ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XVIII (1974)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XVIII (1974)

HORACE H. CHUANG and HIDEGORO NAKANO (Detroit, Mich.)

Simple linear lattices

A function lattice $\mathfrak{F}$ on a space S is said to be *simple* if for any $\varphi \in \mathfrak{F}$ we can find a finite system of characteristic functions $\chi_\nu \in \mathfrak{F}$ and real numbers a_ν $(\nu = 1, 2, \ldots, n)$ such that $\varphi = \sum_{\nu=1}^{n} a_\nu \chi_\nu$, and the totality of characteristic functions in $\mathfrak{F}$ is called the *character of* $\mathfrak{F}$.

An ordered space $\mathfrak{B}$ is called a *Boolean lattice* if for any $x, y, z \in \mathfrak{B}$ we have $x \wedge y \in \mathfrak{B}$ and $x \vee y \in \mathfrak{B}$, there exists the minimum of $\mathfrak{B}$ that is denoted by 0 but not necessarily the maximum of $\mathfrak{B}$, $(x \vee y) \wedge z = (x \wedge z) \vee \vee (y \wedge z)$, and for $x \geqslant y$ there exists a unique $w \in \mathfrak{B}$ such that $x = y \vee w$ and $y \wedge w = 0$, and such w is denoted by $x - y$.

The character of a simple function lattice forms a Boolean lattice. Conversely, we prove that for any Boolean lattice $\mathfrak{B}$ there exists a simple function lattice $\mathfrak{F}$ whose character is isomorphic to $\mathfrak{B}$, and the linear lattice of all measures on $\mathfrak{B}$ is isomorphic to the bounded linear functionals of $\mathfrak{F}$. Therefore from the many properties of the bounded linear functionals established by H. Nakano(1) we can derive the corresponding properties of meaures on a Boolean lattice.

In this paper we will use notations and terminologies as in Nakano's paper.

1. Lattice bases of linear lattices. Let L be a linear lattice. A manifold $B \subseteq L$ is called a *lattice basis* of L if

1° *every $x \in B$ is normalable and positive,*

2° *for any $x, y \in B$ we have $x \vee y \in B$, $x - x \wedge y \in B$, and $(x - x \wedge y) \wedge y = 0$,*

3° *for $0 < a \in L$ we can find $0 \neq x \in B$ and $0 < a$* such that $ax \leqslant a$.

Since $x \wedge y = x - (x \wedge (x - x \wedge y))$ for $x, y \geqslant 0$, by 2° we have

(1.1) *$x \wedge y \in B$ for any $x, y \in B$.*

We will prove

(1.2) *$[x]y = x \wedge y$ for any $x, y \in B$.*

Proof. It is clear that $y = y \wedge x + (y - y \wedge x)$ and $y \wedge x \in \{x\}^{\perp\perp}$. Since $y - y \wedge x \in \{x\}^{\perp}$ by 2°, we obtain (1.2) by definition.

(1) H. Nakano, *Linear lattices*, Wayne State University Press, 1966.

From (1.2) we conclude

(1.3) *For $x, y \in B$ we have $[x] \leqslant [y]$ if and only if $x \leqslant y$.*

(1.4) *For $x, y \in B$ we have $[x] = [y]$ if and only if $x = y$.*

We will prove

(1.5) *Every $x \in B$ is archimedean.*

Proof. If $a \in L$ and $0 < a \leqslant (1/n)x$ for all $n = 1, 2, \ldots,$ then by 3° we can find $\alpha > 0$ and $0 < b \in B$ such that $\alpha b \leqslant a$. Consequently we have $b \leqslant 2b \leqslant x$ and hence $(x - x \wedge b) \wedge b \geqslant (2b - b) \wedge b = b > 0$. This is a contradiction by 2°.

Let B be ordered by the order $\leqslant$ induced on B from L. Then we can easily prove

(1.6) *B is a Bolean lattice.*

A linear lattice is said to be *semi-continuous* if each of its elements is normalable. A manifold $\mathfrak{O}$ of a Boolean lattice $\mathfrak{B}$ is called an *orthogonal cover* of $\mathfrak{B}$ if (1) for any $x, y \in \mathfrak{O}$ we have $x = y$ or $x \wedge y = 0$, and (2) for any $a \in \mathfrak{B}$ we can find a finite system $a_\nu \in \mathfrak{O}$ $(\nu = 1, 2, \ldots, n)$ such that $a \leqslant \bigvee_{\nu=1}^{n} a_\nu$.

THEOREM 1.1. *If L is archimedean and semi-continuous, then it has a lattice basis B. Furthermore if the whole projector lattice $\mathfrak{P}$ of L has an orthogonal cover $\mathfrak{O}$, then there is a lattice basis B of L such that $\mathfrak{P} = \{[x] : x \in B\}$.*

Proof. Let K be a maximal orthogonal system of positive elements of L. We will prove that

$$B = \{ \bigvee_{\nu=1}^{n} [x_\nu] a_\nu :\ a_\nu \in K \text{ and } x_\nu \in L\ (\nu = 1, 2, \ldots, n),\ n = 1, 2, \ldots\}$$

is a lattice basis of L. Condition 1° is clearly satisfied by B. Condition 2° is also satisfied: For $x, y \in B$, if $x = \bigvee_{\nu=1}^{n} [x_\nu] a_\nu$ and $y = \bigvee_{\nu=1}^{n} [y_\nu] a_\nu$, where $a_\nu \in K$ $(\nu = 1, 2, \ldots, n)$ is an orthogonal system, then

$$x \vee y = \bigvee_{\nu=1}^{n} [|x_\nu| \vee |y_\nu|] a_\nu \in B,$$

$$x - x \wedge y = \bigvee_{\varrho=1}^{n} [a_\varrho](x - x \wedge y) = \bigvee_{\varrho=1}^{n} ([x_\varrho] a_\varrho - [x_\varrho][y_\varrho] a_\varrho)$$

$$= \bigvee_{\varrho=1}^{n} ([x_\varrho] - [x_\varrho][y_\varrho]) a_\varrho \in B,$$

$$(x - x \wedge y) \wedge y = \bigvee_{\varrho=1}^{n} [a_\varrho]\big((x - x \wedge y) \wedge y\big)$$

$$= \bigvee_{\varrho=1}^{n} ([x_\varrho] a_\varrho - [x_\varrho][y_\varrho] a_\varrho) \wedge [y_\varrho] a_\varrho = 0,$$

because $[a_\varrho]x = [x_\varrho] a_\varrho$ and $[a_\varrho]y = [y_\varrho] a_\varrho$ for $\varrho = 1, 2, \ldots, n$.

Condition 3° is also satisfied by B: For $0 < a \in L$, by the maximality of K there exists $b \in K$ such that $a \wedge b \neq 0$ and $[b]a \neq 0$. Since L is archimedean by assumption we can find $\alpha > 0$ such that $([b]a - \alpha b)^+ \neq 0$. Then $[(a-\alpha b)^+]b \in B$ and

$$a \geqslant [(a-\alpha b)^+]a \geqslant \alpha[(a-\alpha b)^+]b \neq 0$$

because $[(a-\alpha b)^+][b] = \left[([b]a-\alpha b)^+\right] \neq 0$.

Next we suppose that the whole projector lattice $\mathfrak{P}$ of L has an orthogonal cover $\mathfrak{O}$. Setting $K = \{a\colon a = 0, \text{ or } a \geqslant 0 \text{ and } [a] \in \mathfrak{O}\}$ we obtain a maximal orthogonal system K of positive elements. Then as shown just above, the set B is a lattice basis of L. For any $x \in L$ by the definition of orthogonal covers we can find a finite system $a_\nu \in K$ $(\nu = 1, 2, \ldots, n)$ such that $[x] \leqslant \bigvee_{\nu=1}^{n} [a_\nu]$. Then $[x] = \bigvee_{\nu=1}^{n} [x][a_\nu] = \left[\bigvee_{\nu=1}^{n} [x]a_\nu\right]$, and $\bigvee_{\nu=1}^{n} [x]a_\nu \in B$. So we conclude $\mathfrak{P} = \{[x]\colon x \in B\}$.

2. Simple linear lattices. A lattice basis B of L is said to be *simple* if for any $a \in L$ we can find a finite system $x_\nu \in B$ and real numbers α_ν $(\nu = 1, 2, \ldots, n)$ such that $a = \sum_{\nu=1}^{n} \alpha_\nu x_\nu$. A linear lattice is said to be *simple* if it has a simple lattice basis.

We state a partition theorem that is easily proved by induction.

PARTITION THEOREM. *For any finite system x_ν $(\nu = 1, 2, \ldots, n)$ of a Boolean lattice $\mathfrak{B}$ there exists a finite orthogonal system $a_\mu \in \mathfrak{B}$ $(\mu = 1, 2, \ldots, m)$ such that $x_\nu = \bigvee_{a_\mu \leqslant x_\nu} a_\mu$ for each $\nu = 1, 2, \ldots, n$.*

THEOREM 2.1. *If B is a simple lattice basis of L, then*

(1) *for any $a \in L$ we can find $x_\nu \in B$ and α_ν $(\nu = 1, 2, \ldots, n)$ such that $a = \sum_{\nu=1}^{n} \alpha_\nu x_\nu$ and $x_\nu \wedge x_\mu = 0$ for $\nu \neq \mu$,*

(2) *$\{[x]\colon x \in L\} = \{[x]\colon x \in B\}$,*

(3) *$B \ni x_\nu \downarrow_{\nu=1}^{\infty} x \in L$ implies $x \in B$ and $x_\nu \downarrow_{\nu=1}^{\infty} x$ in B, and*

(4) *$B \ni x_\nu \downarrow_{\nu=1}^{\infty} x$ in B implies $x_\nu \downarrow_{\nu=1}^{\infty} x$ in L.*

Proof. (1) Let $L \ni a = \sum_{\mu=1}^{m} \beta_\mu b_\mu$, where $b_\mu \in B$ for all $\mu = 1, 2, \ldots, m$. Since B is a Boolean lattice, by the Partition Theorem there exists an orthogonal system $a_\nu \in B$ $(\nu = 1, 2, \ldots, n)$ such that $b_\mu = \bigvee_{a_\nu \leqslant b_\mu} a_\nu = \sum_{a_\nu \leqslant b_\mu} a_\nu$ for all $\mu = 1, 2, \ldots, m$. Therefore $a = \sum_{\nu=1}^{n} \alpha_\nu a_\nu$ for some α_ν $(\nu = 1, 2, \ldots, n)$, and we can easily prove that a is normalable and archimedean.

(2) follows immediately from (1).

(3) Suppose $B \ni x_\nu \underset{\nu=1}{\overset{\infty}{\downarrow}} x \in L$. Without loss of generality we may assume $x > 0$ and hence $x = \sum_{\mu=1}^{m} \alpha_\mu y_\mu$ for some orthogonal system $0 < y_\mu \in B$ and $0 < \alpha_\mu$ ($\mu = 1, 2, \ldots, m$). Then $[x_\nu] \geqslant [y_\mu]$ for all $\mu = 1, 2, \ldots, m$ and $\nu = 1, 2, \ldots,$ and by (1.2) and (1.3) we have

$$y_\mu = [y_\mu]x_\nu \underset{\nu=1}{\overset{\infty}{\downarrow}} [y_\mu]x = \alpha_\mu y_\mu .$$

This means $\alpha_\mu = 1$ for all μ, hence $x \in B$. Now it is clear by definition that $x_\nu \underset{\nu=1}{\overset{\infty}{\downarrow}} x$ in B.

(4) Let $B \ni x_\nu \underset{\nu=1}{\overset{\infty}{\downarrow}} x$ in B and $L \ni y \leqslant x_\nu$ for all ν. Then $0 \leqslant y^+ \leqslant x_\nu$ for all ν. If $y^+ = \sum_{\mu=1}^{m} \alpha_\mu a_\mu$ for some orthogonal system $a_\mu \in B$ and $\alpha_\mu > 0$ ($\mu = 1, 2, \ldots, m$), then $[a_\mu] \leqslant [x_\nu]$ and $a_\mu \leqslant x_\nu$ by (1.3). Thus by (1.2) we have $\alpha_\mu a_\mu = [a_\mu]y^+ \leqslant [a_\mu]x_\nu = a_\mu \leqslant x_\nu$ for all μ and ν. Since $B \ni \bigvee_{\mu=1}^{m} a_\mu \leqslant x_\nu$ for all ν, we have $\bigvee_{\mu=1}^{m} a_\mu \leqslant x$ by assumption. Hence $y \leqslant y^+ = \bigvee_{\mu=1}^{m} \alpha_\mu a_\mu \leqslant \bigvee_{\mu=1}^{m} a_\mu \leqslant x$, and we conclude that $x_\nu \underset{\nu=1}{\overset{\infty}{\downarrow}} x$ in L.

In this proof we also proved

THEOREM 2.2. *Every simple linear lattice is archimedean and semi-continuous.*

If a linear lattice L has a lattice basis B, then the linear manifold S generated by B is called the *simplicity* of B. S is a linear lattice since for any $y \in S$ if $y = \sum_{\nu=1}^{n} \alpha_\nu a_\nu$ for some orthogonal system $a_\nu \in B$ ($\nu = 1, 2, \ldots$ $\ldots, n$), then $y = \sum_{\alpha_\nu \geqslant 0} \alpha_\nu a_\nu - \sum_{\alpha_\nu < 0} (-\alpha_\nu) a_\nu$, and hence y^+ exists in S and is equal to $\sum_{\alpha_\nu \geqslant 0} \alpha_\nu a_\nu$.

THEOREM 2.3. *If an archimedean linear lattice L has a lattice basis B, then for the simplicity S of B we have $a = \bigvee_{S \ni x \leqslant a} x$ for $0 \leqslant a \in L$.*

Proof. If $y \leqslant a - x$ for all $S \ni x \leqslant a$, then $y \leqslant 0$. For if not, we have $0 < y^+ \leqslant a - x$ for all $S \ni x \leqslant a$. By 3° of Section 1, there exists $\alpha > 0$ and $B \ni b > 0$ such that $\alpha b \leqslant y^+ \leqslant a$. Since $S \ni \alpha b \leqslant a$, we have $\alpha b \leqslant y^+ \leqslant a - \alpha b$, i. e., $S \ni 2\alpha b \leqslant a$. Thus by induction we have $0 < n\alpha b \leqslant a$ for all natural numbers n. This is a contradiction because L is archimedean by assumption. Hence we have $0 = \bigwedge_{S \ni x \leqslant a} (a - x)$, i. e., $a = \bigvee_{S \ni x \leqslant a} x$.

3. Isomorphism. For two Boolean lattices $\mathfrak{B}_1$ and $\mathfrak{B}_2$ we say that $\mathfrak{B}_1$ is *isomorphic* to $\mathfrak{B}_2$ and we write $\mathfrak{B}_1 \approx \mathfrak{B}_2$ if there is a mapping T from $\mathfrak{B}_1$ onto $\mathfrak{B}_2$ such that for any $x, y \in \mathfrak{B}_1$ we have $T(x) \leqslant T(y)$ if and only if $x \leqslant y$. Such a mapping T is called an *isomorphism* from $\mathfrak{B}_1$ to $\mathfrak{B}_2$. Every isomorphism is one-to-one by definition.

A mapping T from an ordered linear space L_1 onto another ordered linear space L_2 is called an *isomorphism* from L_1 onto L_2 and we write $L_1 \approx L_2$ if (1) T is linear, i. e., $T(ax+\beta y) = aT(x)+\beta T(y)$, and (2) $T(x) \geqslant 0$ if and only if $x \geqslant 0$. Thus every isomorphism is one-to-one because $T(x) = 0$ means both $T(x) \geqslant 0$ and $T(-x) \geqslant 0$, and they imply $x = 0$.

THEOREM 3.1. *For a simple linear lattice L, every simple lattice basis B of L is isomorphic to the whole projector lattice $\mathfrak{P}$ of L.*

Proof. By (1.3) and (2) of Theorem 2.1, we can easily prove that by setting $T(b) = [b]$ for every $b \in B$ we obtain an isomorphism T from the Boolean lattice B to the Boolean lattice $\mathfrak{P}$.

We can easily prove

THEOREM 3.2. *Let B be a simple lattice basis of L and M be a mapping from B into another linear space K satisfying the condition that $M(a \vee b) = M(a)+M(b)$ if $a \wedge b = 0$. Then setting*

$$T\Big(\sum_{\nu=1}^{n} \alpha_\nu a_\nu\Big) = \sum_{\nu=1}^{n} \alpha_\nu M(a_\nu),$$

where $a_\nu \in B$ ($\nu = 1, 2, \ldots, n$), we obtain a linear mapping T from L into K.

ISOMORPHISM THEOREM 3.3. *For two simple linear lattices L_1 and L_2, if*

$$\{[x]\colon x \in L_1\} \approx \{[x]\colon x \in L_2\},$$

then for any simple lattice bases B_1 and B_2 of L_1 and L_2 respectively there exists an isomorphism T from L_1 to L_2 such that $TB_1 = B_2$.

Proof. By the assumption and Theorem 3.1, there exists an isomorphism K from B_1 to B_2. By Theorem 3.2 we can define a linear mapping T from L_1 into L_2 by $T\big(\sum_{\nu=1}^{n} \alpha_\nu a_\nu\big) = \sum_{\nu=1}^{n} \alpha_\nu K(a_\nu)$ for all $\sum_{\nu=1}^{n} \alpha_\nu a_\nu \in L_1$, where $0 < a_\nu \in B_1$ ($\nu = 1, 2, \ldots, n$). Then it is clear that T is onto, linear, and $T(B_1) = B_2$.

It is obvious by Theorem 2.1 (1) that $x \geqslant 0$ implies $T(x) \geqslant 0$. Suppose $T(x) \geqslant 0$ and $x = \sum_{\nu=1}^{n} \alpha_\nu a_\nu$ for some orthogonal system $B_1 \ni a_\nu > 0$ ($\nu = 1, 2, \ldots, n$). Then we have $0 \leqslant [K(a_\nu)]\big(T(x)\big) = \alpha_\nu K(a_\nu)$. Since $K(a_\nu) > 0$, we have $0 \leqslant \alpha_\nu$ for all $\nu = 1, 2, \ldots, n$, i. e., $x \geqslant 0$. Therefore $T(x) \geqslant 0$ if and only if $x \geqslant 0$.

4. Simple function lattices. A function lattice $\mathfrak{F}$ on a space S is said to be *simple* if for any $\varphi \in \mathfrak{F}$ we can find a finite system of characteristic functions $\chi_\nu \in \mathfrak{F}$ and real numbers α_ν $(\nu = 1, 2, \ldots, n)$ such that $\varphi = \sum_{\nu=1}^{n} \alpha_\nu \chi_\nu$. For a simple function lattice $\mathfrak{F}$ the system of all characteristic functions in $\mathfrak{F}$ is called the *character* of $\mathfrak{F}$. We can easily prove

THEOREM 4.1. *A simple function lattice $\mathfrak{F}$ is simple as a linear lattice and the character of $\mathfrak{F}$ is a simple lattice basis.*

EXISTENCE THEOREM 4.2. *For any Boolean lattice $\mathfrak{F}$ there exists a simple function lattice whose character is isomorphic to $\mathfrak{B}$.*

Proof. Let $\mathfrak{E}$ be the space of all maximal ideals of $\mathfrak{B}$, as defined in Section 8 in H. Nakano paper. We set $U_a = \{\mathfrak{p}\colon a \in \mathfrak{p} \in \mathfrak{E}\}$ for $a \in \mathfrak{B}$, then we have $U_a \cup U_b = U_{a \vee b}$, $U_a \cap U_b = U_{a \wedge b}$, $U_a - U_b = U_{a-b}$ for $a \geqslant b$, and $\mathfrak{E} = \bigcup_{a \in \mathfrak{B}} U_a$, as proved in H. Nakano paper (see p. 30). Let χ_a be the characteristic function of U_a, i. e., $\chi_a(\mathfrak{p}) = 1$ if $\mathfrak{p} \in U_a$, and $\chi_a(\mathfrak{p}) = 0$ if $\mathfrak{p} \notin U_a$. If we set $B = \{\chi_a\colon a \in \mathfrak{B}\}$ and $\mathfrak{F} = \{\sum_{\nu=1}^{n} \alpha_\nu \chi_{a_\nu}\colon a_\nu \in \mathfrak{B}$ and $\alpha_\nu \in (-\infty, \infty)$ $(\nu = 1, 2, \ldots, n)$, $n = 1, 2, \ldots\}$, then we can easily prove that $\mathfrak{F}$ is a simple function lattice and B its character isomorphic to $\mathfrak{B}$.

The following theorem is an immediate consequence of the definition of a simple linear lattice:

THEOREM 4.3. *Let L_1 and L_2 be linear lattices. If L_1 is simple and $L_1 \approx L_2$, then L_2 is also simple.*

CHARACTERIZATION THEOREM 4.4. *A linear lattice L is isomorphic to some simple function lattice $\mathfrak{F}$ if and only if L is simple.*

Proof. If $L \approx \mathfrak{F}$, then by Theorem 4.1 $\mathfrak{F}$ is simple as a linear lattice. Hence by Theorem 4.3 L is also simple.

Conversely, let L be simple with a simple lattice basis B. Since B is a Boolean lattice, by the Existence Theorem 4.2, there exists a simple function lattice $\mathfrak{F}$ such that $B \approx \{[f]\colon f \in \mathfrak{F}\}$. By Theorem 3.1 we also have $B \approx \{[x]\colon x \in L\}$. Hence $L \approx \mathfrak{F}$ by Isomorphism Theorem 3.3.

5. Measures. A function m on a Boolean lattice $\mathfrak{F}$ is called an *additive function* if

(1) $m(x \vee y) = m(x) + m(y)$ *for* $x \wedge y = 0$.

An additive function m is called a *measure* if

(2) $\sup_{z \leqslant x} |m(z)| < \infty$ *for every* $x \in \mathfrak{B}$.

For any additive functions m and n on $\mathfrak{B}$ and real numbers α and β we define

(3) $(\alpha m + \beta n)(x) = \alpha m(x) + \beta n(x)$ *for every* $x \in \mathfrak{B}$, *and*

(4) $m \leqslant n$ *if* $m(x) \leqslant n(x)$ *for every* $x \in \mathfrak{B}$.

We can easily prove that the measures on $\mathfrak{B}$ form a linear lattice $\tilde{\mathfrak{B}}$ such that for $m \in \tilde{\mathfrak{B}}$ and $x \in \mathfrak{B}$ we have

(5) $m^{+}(x) = \sup_{z \leqslant x} m(z)$,

(6) $m^{-}(x) = -\inf_{z \leqslant x} m(z)$, *and*

(7) $|m|(x) = \sup_{y \vee z \leqslant x} \big(m(y) - m(z)\big)$.

A measure $m \in \tilde{\mathfrak{B}}$ is said to be *continuous* if $x_\nu \overset{\infty}{\underset{\nu=1}{\downarrow}} 0$ implies $|m|(x_\nu) \overset{\infty}{\underset{\nu=1}{\downarrow}} 0$; and it is said to be *universally continuous* if $x_\lambda \underset{\lambda \in \Lambda}{\downarrow} 0$ implies $|m|(x_\lambda) \underset{\lambda \in \Lambda}{\downarrow} 0$.

6. Isomorphism theorem. Let B be a simple lattice basis of a linear lattice L. Let $\hat{L}$ be the collection of all linear functionals on L. Then for every $\hat{x} \in \hat{L}$ setting $\hat{x}^B(x) = \hat{x}(x)$ for $x \in B$ we obtain an additive function $\hat{x}^B$ on B. Conversely, for any additive function m on B there exists a unique $\hat{x} \in \hat{L}$ such that $\hat{x}^B = m$: By Theorem 3.2 $\hat{x}\big(\sum_{\nu=1}^{n} \alpha_\nu a_\nu\big) = \sum_{\nu=1}^{n} \alpha_\nu m(a_\nu)$, where $a_\nu \in B$, defines a linear functional $\hat{x}$ on L and $\hat{x}^B = m$. Such a functional $\hat{x}$ is unique, since if two linear functionals coincide on a simple lattice basis of L they must coincide throughout L. Therefore we can state

Isomorphism Theorem 6.1. *$\hat{x}^B$ ($\hat{x} \in \hat{L}$) is an isomorphism from $\hat{L}$ to the ordered linear space of all additive functions $\hat{B}$ on B.*

Theorem 6.2. *$\hat{x}$ is bounded if and only if $\hat{x}^B$ is a measure.*

Proof. If $\hat{x}$ is bounded, then we have

$$\sup_{B \ni z \leqslant x} |\hat{x}^B(z)| = \sup_{B \ni z \leqslant x} |\hat{x}(z)| \leqslant \sup_{0 \leqslant y \leqslant x, y \in L} |\hat{x}(y)| < \infty.$$

Conversely, we suppose that $\hat{x}^B$ is a measure. For $x \in L$ with $0 \leqslant x \leqslant a \in L$, we have $x = \sum_{\nu=1}^{n} \alpha_\nu x_\nu$ for some orthogonal system $0 \leqslant x_\nu \in B$ and real numbers α_ν ($\nu = 1, 2, \ldots, n$), and by Isomorphism Theorem 6.1 we can find $0 \leqslant \hat{y} \in \hat{L}$ such that $\hat{y}^B = |\hat{x}^B|$ and

$$|\hat{x}(x)| = \Big|\sum_{\nu=1}^{n} \alpha_\nu \hat{x}^B(x_\nu)\Big| \leqslant \sum_{\nu=1}^{n} \alpha_\nu |\hat{x}^B(x_\nu)| \leqslant \sum_{\nu=1}^{n} \alpha_\nu |\hat{x}^B|(x_\nu) = \hat{y}(x) \leqslant \hat{y}(a).$$

Therefore we have $\sup_{0 \leqslant x \leqslant a} |\hat{x}(x)| \leqslant \hat{y}(a) < \infty$.

By Isomorphism Theorem 6.1 and Theorem 6.2 we have

Theorem 6.3. *$\tilde{x}^B$ ($\tilde{x} \in \tilde{L}$) is an isomorphism from the linear lattice $\tilde{L}$ of all bounded linear functionals on L to the linear lattice $\tilde{B}$ of all measures on B.*

Since L is not necessarily continuous, we must slightly change the definitions in Nakano book of the continuity and universal continuity of a bounded linear functional $\tilde{x} \in \tilde{L}$. A functional $\tilde{x} \in \tilde{L}$ is said to be *continuous* if $L \ni x_\nu \downarrow_{\nu=1}^{\infty} 0$ implies $|\tilde{x}|(x_\nu) \downarrow_{\nu=1}^{\infty} 0$. $\tilde{x}$ is said to be *universally continuous* if $L \ni x_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies $|\tilde{x}|(x_\nu) \downarrow_{\lambda \in \Lambda} 0$. In the case that L is continuous we can easily prove by means of Theorem 19.2 of Nakano that these definitions are equivalent respectively to their original ones in Nakano's book (see p. 68 and 81).

The proof of Theorem 19.4 of Nakano is also available for

THEOREM 6.4. *For a semi-continuous linear lattice L, $\tilde{x} \in \tilde{L}$ is continuous if and only if for any $L \ni a \geqslant 0$ and $[x_\nu] \downarrow_{\nu=1}^{\infty} 0$ we have $|\tilde{x}|([x_\nu]a) \downarrow_{\nu=1}^{\infty} 0$.*

Let $\tilde{L}^c$ be the linear lattice of all continuous linear functionals on L, and let $\bar{L}$ be the linear lattice of all universally continuous linear functionals on L.

THEOREM 6.5. *$\tilde{x}^B$ is continuous if and only if $\tilde{x} \in \tilde{L}^c$. $\tilde{x}^B$ is universally continuous if and only if $\tilde{x} \in \bar{L}$.*

Proof. If $\tilde{x} \in \tilde{L}^c$ and $B \ni x_\nu \downarrow_{\nu=1}^{\infty} 0$ in B, then by (4) of Theorem 2.1, it is also true that $x_\nu \downarrow_{\nu=1}^{\infty} 0$ in L. Hence we have

$$|\tilde{x}^B|(x_\nu) = |\tilde{x}|^B(x_\nu) = |\tilde{x}|(x_\nu) \downarrow_{\nu=1}^{\infty} 0,$$

since $|\tilde{x}^B| = |\tilde{x}|^B$ by Isomorphism Theorem 6.1.

Conversely, we assume that $\tilde{x}^B$ is continuous and suppose that $0 \leqslant a = \sum_{\mu=1}^{m} \alpha_\mu a_\mu \in L$ for some orthogonal system $0 \leqslant a_\mu \in B$ and $0 \leqslant \alpha_\mu$ ($\mu = 1, 2, \ldots, m$), and $[x_\nu] \downarrow_{\nu=1}^{\infty} 0$. By Theorem 2.1 we can assume $x_\nu \in B$ for all $\nu = 1, 2, \ldots$ Furthermore we have $x_\nu \downarrow_{\nu=1}^{\infty} 0$ by the isomorphism $[x] (x \in B)$ from B onto the whole projector lattice $\mathfrak{P}$ of L. By (1.2) we have $|\tilde{x}|([x_\nu]a) = \sum_{\mu=1}^{m} \alpha_\mu |\tilde{x}|^B(x_\nu \wedge a_\mu) \downarrow_{\nu=1}^{\infty} 0$. Therefore $\tilde{x}$ is continuous by Theorem 6.4.

The second part of the theorem can be proven similarly.

THEOREM 6.6. *For any Boolean lattice $\mathfrak{B}$, $\tilde{\mathfrak{B}}$ is reflexive. $\tilde{\mathfrak{B}}^c$ is a normal manifold of $\tilde{\mathfrak{B}}$, and $\bar{\mathfrak{B}}$ is a normal manifold of $\tilde{\mathfrak{B}}^c$, where $\tilde{\mathfrak{B}}^c$ consists all continuous measures, and $\bar{\mathfrak{B}}$ consists of all universally continuous measures.*

Proof. Let $\mathfrak{B}$ be isomorphic to a simple lattice basis B of a simple linear lattice L. Then by Theorem 6.3 we need only to prove the corresponding theorem for $\tilde{L}$.

Since $\tilde{L}$ is universally continuous, to show $\tilde{L}$ is reflexive, by Theorem 24.4 of Nakano, we only have to show that $\tilde{L}$ is monotone complete. (A linear lattice S is said to be *monotone complete* if $S \ni a_\lambda \uparrow_{\lambda \in \Lambda}$ and $\sup_{\lambda \in \Lambda} \bar{x}(a_\lambda) < \infty$ for all $\bar{S} \ni \bar{x} \geqslant 0$, then there exists some $a \in S$ such that $a_\lambda \uparrow_{\lambda \in \Lambda} a$.)

So suppose $\tilde{L} \ni \bar{x}_\lambda \uparrow_{\lambda \in \Lambda}$ and $\sup_{\lambda \in \Lambda} \bar{a}(\bar{x}_\lambda) < \infty$ for all $0 \leqslant \bar{a} \in \bar{\tilde{L}}$. For every $0 \leqslant x \in L$ we can consider $x \in \bar{\tilde{L}}$ by the duality $x(\tilde{x}) = \tilde{x}(x)$ for $\tilde{x} \in \tilde{L}$. Then for $0 \leqslant x \in L$ we have $\sup_{\lambda \in \Lambda} \tilde{x}_\lambda(x) < \infty$. Therefore by Theorem 18.2 of Nakano there exists $\tilde{a} \in \tilde{L}$ such that $\tilde{x}_\lambda \uparrow_{\lambda \in \Lambda} \tilde{a}$.

By the definition of $\tilde{L}^c$ it is clear that $\tilde{L}^c$ is a semi-normal manifold of $\tilde{L}$. To prove that $\tilde{L}^c$ is a normal manifold of $\tilde{L}$, by Theorem 4.9 of Nakano, we only have to prove that $0 \leqslant \tilde{a}_\lambda \in \tilde{L}^c$ and $\tilde{a}_\lambda \downarrow_{\lambda \in \Lambda} \tilde{a}$ imply $\tilde{a} \in \tilde{L}^c$: For any $L \ni a_\nu \uparrow_{\nu=1} 0$ and $\varepsilon > 0$ we can find $\lambda_0 \in L$, by (5) of Theorem 18.2 of Nakano, such that $\tilde{a}(a_1) \leqslant \tilde{a}_{\lambda_0}(a_1) + \varepsilon$. Since $\tilde{a} - \tilde{a}_{\lambda_0} \geqslant 0$, we have

$$\tilde{a}(a_\nu) = \tilde{a}_{\lambda_0}(a_\nu) + (\tilde{a} - \tilde{a}_{\lambda_0})(a_\nu) \leqslant \tilde{a}_{\lambda_0}(a_\nu) + \varepsilon$$

for $\nu = 1, 2, \ldots$ Since $\bar{a}_{\lambda_0}(a_\nu) \downarrow_{\nu=1}^{\infty} 0$, we obtain $\inf_{\nu \geqslant 1} \tilde{a}(a_\nu) \leqslant \varepsilon$. Since $\varepsilon > 0$ is arbitrary we conclude that $\tilde{a} \in \tilde{L}^c$.

Similarly we can prove that $\bar{L}$ is a normal manifold of $\tilde{L}$. From this and the fact that $\tilde{L}^c$ is a normal manifold of $\tilde{L}$ we conclude that $\bar{L}$ is a normal manifold of $\tilde{L}^c$.

ALBANY STATE COLLEGE
WAYNE STATE UNIVERSITY