COMMENTATIONES MATHEMATICAE Vol. 50, No. 2 (2010), 161-174

A. Boccuto*, X. Dimitriou, N. Papanastassiou†

Unconditional convergence in lattice groups with respect to ideals

Abstract. We deal with unconditional convergence of series and some special classes of subsets of $\mathbb{N}$.

2010 Mathematics Subject Classification: 28B15, 54A20..

Key words and phrases: (ℓ)-group, ideal, ideal order and (D)-convergence, limit theorem, matrix theorem, Schur theorem, unconditional convergence.

1. Introduction. In this paper we extend to the context of (ℓ)-groups and ideal convergence some matrix theorems and applications to unconditional convergence of series, proved for the real case in [2]. About the matrix theorems existing in the literature, we quote the famous Basic Matrix Theorem (see [3]), which in the real case was extended in [1] to the setting of the statistical convergence, and to the context of (ℓ)-groups and $\mathcal{I}$-convergence generalized in [6]. Recall that there are Riesz spaces such that order and (D)-convergence are not generated by *any* topology: for example, $L^0(X, \mathcal{B}, \mu)$, where μ is a σ-additive and σ-finite non-atomic positive $\widetilde{\mathbb{R}}$-valued measure, endowed with the almost everywhere convergence (see [9, 11, 13]).

As an application of our main result, we also present a corollary, which is a consequence of the Basic Matrix Theorem for (ℓ)-group-valued double sequences involving P-ideals, proved in [6].

2. Preliminaries.

Definition 2.1 An abelian group $(R, +)$ is an (ℓ)-*group* iff it is a lattice and the following implication holds:

(1) $$a \leqslant b \Longrightarrow a + c \leqslant b + c \quad \text{for all } a, b, c \in R.$$

*corresponding author
†Supported by Universities of Perugia and Athens

An (ℓ)-group R is said to be *Dedekind complete* iff every nonempty subset of R, bounded from above, has supremum in R. A Dedekind complete (ℓ)-group is said to be *super Dedekind complete* iff every subset $R_1 \subset R$, $R_1 \neq \emptyset$ bounded from above contains a countable subset having the same supremum as R_1.

Let R be an (ℓ)-group. We say that a sequence $(p_n)_n$ of positive elements of R is an (O)*-sequence* iff it is decreasing and $\wedge_n p_n = 0$. A sequence $(x_n)_n$ in R is said to be *order-convergent* (or (O)*-convergent*) to $x \in R$ iff there exists an (O)-sequence $(p_n)_n$ in R with $|x_n - x| \leqslant p_n$, $\forall\, n \in \mathbb{N}$, and in this case we will write $(O) \lim_n x_n = x$. If Λ is any nonempty set, $(x_n^{(\lambda)})_n$ are sequences in R and $(x^{(\lambda)})$ are in R for all $\lambda \in \Lambda$, we say that $(O) \lim_n x_n^{(\lambda)} = x^{(\lambda)}$ *uniformly with respect to* $\lambda \in \Lambda$ iff there exists an (O)-sequence $(q_n)_n$ in R with $|x_n^{(\lambda)} - x^{(\lambda)}| \leqslant q_n$ for all $n \in \mathbb{N}$ and $\lambda \in \Lambda$. We say that the sequence $(x_n)_n$ is (O)*-Cauchy* iff $(O) \lim_n (x_n - x_{n+p}) = 0$ uniformly with respect to $p \in \mathbb{N}$.

A bounded double sequence $(a_{t,l})_{t,l}$ in R is called (D)*-sequence* or *regulator* iff for all $t \in \mathbb{N}$ we have $a_{t,l} \downarrow 0$ as $l \to +\infty$. A sequence $(x_n)_n$ in R is said to be (D)*-convergent* to $x \in R$ (and we write $(D) \lim_n x_n = x$) iff there exists a (D)-sequence $(a_{t,l})_{t,l}$ in R, such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $n_0 \in \mathbb{N}$ such that $|x_n - x| \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for all $n \in \mathbb{N}$, $n \geqslant n_0$. If $(x_n^{(\lambda)})_n$ and $(x^{(\lambda)})$ are as above, we say that $(D) \lim_n x_n^{(\lambda)} = x^{(\lambda)}$ *uniformly with respect to* $\lambda \in \Lambda$ iff there exists a (D)-sequence $(a_{t,l})_{t,l}$ in R, such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ such that $|x_n^{(\lambda)} - x^{(\lambda)}| \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for all $n \in \mathbb{N}$, $n \geqslant n_0$ and $\lambda \in \Lambda$. The sequence $(x_n)_n$ is said to be (D)*-Cauchy* iff $(D) \lim_n (x_n - x_{n+p}) = 0$ uniformly with respect to $p \in \mathbb{N}$.

We say that an (ℓ)-group is (O)*-complete* iff every (O)-Cauchy sequence is (O)-convergent, and (D)*-complete* iff every (D)-Cauchy sequence is (D)-convergent. We recall that every Dedekind complete (ℓ)-group is (O)-complete and (D)-complete (see also [9, Chapter 2]).

An (ℓ)-group R is said to be *weakly* σ*-distributive* iff for every (D)-sequence $(a_{t,l})_{t,l}$ we have:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \right) = 0.$$

In general, the limit of a sequence (with respect to (D)-convergence) is not unique. However, (O)-convergence of sequences implies always (D)-convergence; moreover, if R is weakly σ-distributive, then a sequence is (D)-convergent if and only if it is (O)-convergent, and in this case we get uniqueness of the limit.

We now denote by $l^1(R)$ the set of all sequences of the type $(x_j)_j$, with $x_j \in R$ for all $j \in \mathbb{N}$ and such that $\bigvee_q \left(\sum_{j=1}^{q} |x_j| \right) \in R$. As R is complete, if $(x_j)_j$ belongs to

l^1, then $S := (O)\lim_n \sum_{j=1}^{n} x_j$ exists in R (see [8]). For every element $(x_j)_j$ in $l^1(R)$, we shall also write $S = (O)\lim_n \sum_{j=1}^{n} x_j = \sum_{j=1}^{\infty} x_j$, and say that S is the *sum* of the sequence $(x_j)_j$. Similarly as in the classical case, it is easy to check that, if the sum of a series $\sum_{j=1}^{\infty} x_j$ exists in R, then $(D)\lim_j x_j = 0$.

A series $\sum_{j=1}^{\infty} x_j$ in R is said to be *unconditionally convergent* iff there is a regulator $(a_{t,l})_{t,l}$ with the property that to every $\varphi : \mathbb{N} \to \mathbb{N}$ there corresponds a finite set $A_0 \subset \mathbb{N}$ such that

$$\left| \sum_{i \in A_1} a_i - \sum_{i \in A_2} a_i \right| \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever we take two finite subsets of $\mathbb{N}$, A_1, A_2 with $A_1, A_2 \supset A_0$.

The following well-known result will be useful in the sequel (see [9, 14, 15]).

LEMMA 2.2 *Let R be a Dedekind complete (ℓ)-group (not necessarily weakly σ-distributive), $(a^{(n)}_{t,l})_{t,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geqslant 0$ there exists a (D)-sequence $(a_{t,l})_{t,l}$ in R such that:*

$$u \wedge \left[\sum_{n=1}^{\infty} \left(\bigvee_{t=1}^{\infty} a^{(n)}_{t,\varphi(t+n)} \right) \right] \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \textit{for all } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We now recall the Maeda-Ogasawara-Vulikh representation theorem in its (ℓ)-group version (see [4, 10]).

THEOREM 2.3 *Given a Dedekind complete (l)-group R, there exists a compact Hausdorff extremely disconnected topological space Ω, unique up to homeomorphisms, such that R can be embedded as a solid subgroup of $\mathcal{C}_\infty(\Omega) = \{f \in \widetilde{\mathbb{R}}^\Omega : f$ is continuous, and $\{\omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if $(a_\lambda)_{\lambda \in \Lambda}$ is any family such that $a_\lambda \in R$ for all λ, and $a = \vee_\lambda a_\lambda \in R$ (where the supremum is taken with respect to R), then $a = \vee_\lambda a_\lambda$ with respect to $\mathcal{C}_\infty(\Omega)$, and the set $\{\omega \in \Omega : (\vee_\lambda a_\lambda)(\omega) \neq \sup_\lambda a_\lambda(\omega)\}$ is meager in Ω.*

From now on we denote by $\vee$ and $\wedge$ (resp. sup and inf) the lattice (pointwise) supremum and infimum respectively.

DEFINITION 2.4 Let X be any nonempty set. A family of sets $\mathcal{I} \subset \mathcal{P}(X)$ is called an *ideal* of X iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$. An ideal is said to be *non-trivial* iff $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is said to be *admissible* iff it contains all singletons.

An admissible ideal $\mathcal{I}$ is said to be a *P-ideal* iff for any sequence $(A_j)_j$ in $\mathcal{I}$ there are sets $B_j \subset X$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$ (see also [12]).

Let $X = \mathbb{N}$, and for every $A \subset \mathbb{N}$ and $j \in \mathbb{N}$ set

$$d_j(A) = \frac{\operatorname{card}(A \cap \{1, \dots, j\})}{j}.$$

The limit $d(A) := \lim_j d_j(A)$, if it exists, is called the *(asymptotic) density* of A. It is known that the ideal $\mathcal{I}_d := \{A \subset \mathbb{N} : d(A) = 0\}$ is a P-ideal, as well as the ideal $\mathcal{I}_{\text{fin}}$ of all finite subsets of $\mathbb{N}$. For other examples of P-ideals see [12].

Now, given a *fixed* admissible ideal $\mathcal{I}$, together with its dual filter

$$\mathcal{F} = \mathcal{F}(\mathcal{I}) := \{X \setminus I : I \in \mathcal{I}\},$$

we recall the order and the (D)-convergence related with it introduced in [6].

When we deal with an ideal $\mathcal{I}$, we always suppose that $\mathcal{I}$ is admissible, without saying it explicitly.

If $\mathcal{I}$ is an ideal of $\mathbb{N}$, we say that a sequence $(x_n)_n$ in R *$(O\mathcal{I})$-converges to $x \in R$* iff there exists an (O)-sequence $(\sigma_p)_p$ with the property that

$$\{n \in \mathbb{N} : |x_n - x| \leqslant \sigma_p\} \in \mathcal{F} \tag{2}$$

for all $p \in \mathbb{N}$. Similarly, if $\mathcal{I}$ is an ideal of $\mathbb{N}^2$, a double sequence $(x_{i,j})_{i,j}$ in R is *$(O\mathcal{I})$-convergent to $\xi \in R$* iff there is an (O)-sequence $(\sigma_p)_p$ with the property that

$$\{(i,j) \in \mathbb{N}^2 : |x_{i,j} - \xi| \leqslant \sigma_p\} \in \mathcal{F}$$

for all $p \in \mathbb{N}$.

A sequence $(x_n)_n$ in R *$(D\mathcal{I})$-converges to $x \in R$* iff there exists a (D)-sequence $(a_{t,l})_{t,l}$ with the property that

$$\{n \in \mathbb{N} : |x_n - x| \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\} \in \mathcal{F} \tag{3}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$. A double sequence $(x_{i,j})_{i,j}$ in R is *$(D\mathcal{I})$-convergent to $\xi \in R$* iff there is a regulator $(a_{t,l})_{t,l}$ such that

$$\{(i,j) \in \mathbb{N}^2 : |x_{i,j} - \xi| \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\} \in \mathcal{F}$$

for any $\varphi \in \mathbb{N}^{\mathbb{N}}$.

In [6] the following result is proved.

PROPOSITION 2.5 *Every* $(O\mathcal{I})$*-convergent (double) sequence is* $(D\mathcal{I})$*-convergent to the same limit. Moreover, if* R *is a super Dedekind complete and weakly* σ*-distributive* (ℓ)*-group, then the converse implication holds too.*

From now on, we always suppose that R is a super Dedekind complete weakly σ-distributive (ℓ)-group. Examples of such spaces are $\mathbb{R}^{\mathbb{N}}$ and $L^0(X, \mathcal{B}, \mu)$, where μ is a positive, σ-additive, σ-finite and non-atomic $\widetilde{\mathbb{R}}$-valued measure (see also [9]).

If $R = \mathbb{R}$, instead of $(O\mathcal{I})$ and $(D\mathcal{I})$ we will write simply $(\mathcal{I})$, since in this setting these two concepts of convergence coincide.

Moreover, let us define

$$(\mathcal{I}) \sum_{j=1}^{\infty} x_j := (O\mathcal{I})[(D\mathcal{I})] \lim_n \sum_{j=1}^{n} x_j.$$

In [6] the following result is proved.

PROPOSITION 2.6 *Let* $\mathcal{I}$ *be any fixed admissible ideal of* $\mathbb{N}$*. If* $(D) \lim_n x_n = x$*, then* $(D\mathcal{I}) \lim_n x_n = x$*.*

Moreover, if $(x_n)_n$ *is a monotone sequence in* R *and* $x \in R$*, then* $(D\mathcal{I}) \lim_n x_n = x$ *if and only if* $(D) \lim_n x_n = x$*.*

A consequence of Proposition 2.6 is that, if a series $\sum_{j=1}^{\infty} x_j$ is of positive terms in R and S is its sum, then $(\mathcal{I}) \sum_{j=1}^{\infty} x_j = S$ (and vice-versa).

Also the following results were proved in [6].

PROPOSITION 2.7 *Let* $\mathcal{I}$ *be a* P*-ideal, and* $(x_n)_n$ *be a sequence in* R*, such that* $(D\mathcal{I}) \lim_n x_n = x \in R$*. Then there exists a subsequence* $(x_{n_q})_q$ *of* $(x_n)_n$*, such that* $(D) \lim_q x_{n_q} = x$*.*

DEFINITION 2.8 We say that a sequence $(x_n)_n$ in R $(O\mathcal{I}^*)$- $[(D\mathcal{I}^*)]$-*converges to* $\xi \in R$ iff there exists $A \in \mathcal{F}(\mathcal{I})$ with

$$(O) \lim_{n \to +\infty, n \in A} x_n = \xi \quad [(D) \lim_{n \to +\infty, n \in A} x_n = \xi].$$

PROPOSITION 2.9 *Suppose that* $(D\mathcal{I}^*) \lim_n x_n = \xi$*. Then* $(D\mathcal{I}) \lim_n x_n = \xi$*.*

PROPOSITION 2.10 *Let* R *be a super Dedekind complete weakly* σ*-distributive* (ℓ)*-group,* $(x_n)_n$ *be a sequence in* R*,* $(D\mathcal{I})$*-convergent to* $\xi \in R$*. If* $\mathcal{I}$ *is a* P*-ideal, then* $(x_n)_n$ $(D\mathcal{I}^*)$*-converges to* ξ*.*

Proposition 2.11 *Let $(x_{i,j})_{i,j}$ be a bounded double sequence in R, $\mathcal{I}$ be any P-ideal, $\mathcal{F} = \mathcal{F}(\mathcal{I})$ be its dual filter, and let us suppose that $(D\mathcal{I})\lim_i x_{i,j} = x_j$ for every $j \in \mathbb{N}$.*

Then there exists $B_0 \in \mathcal{F}$ such that $(D)\lim_{h\to+\infty, h\in B_0} x_{h,j} = x_j$ for all $j \in \mathbb{N}$ and with respect to a same (D)-sequence $(\widehat{a}_{t,l})_{t,l}$.

Definition 2.12 A sequence $(x_n)_n$ is said to be $(D\mathcal{I})$-*Cauchy* iff there exists a regulator $(a_{t,l})_{t,l}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $\nu \in \mathbb{N}$ with

$$\{n \in \mathbb{N} : |x_n - x_\nu| \not\leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\} \in \mathcal{I}.$$

The following Cauchy-type condition, proved in [6], will be useful in the sequel.

Proposition 2.13 *Let $\mathcal{I} \subset \mathbb{N}$ be any admissible ideal. A sequence $(x_n)_n$ is $(D\mathcal{I})$-convergent if and only if it is $(D\mathcal{I})$-Cauchy.*

We now introduce the R-valued measures (see also [7]).

Definition 2.14 a) Given a finitely additive order bounded set function $m : \mathcal{A} \to R$, we define the *semivariation* of m (on $\mathcal{A}$), $v(m) : \mathcal{A} \to R$, as follows: $v(m)(A) = \vee_{B\in\mathcal{A}, B\subset A} |m(B)|$, $A \in \mathcal{A}$.

b) A finitely additive set function $m : \mathcal{A} \to R$ is said to be (s)-*bounded* iff for every disjoint sequence $(H_n)_n$ in $\mathcal{A}$ we have $(O)\lim_n v(m)(H_n) = 0$. The maps $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are called *uniformly* (s)-*bounded* iff $(O)\lim_n[\vee_j v(m_j)(H_n)] = 0$ whenever $(H_n)_n$ is a sequence of pairwise disjoint elements of $\mathcal{A}$.

c) A finitely additive map $m : \mathcal{A} \to R$ is said to be σ-*additive* iff for every disjoint sequence $(H_n)_n$ in $\mathcal{A}$ we get: $(O)\lim_n v(m)(\bigcup_{l=n}^{\infty} H_l) = 0$. The set functions $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are called *uniformly* σ-*additive* iff for each disjoint sequence $(H_n)_n$ in $\mathcal{A}$, $(O)\lim_n[\vee_j v(m_j)(\bigcup_{l=n}^{\infty} H_l)] = 0$.

d) A sequence of set functions $(m_i)_i$ is said to be (RO)-*convergent* to m_0 in $\mathcal{A}$ iff there exists an (O)-sequence $(p_l)_l$ such that for every $l \in \mathbb{N}$ and $A \in \mathcal{A}$ there is $i_0 \in \mathbb{N}$ with $|m_i(A) - m_0(A)| \leqslant p_l$ for all $i \geqslant i_0$. In this case we say that $(RO)\lim_i m_i = m_0$.

e) A sequence of set functions $(m_i)_i$ (RD)-*converges* to m_0, or shortly $(RD)\lim_j m_j = m_0$, iff there exists a (D)-sequence $(b_{t,l})_{t,l}$ such that $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$, $\forall\, A \in \mathcal{A}$, $\exists\, i_0 \in \mathbb{N}$ such that

$$|m_i(A) - m_0(A)| \leqslant \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \quad \forall\, i \in \mathbb{N}, i \geqslant i_0. \tag{4}$$

f) We say that $(m_i)_i$ is (RD)-*Cauchy* iff there is a (D)-sequence $(b_{t,l})_{t,l}$ such that to each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $A \in \mathcal{A}$ there corresponds an integer j_0 with

$$|m_i(A) - m_{i+p}(A)| \leqslant \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \quad \forall\, i \in \mathbb{N}, i \geqslant i_0,\ \forall\, p \in \mathbb{N}. \tag{5}$$

Remark 2.15 Observe that a sequence $(m_i)_i$ is (RD)-convergent iff it is (RD)-Cauchy (see [9]).

3. The main results. In this section we will prove the main theorem and we give a corollary. We begin with the following

Definition 3.1 We say that a subset $\mathcal{W}$ of $\mathcal{P}(\mathbb{N})$ which contains the ideal $\mathcal{I}_{\text{fin}}$ of all finite subsets of $\mathbb{N}$ *has property* (M) iff for every disjoint sequence $(F_n)_n$ of elements of $\mathcal{I}_{\text{fin}}$ there exist $B \in \mathcal{W}$ and an infinite subset $M \subset \mathbb{N}$ with

$$\cup_{n \in M} F_n \subset B \subset \cup_{n \in \mathbb{N}} F_n.$$

Given a series $\sum_{i=1}^{\infty} a_i$ in R and $B \subset \mathbb{N}$, let us denote by $S_n^{(B)}$ the quantity $\sum_{i=1,\ldots,n;\, i \in B} a_i$.

Let $\mathcal{W}$ satisfy property (M). We say that the series $\sum_{i=1}^{\infty} a_i$ satisfies *property* (A) with respect to $\mathcal{I}$ iff there exists a sequence $(x_n)_n$, $(O\mathcal{I})$-convergent to 0 and such that to every infinite set $B \in \mathcal{W}$ there corresponds $S^{(B)}$ in R with

$$|S_n^{(B)} - S^{(B)}| \leqslant x_n \tag{6}$$

whenever $n \in \mathbb{N}$. We often denote the quantity $S^{(B)}$ by the symbol $(\mathcal{I}) \sum_{i \in B} a_i$.

We now turn to our main theorem.

Theorem 3.2 *Suppose that* $\mathcal{I}$ *is a* P*-ideal of* $\mathbb{N}$, $\mathcal{W}$ *satisfies property* (M) *and* $(a_{i,j})_{i,j}$ *is such that:*

i) the series $\sum_{j=1}^{\infty} a_{i,j}$ *satisfies property* (A) *for all* $i \in \mathbb{N}$ *with respect to a same sequence* $(x_n)_n$*, independent on* i*;*

ii) the family $\left(\sum_{j \in A} a_{i,j} \right)_{i \in \mathbb{N}, A \subset \mathbb{N}}$ *is order equibounded.*

Moreover, assume that:

iii) $(a_{i,j})_i$ *is an order convergent sequence for all* j*;*

iv) for all $i \in \mathbb{N}$ *and for each infinite subset* $B \in \mathcal{W}$ *the series* $\left((\mathcal{I}) \sum_{j \in B} a_{i,j} \right)_i$ (O)*-converges (with respect to a same* (O)*-sequence independent of* B *and* i*).*

Then for each disjoint sequence $(A_n)_n$ *in* $\mathcal{P}(\mathbb{N})$ *the sequence* $\left(\sum_{j \in A_n} a_{i,j} \right)_i$ *is order convergent uniformly in* $n \in \mathbb{N}$ *and the sequence* $\left(\sum_{j \in A_n} a_{i,j} \right)_n$ (O)*-converges to* 0 *uniformly in* $i \in \mathbb{N}$*.*

Proof First of all we show that, if $\sum_{i=1}^{\infty} a_i$ is a series of having property (A), then to the sequence $(x_n)_n$, existing by property (A), there corresponds an (O)-sequence $(\sigma_p)_p$ such that $\{n \in \mathbb{N} : |x_n| \leqslant \sigma_p\} \in \mathcal{F}$ for all $p \in \mathbb{N}$, where $\mathcal{F}$ is the dual filter associated with $\mathcal{I}$. Hence, since $(O\mathcal{I})$-convergence implies $(D\mathcal{I})$-convergence, then a regulator $(z_{t,l})_{t,l}$ can be found, with

$$\{n \in \mathbb{N} : |x_n| \leqslant \bigvee_{t=1}^{\infty} z_{t,\varphi(t)}\} \in \mathcal{F} \tag{7}$$

for each $\varphi \in \mathbb{N}^{\mathbb{N}}$. Let Ω be as in Theorem 2.3 and $N \subset \Omega$ be a meager set such that the sequence $(\sigma_p(\omega))_p$ is an (O)-sequence for all $\omega \in \Omega \setminus N$. For such ω's we get:

$$\{n \in \mathbb{N} : |x_n| \leqslant \sigma_p\} \subset \{n \in \mathbb{N} : |x_n(\omega)| \leqslant \sigma_p(\omega)\} \in \mathcal{F} : \tag{8}$$

indeed, since $\mathcal{F}$ is a filter and the left hand of (8) belongs to $\mathcal{F}$, then a fortiori the right hand of (8) belongs to $\mathcal{F}$ too. This implies that for each $\omega \in \Omega \setminus N$ and for every infinite set $B \in \mathcal{W}$ the real series $\sum_{i \in B} a_i(\omega)$ $\mathcal{I}$-converges. Now, since $\mathcal{I}$ is a P-ideal, we can argue analogously as in the first part of the proof of [2, Lemma 2.2] (with the difference that we have to consider a set of the dual filter $\mathcal{F}$ associated to $\mathcal{I}$ rather than a set of asymptotical density one): so, we deduce that $\sum_{i=1}^{\infty} a_i(\omega)$ is unconditionally convergent for all $\omega \notin N$, and so in particular that the series $\sum_{i \in C} a_i(\omega)$ converges uniformly with respect to $C \subset \mathbb{N}$ for such ω's. By the Maeda-Ogasawara-Vulikh Theorem 2.3, this implies that $\sum_{i=1}^{\infty} a_i$ (O)-converges (and a fortiori (D)-converges) uniformly with respect to the parameter $C \subset \mathbb{N}$, with a corresponding associated regulator $(\alpha_{t,l})_{t,l}$.

Let now $(a_{i,j})_{i,j}$ be a double sequence in R, satisfying ii), such that the series $\sum_{j=1}^{\infty} a_{i,j}$ has property (A) for every $i \in \mathbb{N}$ as in i). So for all i it is possible to find a regulator $(\alpha^{(i)}_{t,l})_{t,l}$, having the same role as above. By the Fremlin Lemma 2.2, thanks to ii), these regulators can be żeplaced"by one regulator $(b_{t,l})_{t,l}$, playing still the same role: that is, in another words, $(b_{t,l})_{t,l}$ is such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $p, q \in \mathbb{N}$ there corresponds a positive integer $\overline{h}$ with

$$\left| \sum_{j \in C} (a_{p,j} - a_{q,j}) \right| \leqslant \bigvee_{i=1}^{\infty} b_{t,\varphi(t)} \tag{9}$$

whenever $C \subset \{\overline{h}+1, \overline{h}+2, \ldots\}$.

Now, since by hypothesis $(a_{i,j})_i$ is (O)-convergent (or equivalently (D)-convergent) for every $j \in \mathbb{N}$, then for each j there exists a corresponding regulator $(\beta^{(j)}_{t,l})_{t,l}$. By equiboundedness and applying the Fremlin Lemma again, these regulators can be żeplaced"with a unique regulator $(c_{t,l})_{t,l}$, with the property that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $s \in \mathbb{N}$ there corresponds a positive integer $\overline{h}$ with

$$\left| \sum_{j \in D} (a_{p,j} - a_{q,j}) \right| \leqslant \bigvee_{i=1}^{\infty} c_{t,\varphi(t)} \tag{10}$$

whenever $p, q \geqslant \overline{h}$ and $D \subset \{1, \ldots, s\}$. For all $t, l \in \mathbb{N}$ set $e_{t,l} := 2\,(z_{t,l} + b_{t,l} + c_{t,l})$ and $d_{t,l} := 2\,e_{t,l}$.

We now prove that the regulator $(d_{t,l})_{t,l}$ is such that for every subset $A \subset \mathbb{N}$ the sequence $\left(\sum_{j \in A} a_{i,j} \right)_i$ is (D)-Cauchy with respect to the regulator $(d_{t,l})_{t,l}$. Otherwise, A is an infinite set and there exists $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that for all $i \in \mathbb{N}$ there is $k \in \mathbb{N}$, $k > i$, such that

$$\left| \sum_{j \in A} (a_{i,j} - a_{k,j}) \right| \not\leqslant \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}. \tag{11}$$

At the first step, let $i_1 = 1$ and $k_1 > 1$ such that

$$\left| \sum_{j \in A} (a_{i_1,j} - a_{k_1,j}) \right| \not\leqslant \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}.$$

There is $l_1 \in \mathbb{N} \setminus \{1\}$ with

$$\left| \sum_{j \in C} (a_{i_1,j} - a_{k_1,j}) \right| \leqslant \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \tag{12}$$

whenever $C \subset \{l_1 + 1, l_1 + 2, \ldots\}$. Then

$$\left| \sum_{j \in A \cap \{1, \ldots, l_1\}} (a_{i_1,j} - a_{k_1,j}) \right| \not\leqslant \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}, \tag{13}$$

because otherwise

$$\left|\sum_{j\in A}(a_{i_1,j}-a_{k_1,j})\right| \leqslant \left|\sum_{j\in A\cap\{1,\ldots,l_1\}}(a_{i_1,j}-a_{k_1,j})\right| + \left|\sum_{j\in A, j>l_1}(a_{i_1,j}-a_{k_1,j})\right|$$
$$\leqslant \bigvee_{t=1}^{\infty} e_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \leqslant \bigvee_{t=1}^{\infty} d_{t,\varphi(t)},$$

which contradicts (11). Thus (13) is satisfied. Furthermore, by (10), the regulator $(c_{t,l})_{t,l}$ is such that there is a natural number $r_1 > k_1$ such that

$$\left|\sum_{j\in D}(a_{p,j}-a_{q,j})\right| \leqslant \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \tag{14}$$

whenever $p, q \geqslant r_1$ and $D \subset \{1,\ldots,l_1\}$.

At the second step, let $i_2 > r_1$ and $k_2 > i_2$ with

$$\left|\sum_{j\in A}(a_{i_2,j}-a_{k_2,j})\right| \not\leqslant \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}. \tag{15}$$

There is an integer $l_2 > l_1$ with

$$\left|\sum_{j\in C}(a_{i_2,j}-a_{k_2,j})\right| \leqslant \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$$

whenever $C \subset \{l_2+1, l_2+2,\ldots\}$. Then

$$\left|\sum_{j\in A\cap\{l_1+1,\ldots,l_2\}}(a_{i_2,j}-a_{k_2,j})\right| \not\leqslant \bigvee_{t=1}^{\infty} e_{t,\varphi(t)} : \tag{16}$$

otherwise we should have

$$\left|\sum_{j\in A}(a_{i_2,j}-a_{k_2,j})\right| \leqslant \left|\sum_{j\in A\cap\{l_1+1,\ldots,l_2\}}(a_{i_2,j}-a_{k_2,j})\right| + \left|\sum_{j\in A, j\leqslant l_1}(a_{i_2,j}-a_{k_2,j})\right| +$$
$$+ \left|\sum_{j\in A, j>l_2}(a_{i_2,j}-a_{k_2,j})\right| \leqslant \bigvee_{t=1}^{\infty} e_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \leqslant \bigvee_{t=1}^{\infty} d_{t,\varphi(t)},$$

which contradicts (15). Thus (16) is fulfilled.

Proceeding by induction, we get the existence of three strictly increasing sequences in $\mathbb{N}$: $(i_r)_r$, $(k_r)_r$, $(l_r)_r$, with the properties that $i_r < k_r < i_{r+1}$ for all $r \in \mathbb{N}$, and:

j) $\left|\sum_{j\in D}(a_{i_r,j}-a_{k_r,j})\right| \leqslant \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$ whenever $D \subset \{1,\ldots,l_{r-1}\}$;

jj) $\left|\sum_{j\in C}(a_{i_r,j}-a_{k_r,j})\right| \leqslant \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$ whenever $C\subset\{l_r+1, l_r+2,\dots\}$;

jjj) $\left|\sum_{j\in F_r}(a_{i_r,j}-a_{k_r,j})\right| \not\leqslant \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}$ if $F_r = A\cap\{l_{r-1}+1,\dots,l_r\}$

for $r\geqslant 2$. If we consider the disjoint sequence $(F_r)_r$, then by property (M) there are $W\in\mathcal{W}$ and an infinite subset $P\subset\mathbb{N}$ with the property that $\cup_{r\in P}F_r\subset W\subset\cup_{r\in\mathbb{N}}F_r$. Note that, by virtue of iv) and (7), we get that the sequence $\left((\mathcal{I})\sum_{j\in W} a_{i,j}\right)_i$ is (D)-Cauchy with respect to the regulator $(z_{t,l})_{t,l}$. Moreover, thanks to unconditional convergence, the quantities $\sum_{j\in W} a_{i,j}$, $i\in\mathbb{N}$, do exist in R, and hence do coincide with the corresponding ones $(\mathcal{I})\sum_{j\in W} a_{i,j}$, $i\in\mathbb{N}$. Thus the sequence $\left(\sum_{j\in W} a_{i,j}\right)_i$ is (D)-Cauchy with respect to the regulator $(z_{t,l})_{t,l}$. From this and j), jj), for $r\in P$ with r large enough we get:

$$\begin{aligned}\left|\sum_{j\in F_r}(a_{i_r,j}-a_{k_r,j})\right| &\leqslant \left|\sum_{j\in W}(a_{i_r,j}-a_{k_r,j})\right| + \\ &+ \left|\sum_{j\in W, j\leqslant l_{r-1}}(a_{i_r,j}-a_{k_r,j})\right| + \left|\sum_{j\in W, j>l_r}(a_{i_r,j}-a_{k_r,j})\right| \\ &\leqslant \bigvee_{t=1}^{\infty} z_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \leqslant \bigvee_{t=1}^{\infty} e_{t,\varphi(t)},\end{aligned}$$

obtaining a contradiction with jjj). Thus we proved that for every subset $A\subset\mathbb{N}$ the sequence $\left(\sum_{j\in A} a_{i,j}\right)_i$ is (D)-Cauchy with respect to a same regulator $(d_{t,l})_{t,l}$, independent on the choice of A.

Now for every $i\in\mathbb{N}$ let us define $m_i:\mathcal{P}(\mathbb{N})\to R$ by setting $m_i(A):=\sum_{j\in A} a_{i,j}$, $A\in\mathcal{P}(\mathbb{N})$. By the above, the sequence $(m_i)_i$ is (RD)-Cauchy and by Remark 2.15 it is (RD)-convergent. By [5, Theorem 3.4], $(m_i)_i$ is (RO)-convergent to the same limit.

By unconditional convergence of the series $m_i(A):=\sum_{j\in A} a_{i,j}$, $i\in\mathbb{N}$, the m_i's are σ-additive, and σ-additivity can be intended with respect to the semivariation $v(\cdot)$.

Since, by hypothesis, the family $\left(\sum_{j\in A} a_{i,j}\right)_{i\in\mathbb{N}, A\subset\mathbb{N}}$ is (O)-equibounded, then the m_i's turn to be (O)-equibounded.

Thus, by virtue of the Schur theorem [7, Theorem 3.7], the m_i's turn to be uniformly σ-additive (with respect to the semivariation $v(\cdot)$). Hence, the limit measure m_0 is σ-additive too. and the sequence $(m_i)_i$ order converges uniformly to m_0. From this the assertion of the theorem follows.

We include here a new proof of this last fact.

We know from σ-additivity of m_i, $i \geqslant 0$, that to every disjoint sequence $(H_k)_k$ in $\mathcal{P}(\mathbb{N})$ there corresponds a regulator $(A_{t,l})_{t,l}$ such that for each map $\varphi : \mathbb{N} \to \mathbb{N}$ there is an integer k_0, such that

$$v(m_n)\left(\bigcup_{k\geqslant k_0} H_k\right) \leqslant \bigvee_{t=1}^{\infty} A_{t,\varphi(t)}$$

for all $n \in \mathbb{N} \cup \{0\}$. Choose now $H_k := \{k\}$, $k \in \mathbb{N}$, and let $(A_{t,l})_{t,l}$ be the corresponding (D)-sequence. Moreover, thanks to Lemma 2.2 there exists a regulator $(B_{t,l})_{t,l}$ such that, for every $h \in \mathbb{N}$ and $\varphi : \mathbb{N} \to \mathbb{N}$ an integer n_0 corresponds, such that

$$\sum_{q\leqslant h} |m_n(\{q\}) - m_0(\{q\})| \leqslant \bigvee_{t=1}^{\infty} B_{t,\varphi(t)}$$

whenever $n \geqslant n_0$. Setting $C_{t,l} = A_{t,l} \vee B_{t,l}$, $t, l \in \mathbb{N}$, we shall prove that for each $\varphi : \mathbb{N} \to \mathbb{N}$ there exists $n^* \in \mathbb{N}$ such that

$$|m_n(F) - m_0(F)| \leqslant 3 \bigvee_{t=1}^{\infty} C_{t,\varphi(t)}$$

for all $F \subset \mathbb{N}$ and $n \geqslant n^*$.
Indeed, fix $\varphi : \mathbb{N} \to \mathbb{N}$. By uniform σ-additivity, an integer k_0 corresponds, such that

$$v(m_n)(\{k_0, k_0+1, ...\}) \leqslant \bigvee_{t=1}^{\infty} A_{t,\varphi(t)}$$

for any $n \in \mathbb{N} \cup \{0\}$. Now, there is an integer n^*, such that

$$\sum_{q\leqslant k_0} |m_n(\{q\}) - m_0(\{q\})| \leqslant \bigvee_{t=1}^{\infty} B_{t,\varphi(t)}$$

holds, as soon as $n \geqslant n^*$. Thus, fixed arbitrarily $F \subset \mathbb{N}$, we have, for each $n \geqslant n^*$:

$$|m_n(F) - m_0(F)| \leqslant |m_n(F \cap \{1, ..., k_0\}) - m_0(F \cap \{1, ..., k_0\})| +$$
$$+ v(m_n)(F \cap \{k_0+1, k_0+2, ...\}) + v(m_0)(F \cap \{k_0+1, k_0+2, ...\}) \leqslant$$
$$\leqslant \bigvee_{t=1}^{\infty} B_{t,\varphi(t)} + 2\bigvee_{t=1}^{\infty} A_{t,\varphi(t)} \leqslant 3\bigvee_{t=1}^{\infty} C_{t,\varphi(t)}. \quad \square$$

In the following corollary, we employ a technique, involving fundamental properties of P-ideals and theorems on interchange of limits, used to prove our version of the Basic Matrix Theorem given in [6].

COROLLARY 3.3 *Under the same hypotheses and notations as in the previous theorem, let us replace iii) with the following condition:*
iii') $(a_{i,j})_i$ is $(O\mathcal{I})$-convergent for all $j \in \mathbb{N}$.

Then there exist $(a_j)_j$ in R and a set K belonging to the dual filter of $\mathcal{I}$ such that:

$$(D) \lim_{i \in K} [\vee_{j \in \mathbb{N}} |a_{i,j} - a_j|\,] = 0; \tag{17}$$

$$(D) \lim_{j \in \mathbb{N}} [\vee_{i \in K} |a_{i,j}|\,] = 0. \tag{18}$$

Moreover, if $\mathcal{J} \subset \mathcal{P}(\mathbb{N}^2)$ is the ideal of $\mathbb{N}^2$ generated by the finite unions of the Cartesian products of the elements of $\mathcal{I}$, then $(D\mathcal{J}) \lim_{i,j} a_{i,j} = 0$.

Proof: By Proposition 2.11 there exists a set K belonging to the dual filter $\mathcal{F}$ of $\mathcal{I}$ such that the double sequence $(a_{i,j})_{i \in K, j \in \mathbb{N}}$ satisfies iii) with respect to a same regulator. By proceeding analogously as in Theorem 3.2 and [6, Theorem 3.1], we get in particular the relations (17) and (18).

The last part of the thesis is a consequence of this and [6, Lemma 2.15]. ∎

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A. Boccuto
Department of Mathematics and Computer Sciences, University of Perugia
via Vanvitelli 1, I-06123 Perugia, Italy
E-mail: boccuto@yahoo.it, boccuto@dmi.unipg.it

X. Dimitriou
Department of Mathematics, University of Athens
Panepistimiopolis, Athens 15784, Greece
E-mail: x.dimitriou@yahoo.gr, dxenof@math.uoa.gr

N. Papanastassiou
Department of Mathematics, University of Athens
Panepistimiopolis, Athens 15784, Greece
E-mail: npapanas@math.uoa.gr

(Received: 23.08.2010)