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Quasi Locally Connected Spaces and Pseudo Locally Connected Spaces

Abstract. Two new generalizations of locally connected spaces called 'quasi locally connected spaces' and 'pseudo locally connected spaces' are introduced and their basic properties are studied. The class of quasi locally connected spaces properly contains the class of almost locally connected spaces (J. Austral. Math. Soc. 31(1981), 421–428) and is strictly contained in the class of pseudo locally connected spaces which in its turn is properly contained in the class of sum connected spaces (Math. Nachrichten 82(1978), 121-129; Ann. Acad. Sci. Fenn. A I Math. 3(1977), 185–205). Product and subspace theorems for quasi (pseudo) locally connected spaces are discussed. Their preservation under mappings and their interplay with mappings are outlined. Function spaces of quasi (pseudo) locally connected spaces are considered. Change of topology of a quasi (pseudo) locally connected space is considered so that it is simply a locally connected space in the coarser topology. In contradistinction with almost locally connected spaces, quasi (pseudo) locally connected spaces constitute a coreflective subcategory of TOP.

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1. Introduction. Vincent J. Mancuso [27] introduced and studied the class of almost locally connected spaces. Later on Noiri [33] obtained improvements of certain results of Mancuso. While the category of locally connected spaces constitutes a coreflective[1] subcategory of TOP ($\equiv$ the category of topological spaces and continuous maps) [5], the larger category of almost locally connected spaces is not a coreflective subcategory of TOP. So a natural question arises: Are there coreflective subcategories of TOP containing all almost locally connected spaces? An affirmative answer is provided by the known category of sum connected spaces [10]. In this

[1]For the definitions of coreflective subcategory and other categorical terms (see [7, 8]).

paper we introduce two generalizations of local connectedness called 'quasi (pseudo) local connectedness' such that the category of quasi (pseudo) locally connected spaces contains all almost locally connected spaces and is a coreflective subcategory of TOP. A regular quasi locally connected space as well as a D_δ-completely regular [17] pseudo locally connected space is locally connected. Product and subspace theorems for quasi (pseudo) locally connected spaces are discussed. It is shown that quasi local connectedness is preserved in the passage to θ-open subspaces and pseudo local connectedness is hereditary for regular F_s-sets. The interplay between quasi (pseudo) locally connected spaces and mappings is considered.

Organization of the paper is as follows: Section 2 is devoted to preliminaries, basic definitions and almost locally connected spaces. In Section 3 we introduce the notions of a 'quasi (pseudo) locally connected space'. Basic properties of quasi locally connected spaces are dealt with in Section 4 while Section 5 is devoted to the study of pseudo locally connected spaces. Section 6 is devoted to the interplay between quasi (pseudo) locally connected spaces and mappings. Function spaces of quasi (pseudo) locally connected spaces are considered in Section 7 and the conditions for their closedness/compactness in the topology of pointwise convergence are given. In Section 8 we consider change of topology of a quasi (pseudo) locally connected space such that it is just a locally connected space in the coarser topology.

2. Preliminaries, basic definitions and almost locally connected spaces. A subset A of a space X is said to be **regular open** if it is the interior of its closure, i.e., $A = \overline{A}^{\circ}$. The complement of a regular open set is referred to as a **regular closed** set. A union of regular open sets is called **δ-open** [44]. The complement of a δ-open set is referred to as a **δ-closed** set. A subset A of a space X is called **regular G_δ-set** [26] if A is an intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n^0$, where each F_n is a closed subset of X (here F_n^0 denotes the interior of F_n). The complement of a regular G_δ-set is called a **regular F_σ-set**. A subset A of a space X is called d_δ-open [14] if for each $x \in A$, there exists a regular F_σ-set H such that $x \in H \subseteq A$; equivalently A is expressible as a union of regular F_σ-sets. A point $x \in X$ is called a **θ-adherent point** [44] of $A \subset X$ if every closed neighbourhood of x intersects A. Let $\mathrm{cl}_\theta A$ denote the set of all θ-adherent points of A. The set A is called θ-closed if $A = \mathrm{cl}_\theta A$. The complement of a θ-closed set is referred to as a θ-open set.

Lemma 2.1 ([12, 13]) *A subset U of a space X is θ-open if and only if for each $x \in U$ there exists an open set V containing x such that $\overline{V} \subset U$.*

Definitions 2.2 A space X is said to be

(i) mildly compact [42][2] if every clopen cover of X has a finite subcover.

(ii) D_δ-completely regular ([16, 17]) if it has a base of regular F_σ-sets.

[2] Sostak [41] refers mildly compact spaces as clustered spaces.

(iii) D-regular [6] if it has a base of open F_σ-sets.

The class of D_δ-completely regular spaces properly contains the class of completely regular spaces and is strictly contained in each of the classes of D-regular spaces and regular spaces (see [17]).

Definitions 2.3 A space X is said to be

(i) almost locally connected [27] if for each $x \in X$ and each regular open set U containing x there exists a connected open set V containing x such that $V \subset U$.

(ii) sum connected [10] if each $x \in X$ has a connected neighbourhood, or equivalently each component of X is open.

The notion of almost local connectedness represents a generalization of local connectedness while the notion of sum connectedness represents a simultaneous generalization of connectedness as well as local connectedness. The category of sum connected spaces is precisely the coreflective hull of the category of connected spaces and contains all connected as well as all locally connected spaces. The disjoint topological sum of two copies of topologist's sine curve [43] is a sum connected space which is neither connected nor locally connected. Sum connected spaces have also been referred to as weakly locally connected spaces by some authors (see [28, 31]).

Definitions 2.4 A function $f : X \to Y$ from a topological space X into a topological space Y is said to be

(i) **strongly continuous** [24] if $f(\overline{A}) \subset f(A)$ for each subset A of X.

(ii) **perfectly continuous** ([34, 21]) if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.

(iii) **cl-supercontinuous** [40] ($\equiv$ **clopen continuous** [39]) if for each $x \in X$ and each open set V containing $f(x)$, there is a clopen set U containing x such that $f(U) \subset V$.

(iv) **connected** ($\equiv$ **Darboux**) [38] if $f(C)$ is connected in Y whenever C is connected in X.

(v) (**almost**) **strongly θ-continuous** ([36, 32]) if for each $x \in X$ and each (regular open) open set V containing $f(x)$ there is an open set U containing x such that $f(\overline{U}) \subset V$.

(vi) d_δ**-map** [15] if $f^{-1}(V)$ is a regular F_σ-set in X for every regular F_σ-set $V \subset Y$.

(vii) D_δ**-continuous** [16] if $f^{-1}(V)$ is open in X for every regular F_σ-set $V \subset Y$.

(viii) (**quasi**) D_δ**-supercontinuous** ([22, 14]) if $f^{-1}(V)$ is d_δ-open in X for every (θ-open) open set $V \subset Y$.

(ix) **faintly (quasi θ-)continuous** ([25, 37]) if for each $x \in X$ and each θ-open set V containing $f(x)$, there is an open (θ-open) set U containing x such that $f(U) \subset V$.

(x) **(almost)**[3] **completely continuous** ([2, 19]) if $f^{-1}(V)$ is regular-open in X for every (regular) open set $V \subset Y$.

(xi) **supercontinuous** [29] if for each $x \in X$ and each open set V containing $f(x)$, there is a regular open set U containing x such that $f(U) \subset V$.

(xii) δ-continuous [32] if for each $x \in X$ and each regular open set V containing $f(x)$ there is a regular open set U containing x such that $f(U) \subset V$.

Mancuso [27] showed that almost local connectedness is not preserved under quotient maps. However, almost local connectedness is preserved under quotient maps which are δ-continuous as enshrined in the following.

THEOREM 2.5 *Let $f : X \to Y$ be a δ-continuous, quotient map. If X is almost locally connected, then so is Y.*

PROOF Let $y \in Y$ and let V be a regular open set containing y. Since f is δ-continuous, $f^{-1}(V)$ is a δ-open set in X. Let $f^{-1}(V) = \bigcup_{\alpha \in \Lambda} U_\alpha$, where each U_α is a regular open set in X. Let $x \in f^{-1}(y)$. Then there exists $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$. In view of almost local connectedness of X, there exists an open connected set N_x such that $x \in N_x \subset U_{\alpha_0}$. Then $f(N_x)$ is a connected set such that $y \in f(N_x) \subset V$. Let $N = \cup\{N_x : x \in f^{-1}(y)\}$. Then $f^{-1}(y) \subset N \subset f^{-1}(V)$. Since f is a quotient map, $f(N)$ is a neighbourhood of y contained in V. The set $f(N)$ is connected being a union of connected sets having the common point y. Thus Y is locally connected. ■

COROLLARY 2.6 *Let $f : X \to Y$ be a quotient map which is almost completely continuous. If X is almost locally connected, then so is Y.*

PROOF Every almost completely continuous function is δ-continuous. ■

Moreover, if in addition quotient map is supercontinuous, then we have the following strong consequence.

THEOREM 2.7 *Let $f : X \to Y$ be a quotient map which is supercontinuous. If X is almost locally connected, then Y is locally connected.*

REMARK 2.8 The function f in Theorem 2.7 is precisely the δ-quotient map and Y is equipped with the δ-quotient topology (see [23, 29]). That is δ-quotient image of an almost locally connected space is locally connected.

[3]Carnahan [3] refers almost completely continuous functions as R-maps in his doctoral dissertation (also see [11, 35])

Corollary 2.9 *Let $f : X \to Y$ be a quotient map which is completely continuous. If X is almost locally connected, then Y is locally connected.*

Proof Every completely continuous function is supercontinuous. ■

3. Quasi (pseudo) locally connected spaces. A space X is said to be quasi (pseudo) locally connected at $x \in X$ if for each θ-open set (regular F_σ-set) U containing x there is an open connected set V such that $x \in V \subset U$. The space X is said to be quasi (pseudo) locally connected if it is quasi (pseudo) locally connected at each $x \in X$.

Remark 3.1 Every almost locally connected space is quasi locally connected. For if U is a θ-open set containing x. Then U is a δ-open set. Let $U = \cup V_\alpha$, where each V_α is a regular open set. So there exists an α_0 such that $x \in V_{\alpha_0}$. Since X is an almost locally connected space, there exists an open connected set V such that $x \in V \subset V_{\alpha_0} \subset U$. So X is quasi locally connected.

Again since every regular F_σ-set is θ-open, every quasi locally connected space is pseudo locally connected. Thus we have

$$\begin{array}{ccccc}
\text{locally connected} & \rightarrow & \text{almost locally connected} & \rightarrow & \text{quasi locally connected} \\
 & & & & \downarrow \\
 & & & & \text{pseudo locally connected} \\
 & & & & \downarrow \\
 & & & & \text{sum connected} \\
 & & & & (\equiv\text{weakly locally connected})
\end{array}$$

However, none of the above implications is reversible.

Mancuso gave an example of space X which is not almost locally connected. However the same space is quasi locally connected. For completeness and for clarity of presentation we include its details.

Example 3.2 ([27, Example 3.20]) Consider the space R of reals with countable complement extension topology τ [43, Example 63, p. 85]. Let $R \setminus Q$ be the quotient space obtained from (R, τ) by identifying the set Q of rationals to a point. Since Q is closed in (R, τ), the space $R \setminus Q$ is T_1 and the quotient map $p : (R, \tau) \to R \setminus Q$ is a closed map. The space $R \setminus Q$ is not almost locally connected as shown in [27] but $R \setminus Q$ is a quasi locally connected space, since $R \setminus Q$ is the only θ-open set in $R \setminus Q$ and $R \setminus Q$ is connected.

Proposition 3.3 *Every regular, quasi locally connected space is locally connected.*

Proof In view of Lemma 2.1 in a regular space every open set is θ-open. ■

PROPOSITION 3.4 *Every D_δ-completely regular, pseudo locally connected space is locally connected.*

PROOF This is immediate in view of the fact that in a D_δ-completely regular space regular F_σ-sets constitute a base for the topology. ■

PROPOSITION 3.5 *Every mildly compact, quasi (pseudo) locally connected space has at most finitely many components.*

PROOF Let X be a mildly compact quasi locally connected space. Since the space X is a θ-open subset of itself, its components are open and so clopen and constitute a partition of X. Since X is a mildly compact space it has at most finitely many components. ■

PROPOSITION 3.6 *Every quasi (pseudo) locally connected space is sum connected.*

PROOF This follows since a space X is a θ-open subset of itself and since in a quasi locally connected space, every θ-open set is expressible as the disjoint union of open connected sets. Thus every component in X is open and so clopen.

However, the converse of Proposition 3.6 is false. For example topologist's sine curve [43] is a connected space and hence sum connected but not pseudo locally connected. ■

PROPOSITION 3.7 *In a quasi (pseudo) locally connected space, components and quasicomponents coincide in every θ-open (regular F_σ) subset.*

PROOF Let X be a quasi (pseudo) locally connected space and let U be θ-open (regular F_σ) set in X. Then any component of U is open in X. Thus every quasicomponent of U, being union of components is open. Result follows since every open quasicomponent is a component. ■

4. Basic properties of quasi locally connected spaces.

THEOREM 4.1 *For a regular space X the following statements are equivalent.*

(a) *X is locally connected.*

(b) *X is almost locally connected.*

(c) *X is quasi locally connected.*

PROOF Since every regular space is semiregular, the equivalence of (a) and (b) is due to Mancuso [27]. The implication (b)⇒(c) is inherent in Remark 3.1. For (c)⇒(a): we need only note that in view of Lemma 2.1 it follows that in a regular space every open set is θ-open. ■

THEOREM 4.2 *A space X is quasi locally connected if and only if for every θ-open subset U of X, each component of U is open in X.*

PROOF Suppose that X is quasi locally connected. Let U be a θ-open set in X and let C be a component of U. Let $x \in C$. In view of quasi local connectedness of X there exists an open connected set V such that $x \in V \subset U$. Since V is connected, $V \subset C$, in view of maximality of C. Thus C is open being a neighbourhood of each of its points. Conversely, suppose that components of θ-open sets in X are open. Let U be a θ-open set in X containing x and let C be the component of x in U. By hypothesis C is an open connected set and so X is quasi locally connected at x. ∎

THEOREM 4.3 *Every θ-open subspace of a quasi locally connected space X is quasi locally connected.*

PROOF Let A be a θ-open subspace of a quasi locally connected space X and let B be a θ-open subset of A. We first show that B is θ-open in X. To this end, let $a \in B$. Since B is a θ-open set in A containing a, there exists an open set V in A such that $a \in V \subset \mathrm{cl}_A V \subset B$. Now, since A is open in X, V is also open in X. Again, since A is θ-open in X and $a \in A$, there exists an open set W in X such that $a \in W \subset \overline{W} \subset A$. Then $V \cap W$ is open in X and contains a. Since $V \cap W \subset W$, $\overline{V \cap W} \subset \overline{W} \subset A$ and hence $\overline{V \cap W} = \mathrm{cl}_A(V \cap W)$. Now, since $W \subset A$, $V \cap W \subset V \cap A = V$. This implies that $\mathrm{cl}_A(V \cap W) \subset \mathrm{cl}_A V$ and hence $\overline{V \cap W} \subset \mathrm{cl}_A V$. Therefore, $a \in V \cap W \subset \overline{V \cap W} \subset B$ and so B is θ-open in X. Now, since X is quasi locally connected, there exists a connected open set C containing a in X such that $C \subset B$. Clearly, C is a connected open subset of A and so A is quasi locally connected. ∎

REMARK 4.4 An open subspace of a quasi locally connected space need not be quasi locally connected. Consider the countable complement extension topology on R [43, Example 63, p. 85]. Then $R \setminus Q$ is an open subspace of R; however this subspace is not quasi locally connected.

THEOREM 4.5 *A Hausdorff space X is quasi locally connected if and only if every θ-open cover of X has a refinement consisting of open connected sets.*

PROOF Let X be a quasi locally connected space and let $\wp = \{V_\alpha : \alpha \in \Lambda\}$ be a θ-open cover of X. Since components of θ-open sets in a quasi locally connected space are open, components of members of $\wp$ constitute a refinement of $\wp$ consisting of open connected sets.

Conversely, let X be a Hausdorff space in which every θ-open cover of X has a refinement consisting of open connected sets. We shall show that X is quasi locally connected. Let $x \in X$ and let U be a θ-open set containing x. Since X is Hausdorff, $X \setminus \{x\}$ is a θ-open set. Then $\{U \cup \{X \setminus \{x\}\}\}$ is a θ-open cover of X. By hypothesis, it has a refinement $\{W_\alpha : \alpha \in \Lambda\}$ consisting of open connected sets. So there exists an $\alpha_0 \in \Lambda$ such that $x \in W_{\alpha_0} \subset U$. Hence X is a quasi locally connected space. ∎

LEMMA 4.6 *Every projection map carries θ-open sets to θ-open sets.*

PROOF Let $X = \prod X_\alpha$ and let $\Pi_\alpha : X \to X_\alpha$ be the projection map onto the α-th coordinate space X_α. Let U be a θ-open set in the product space X and let $x = (x_\alpha) \in U$. To prove that $\Pi_\alpha(U)$ is θ-open, let $x_\alpha \in \Pi_\alpha(U)$. Then there is a point $y \in U$ such that $\Pi_\alpha(y) = x_\alpha$. Since U is θ-open, by Lemma 2.1 there exists an open set and hence a basic open set V containing y such that $\overline{V} \subset U$. Then $x_\alpha \in \Pi_\alpha(V) \subset \Pi_\alpha(\overline{V}) \subset \Pi_\alpha(U)$. Again since V is a basic open set in X, $\Pi_\alpha(\overline{V}) = \overline{\Pi_\alpha(V)}$ and so $x_\alpha \in \Pi_\alpha(V) \subset \overline{\Pi_\alpha(V)} \subset \Pi_\alpha(U)$. Again in view of Lemma 2.1, $\Pi_\alpha(U)$ is θ-open. ∎

In the proof of following theorem we use the fact that a continuous open image of a quasi locally connected space is quasi locally connected (Corollary 6.2).

THEOREM 4.7 *Let $\{X_\alpha : \alpha \in \Lambda\}$ be any collection of quasi locally connected spaces. Then their product $X = \prod X_\alpha$ is quasi locally connected if and only if all except finitely many spaces are connected.*

PROOF Suppose that the product space $X = \prod X_\alpha$ is quasi locally connected. Since projection maps are continuous open maps and a continuous open image of a quasi locally connected space is quasi locally connected, each X_α is quasi locally connected. Again, since if V is any connected open set in X, then $\Pi_\alpha(V) = X_\alpha$ for all but at most finitely many α, and so all except finitely many X_α are connected.

Conversely, let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of quasi locally connected spaces such that each X_α is connected except for $\alpha \neq \alpha_1, \ldots, \alpha_n$ and let $X = \prod X_\alpha$ be the product space. Let $S = \{\alpha_1, \ldots, \alpha_n\}$. For each $\alpha \in S$, let U_α be a connected open subset of X_α. Let $x = (x_\alpha) \in X$ and let G be a θ-open set in X containing x. Then

$$\Pi_\alpha(G) = \begin{cases} X_\beta, & \text{except for finitely many} \quad \beta \neq \beta_1, \ldots, \beta_m \\ G_\beta, & \text{otherwise}. \end{cases}$$

Let $T = \{\beta_1, \ldots, \beta_m\}$. In view of Lemma 4.6 each G_β is θ-open. Again since each X_β is quasi locally connected, there exists a connected open set V_β such that $x_\beta \in V_\beta \subset G_\beta$. Now for each $\alpha \in \Lambda$, let

$$W_\alpha = \begin{cases} X_\alpha & \text{if} \quad \alpha \notin S \cup T \\ U_\alpha & \text{if} \quad \alpha \in S \\ V_\alpha & \text{if} \quad \alpha \in T \setminus S \end{cases}$$

Then $W = \prod W_\alpha$ is a connected open set containing x such that $W \subset G$. Thus the product space $X = \prod X_\alpha$ is quasi locally connected. ∎

THEOREM 4.8 *Disjoint topological sum of any family of quasi locally connected spaces is quasi locally connected.*

Proof Let $\{X_\alpha : \alpha \in \Lambda\}$ be any family of quasi locally connected spaces and let $X = \oplus_{\alpha\in\Lambda} X_\alpha$ denote their disjoint topological sum. Let U be any θ-open set in X and let $x \in U$. Then $x \in U \cap X_\alpha$ for some $\alpha \in \Lambda$. In view of Lemma 2.1 it is easily verified that $U \cap X_\alpha$ is a θ-open set in X_α. Since X_α is quasi locally connected, there exists a connected open set C in X_α containing x such that $C \subset U \cap X_\alpha$. Clearly C is a connected open set in X and so X is quasi locally connected. ■

Theorem 4.9 *If X is a connected quasi locally connected space and if C is a component of a θ-open set in X such that $X \setminus \overline{C}$ is nonempty, then $\overline{C} \setminus C$ is not empty and separates C and $X \setminus \overline{C}$ in X.*

Proof If $\overline{C} \setminus C$ is empty, then C is closed. By Theorem 4.2 C is open and so $X \setminus \overline{C} = X \setminus C$ is a nonempty proper clopen set in X contradicting the fact that X is connected. Thus $\overline{C} \setminus C$ is nonempty. Since $X \setminus (\overline{C} \setminus C) = C \cup (X \setminus \overline{C})$, it follows that C and $X \setminus \overline{C}$ constitute a partition of $X \setminus (\overline{C} \setminus C)$. ■

5. Properties of pseudo locally connected spaces. Combining Theorem 4.1 and Proposition 3.4 we have the following.

Theorem 5.1 *For a D_δ-completely regular space X the following statements are equivalent.*

(a) *X is locally connected.*
(b) *X is almost locally connected.*
(c) *X is quasi locally connected.*
(d) *X is pseudo locally connected.*

Theorem 5.2 *Every regular F_σ-subspace of a pseudo locally connected space X is pseudo locally connected.*

Proof Let A be a regular F_σ-subspace of a pseudo locally connected space X. Then $A = \bigcup_n U_n = \bigcup_n \overline{U}_n$, where each U_n is open in X. Let B be a regular F_σ-set in A. Then $B = \bigcup_n V_n = \bigcup_n \mathrm{cl}_A V_n$, where each V_n is open in A. Let α be the collection of all sets of the form $U_k \cap V_n$. Then $B = \cup\{W : W \in \alpha\}$. Now each $U_k \cap V_n$ is open in X since A is open in X. Thus α is a countable collection of open sets in X. Now each W in α is such that $W \subset U_k$ for some k and $W \subset V_n$ for some n and so $\overline{W} \subset \overline{U}_k \subset A$. Therefore, $\overline{W} = \mathrm{cl}_A W \subset \mathrm{cl}_A V_n$. This means that $\bigcup_{W\in\alpha} \overline{W} \subset \bigcup_n \mathrm{cl}_A V_n = B$. Thus, B is a regular F_σ-set in X. Let $x \in B$. Since X is pseudo locally connected, there exists a connected open set V containing x and contained in B. Then V is a connected open set in A and it is contained in B. So A is pseudo locally connected. ■

Theorem 5.3 *Disjoint topological sum of any family of pseudo locally connected spaces is pseudo locally connected.*

6. Preservation/Interplay with mappings.

THEOREM 6.1 *Every quotient of a quasi (pseudo) locally connected space is quasi locally (pseudo) connected.*

PROOF We shall prove the result only in case of quasi local connectedness. To this end, let $f : X \to Y$ be a quotient map from a quasi locally connected space X onto Y. To prove that Y is quasi locally connected, let V be θ-open set in Y. Then $f^{-1}(V)$ is a θ-open set in X. Let C be a component of V. It suffices to show that C is open in Y or equivalently $f^{-1}(C)$ is open in X. To this end, let $x \in f^{-1}(C)$, and let C_x be the component of x in $f^{-1}(V)$. Since $f(C_x)$ is a connected set containing $f(x)$, $f(C_x) \subset C$ and so $x \in C_x \subset f^{-1}(C)$. Since X is quasi locally connected, and since $f^{-1}(V)$ is θ-open, by Theorem 4.2 C_x is open in X and so $f^{-1}(C)$ is open being a neighbourhood of each of its points. ■

COROLLARY 6.2 *An open (closed) continuous image of a quasi (pseudo) locally connected space is quasi (pseudo) locally connected.*

PROOF Every open(closed) continuous surjection is a quotient map. ■

THEOREM 6.3 *Let $f : X \to Y$ be an open, connected, quasi θ-continuous (d_δ-map) function onto Y. If X is quasi (pseudo) locally connected, then so is Y.*

PROOF Let $y \in Y$ and let U be a θ-open set in Y containing y. Since f is quasi θ-continuous, $f^{-1}(U)$ is θ-open in X. Again, since X is quasi locally connected, for each $x \in f^{-1}(y)$ there exists a connected open set $N(x)$ of x such that $N(x) \subset f^{-1}(U)$. Since f is a connected open map, $f(N(x))$ is an open connected set containing y which is contained in U. So Y is quasi locally connected. ■

THEOREM 6.4 *Let $f : X \to Y$ be an open, connected, faintly continuous function from a locally connected space X onto Y. Then Y is quasi locally connected.*

PROOF Let $y \in Y$ and let V be a θ-open set containing y. Since f is faintly continuous, $f^{-1}(V)$ is open in X. Let $x \in f^{-1}(y)$, since X is locally connected, there exists an open connected set U_x such that $x \in U_x \subset f^{-1}(V)$. Then $f(U_x)$ is an open connected set containing y and contained in V. So Y is quasi locally connected. ■

THEOREM 6.5 *Let $f : X \to Y$ be a quotient map which is strongly θ-continuous. If X is quasi locally connected, then Y is locally connected.*

PROOF Let $y \in Y$ and let V be an open set containing y. In view of strong θ-continuity of f, it is easily verified that $f^{-1}(V)$ is a θ-open set in X. Since X is quasi locally connected, for each $x \in f^{-1}(y)$ there exists a connected open set U_x containing x such that $U_x \subset f^{-1}(V)$. Then each $f(U_x)$ is a connected set containing y. Let $N = \cup\{f(U_x) : x \in f^{-1}(y)\}$. Then $N \subset V$ and is a connected set being the union of connected sets having the common point y. Since f is a quotient map, N is a connected neighbourhood of y and so Y is locally connected. ■

Remark 6.6 The function $f : X \to Y$ in above theorem is precisely the θ-quotient map and Y is endowed with the θ-quotient topology (see [23]). That is θ-quotient image of a quasi locally connected space is locally connected.

We omit proof of following theorem which is similar to that of Theorem 6.5.

Theorem 6.7 *Let $f : X \to Y$ be an open, connected, almost strongly θ-continuous function from a quasi locally connected space X onto Y. Then Y is almost locally connected.*

Theorem 6.8 *Let $f : X \to Y$ be a quotient map which is D_δ-supercontinuous function from a pseudo locally connected space X onto Y. Then Y is locally connected.*

Proof Let $y \in Y$ and let V be an open set containing y. Since f is D_δ-supercontinuous, $f^{-1}(V)$ is a d_δ-open set in X. Let $f^{-1}(V) = \cup U_\alpha$, where each U_α is a regular F_σ-set in X. Then each $x \in f^{-1}(y)$ is contained in some U_{α_x}. Since X is pseudo locally connected, there exists an open connected set N_x containing x such that $N_x \subset U_{\alpha_x}$. Then $f(N_x)$is a connected set containing y. Let $N = \cup\{f(N_x) : x \in f^{-1}(y)\}$. Then N is connected being a union of connected sets having the common point y such that $N \subset V$. Again, since f is a quotient map, N is a connected neighbourhood of y and so Y is locally connected. ∎

Remark 6.9 The function $f : X \to Y$ in Theorem 6.8 is precisely the D_δ-quotient map and Y is equipped with the D_δ-quotient topology (see [14, 23]). That is D_δ-quotient image of a pseudo locally connected space is locally connected.

Theorem 6.10 *Let $f : X \to Y$ be an open, connected, D_δ-continuous function from a locally connected space X onto Y. Then Y is pseudo locally connected.*

Theorem 6.11 *Let $f : X \to Y$ be an open, connected, quasi D_δ-supercontinuous function from a pseudo locally connected space X onto Y. Then Y is quasi locally connected.*

Proof Let $y \in Y$ and let V be a θ-open set containing y. Since f is quasi D_δ-supercontinuous, $f^{-1}(V)$ is a d_δ-open set in X. Let $f^{-1}(V) = \cup U_\alpha$, where each U_α is a regular F_σ-set in X. Then each $x \in f^{-1}(y)$ is contained in some U_{α_x}. Since X is pseudo locally connected, there exists an open connected set N_x containing x such that $N_x \subset U_{\alpha_x}$. Then $f(N_x)$ is an open connected set containing y and contained in V. So Y is quasi locally connected. ∎

Since co-products and extremal quotient objects in TOP are disjoint topological sums and quotient maps, respectively a characterization of co-reflective subcategories of TOP ([7, Theorem 6]) together with Theorems 4.8, 5.3 and 6.1 yields the following.

THEOREM 6.12 *The full subcategory of quasi (pseudo) locally connected spaces is a coreflective subcategory of TOP containing the full subcategory of almost locally connected spaces.*

7. Function spaces and quasi (pseudo) locally connected spaces. It is well known that in general, $C(X,Y)$ the set of all continuous functions from a space X into a space Y is not necessarily closed in Y^X in the topology of pointwise convergence. However, it is of considerable significance both from theoretical considerations as well as from applications view point to formulate conditions on the spaces X, Y and subsets of $C(X,Y)$ to be closed/compact in Y^X in the topology of pointwise convergence. Results of this nature and Ascoli type theorems abound in the literature (see [1, 9]). In particular, Naimpally [30] showed that if X is locally connected and Y is Hausdorff, then the set $S(X,Y)$ of all strongly continuous functions from X into Y is closed in Y^X in the topology of pointwise convergence. It seems natural to ask: Is Naimpally's result still true if X is quasi locally/pseudo locally connected instead of locally connected? This question bears an affirmative answer as will become clear in the sequel.

In [18] Naimpally's result is extended to a larger framework wherein it is shown that if X is sum connected and Y is Hausdorff, then the set $P(X,Y)$ of all perfectly continuous functions as well as $L(X,Y)$ the set of all cl-supercontinuous functions from X into Y equals $S(X,Y)$ and is closed in Y^X in the topology of pointwise convergence.

In view of Proposition 3.6 every quasi (pseudo) locally connected space is sum connected. By [18, Theorem 3.7 and Proposition 3.8] we conclude with the following.

COROLLARY 7.1 *If X is a quasi (pseudo) locally connected space and Y is Hausdorff, then $S(X,Y) = P(X,Y) = L(X,Y)$ is closed in Y^X in the topology of pointwise convergence. Further, if in addition Y is compact, then $S = P = L$ is a compact Hausdorff space in the topology of pointwise convergence.*

8. Change of topology. In this section we show that if the topology of a quasi (pseudo) locally connected space is changed in an appropriate way then it is simply a locally connected space.

8.1 Let (X,τ) be a topological space. Let τ_θ denote the collection of all θ-open subsets of the space (X,τ). Since arbitrary union and finite intersections of θ-open sets are θ-open (see [44]), the collection τ_θ is a topology for X. The topology τ_θ has been extensively referred to in the literature (see [4, 25, 20]).

8.2 Let (X,τ) be a topological space and let β_{d_δ} denote the collection of all regular F_σ-subsets of (X,τ). Since the intersection of two regular F_σ-sets is regular F_σ, the collection β_{d_δ} is a base for a topology τ_* on X. The topology τ_* has been described and used in ([14, 20]) and therein it has been denoted by τ_{d_δ}. In general $\tau_* \subset \tau_\theta \subset \tau$. However, in general none of the above inclusions is reversible (see [20]).

Observations

8.3 The spaces (X, τ), (X, τ_θ) and (X, τ_*) have same classes of clopen sets.

Proof clopen $\Rightarrow$ regular F_σ $\Rightarrow$ θ-open $\Rightarrow$ open. ■

8.4 Either all the three spaces (X, τ), (X, τ_θ) and (X, τ_*) are connected or all three are disconnected.

Throughout the section, for a topological space (X, τ) the symbols τ, τ_θ and τ_* will have the same meaning as in the above paragraphs.

Lemma 8.5 *In a topological space (X, τ) if a set U is θ-open and τ-disconnected, then U is τ_θ-disconnected.*

Proof Let U be a θ-open and τ-disconnected subset of (X, τ). Let $U = G \cup H$, where G and H are nonempty disjoint τ-open subsets of X such that neither G contains any τ-limit point of H nor H contains any τ-limit point of G. Let $x \in G \subset U$. Since U is a θ-open subset of (X, τ), by Lemma 2.1 there exists a τ-open set V such that $x \in V \subset \mathrm{cl}_\tau V \subset U$. Then $G \cap V$ is a τ-open set such that $x \in G \cap V \subset \mathrm{cl}_\tau(G \cap V) \subset \mathrm{cl}_\tau G$. Again, since $\mathrm{cl}_\tau(G \cap V) \subset \mathrm{cl}_\tau V \subset U$, $\mathrm{cl}_\tau(G \cap V) \subset \mathrm{cl}_\tau G \cap U = G$. This shows that G is θ-open. Similarly H is θ-open and so U is τ_θ-disconnected. ■

Theorem 8.6 *The space (X, τ) is quasi locally connected if and only if the space (X, τ_θ) is locally connected.*

Proof To prove necessity suppose that (X, τ) is quasi locally connected. Let U be a τ_θ-open set and let C be a τ_θ-component of U. Let $x \in C$. Then $x \in U$. Since (X, τ) is quasi locally connected and since U is a θ-open in (X, τ), there exists a τ-open, τ-connected set V in X such that $x \in V \subset U$. Then V is τ_θ-connected and so $V \subset C$. Since U is a θ-open set in (X, τ) in view of Lemma 2.1 there exists a τ-open set W in X such that $x \in W \subset \mathrm{cl}_\tau W \subset U$. Then $V \cap W$ is a τ-open set containing x and $V \cap W \subset C$ So $\mathrm{cl}_\tau(V \cap W) \subset \mathrm{cl}_\tau W \subset U$. Thus $\mathrm{cl}_\tau(V \cap W) \subset \mathrm{cl}_\tau C \cap U \subset \mathrm{cl}_{\tau_\theta} C \cap U \subset C$. Again in view of Lemma 2.1 it follows that C is a θ-open set in (X, τ). Hence the space (X, τ_θ) is locally connected.

To prove sufficiency, suppose that the space (X, τ_θ) is locally connected. Let $x \in X$ and let U be a θ-open in (X, τ) containing x. By local connectedness of (X, τ_θ), there exists a τ_θ-open, τ_θ-connected set V such that $x \in V \subset U$. Then V is a θ-open set in (X, τ) containing x. By Lemma 8.5, V is τ-connected and so (X, τ) is quasi locally connected. ■

Lemma 8.7 *In a topological space (X, τ) if a set U is τ_*-open and τ-disconnected, then U is τ_*-disconnected.*

Proof Let U be a τ_*-open and τ-disconnected set in X. Let $U = G \cup H$, where G and H are nonempty disjoint τ-open subsets of X such that neither G contains any τ-limit point of H nor H contains any τ-limit point of G. Let $x \in G$. Then $x \in U$ and since U is τ_*-open, there exists a regular F_σ-set W in (X, τ) such that $x \in W \subset U$. Then $W = \bigcup_{i=1}^\infty W_i = \bigcup_{i=1}^\infty \mathrm{cl}_\tau W_i$, where each W_i is τ-open. So $W \cap G = \bigcup_{i=1}^\infty (W_i \cap G) \subseteq \bigcup_{i=1}^\infty \mathrm{cl}_\tau (W_i \cap G) \subset \bigcup_{i=1}^\infty (\mathrm{cl}_\tau W_i \cap \mathrm{cl}_\tau G) = \left(\bigcup_{i=1}^\infty \mathrm{cl}_\tau W_i \right) \cap \mathrm{cl}_\tau G = W \cap \mathrm{cl}_\tau G = W \cap G$. The equality holds, since every τ-limit point of G which is in U must be in G. This proves $W \cap G = \bigcup_{i=1}^\infty (W_i \cap G) = \bigcup_{i=1}^\infty \mathrm{cl}_\tau (W_i \cap G)$. Thus $W \cap G$ is a regular F_σ-set in (X, τ) and so G is τ_*-open. Similarly H is τ_*-open and thus U is τ_*-disconnected. ∎

Theorem 8.8 *A space (X, τ) is pseudo locally connected if and only if the space (X, τ_*) is locally connected.*

Proof Suppose (X, τ) be a pseudo locally connected space. Let $x \in X$ and let U be a τ_*-open set containing x. Let C be the τ_*-component of U containing x. Since U is a τ_*-open set, there exists a regular F_σ-set W in (X, τ) such that $x \in W \subset U$. Again, since (X, τ) is pseudo locally connected, there exists a τ-open, τ-connected set V such that $x \in V \subset W$. Consequently V is τ_*-connected and so $V \subset C$. Now $W = \bigcup_{i=1}^\infty W_i = \bigcup_{i=1}^\infty \mathrm{cl}_\tau W_i$, where each W_i is τ-open. Further, $W \cap C = \bigcup_{i=1}^\infty (W_i \cap C) \subset \bigcup_{i=1}^\infty \mathrm{cl}_\tau (W_i \cap C) \subset \bigcup_{i=1}^\infty (\mathrm{cl}_\tau W_i \cap \mathrm{cl}_\tau C) = \left(\bigcup_{i=1}^\infty \mathrm{cl}_\tau W_i \right) \cap \mathrm{cl}_\tau C = W \cap \mathrm{cl}_\tau C \subset W \cap \mathrm{cl}_{\tau^*} C = W \cap C$. Thus $W \cap C = \bigcup_{i=1}^\infty (W_i \cap C) = \bigcup_{i=1}^\infty \mathrm{cl}_\tau (W_i \cap C)$ and so $W \cap C$ is a regular F_σ-set in (X, τ). So C is τ_*-open and consequently (X, τ_*) is locally connected.

To prove sufficiency, suppose that the space (X, τ_*) is locally connected. Let $x \in X$ and let U be a regular F_σ-set in (X, τ) containing x. Then U is τ_*-open. By local connectedness of (X, τ_*) there exists a τ_*-open, τ_*-connected set V such that $x \in V \subset W$. Since $\tau_* \subset \tau$, V is τ-open. By Lemma 8.7 V is τ-connected and so (X, τ) is quasi locally connected. ∎

9. Open questions.

9.1 Give an example of a pseudo locally connected space which is not quasi locally connected.

9.2 Formulate and prove a product theorem for pseudo locally connected spaces similar to Theorem 4.7.

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