then $A = \{x' | x \in L\}$ *is a Boolean algebra and a function* $f: L \to A$ *defined by* $f(x) = x''$ *is a lattice-epimorphism.*

Proof. Since $x' \leqslant x'$, we have

$$x' \wedge x = 0 \tag{2}$$

by (1). This implies

$$x \leqslant x''. \tag{3}$$

Since $y \leqslant x$ implies $x' \wedge y \leqslant x' \wedge x$ which implies $x' \wedge y = 0$ by (2), we have $x' \leqslant y'$ by (1). Therefore

$$y \leqslant x \text{ implies } x' \leqslant y'. \tag{4}$$

By (3) and (4), we conclude

$$x''' = x'. \tag{5}$$

Since $x \vee y \geqslant x, y$, we obtain $(x \vee y)' \leqslant x' \wedge y'$ by (4). Since

$$(x' \wedge y') \wedge (x \vee y) = (x' \wedge y' \wedge x) \vee (x' \wedge y' \wedge y) = 0$$

by (2), we obtain $x' \wedge y' \leqslant (x \vee y)'$ by (1). Therefore

$$(x \vee y)' = x' \wedge y'. \tag{6}$$

According to (5), $x \in A$ if and only if $x'' = x$.

It is easily shown that A is a lattice under the same order as L with meet "$\vartriangle$" and join "$\triangledown$", where for $a, b \in A$, $a \vartriangle b = a \wedge b$ and $a \triangledown b = (a \vee b)''$. Also $0 \leqslant x$ implies $x' \leqslant 0'$ by (4). So

$$0'' = 0'' \wedge 0' = 0,$$

the last equality following from (2). Thus $0 \in A$ and A is a lattice with 0 and satisfies (i) by (5), (ii) by (1), and (iii) by the following. For $a, b \in A$, we have

$$(a \triangledown b)' = (a \vee b)''' = (a \vee b)' = a' \wedge b' = a' \vartriangle b'$$

by (6). Thus by the lemma, A is a Boolean algebra.

Let $x, y \in L$. By (4) $x \leqslant y$ implies $y' \leqslant x'$ which in turn implies $x'' \leqslant y''$. Thus

$$x \leqslant y \text{ implies } f(x) \leqslant f(y). \tag{7}$$

By (5) and (6),

$$\begin{aligned} f(x \vee y) &= (x \vee y)'' = (x' \wedge y')' = (x''' \wedge y''')' \\ &= (x'' \vee y'')'' = x'' \triangledown y'' = f(x) \triangledown f(y). \end{aligned}$$

Since $x \wedge y \leqslant x, y$, we have that $f(x \wedge y) \leqslant f(x), f(y)$ by (7).

Thus

$$f(x \wedge y) \leqslant f(x) \vartriangle f(y). \tag{8}$$

On the other hand, by setting $z = (x \wedge y)' \wedge (x'' \wedge y'')$ we have

$$z \wedge x \wedge y = 0, \quad z \wedge x' = 0, \quad \text{and} \quad z \wedge y' = 0$$

by (2). Thus

$$z \wedge x \leqslant y', \quad z \leqslant x'', \quad \text{and} \quad z \leqslant y''$$

so that

$$z \wedge x \leqslant y' \wedge y'' = 0,$$

which implies $z \leqslant x'$ and thus

$$z \leqslant x' \wedge x'' = 0.$$

Consequently, $z = 0$ which implies $x'' \wedge y'' \leqslant (x \wedge y)''$. Therefore

$$f(x \wedge y) = (x \wedge y)'' \geqslant x'' \wedge y'' = f(x) \vartriangle f(y)$$

By (8) and the above $f(x \wedge y) = f(x) \vartriangle f(y)$ and f is a lattice-homomorphism. By (5) f is trivially onto and thus f is a lattice-epimorphism.

As an application, let S be a linear lattice. As defined in [2], a manifold M is called a *star* if $x \in M$ and $0 \leqslant y \leqslant |x|$ implies $y \in M$. For any system of stars M_λ $(\lambda \in \Lambda)$ it is obvious that $\bigcup_{\lambda \in \Lambda} M_\lambda$ and $\bigcap_{\lambda \in \Lambda} M_\lambda$ are also stars. Therefore the stars of S form a distributive lattice under the inclusion order with $\{0\}$ as the zero of this lattice. According to the lemma in [2], we have $M \cap N = \{0\}$ if and only if $M \subset N^\perp$ for M and N stars. Thus the clusters of S ($\emptyset \neq M \subset S$ is called a *cluster* if $M = M^{\perp\perp}$) form a Boolean algebra and the map $f(M) = M^{\perp\perp}$ is a lattice-epimorphism by our generalization of Glivenko's theorem.

References

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