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A survey on Lipschitz-free Banach spaces

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Summary. This article is a survey of Lipschitz-free Banach spaces and recent progress in the understanding of their structure. The results we present have been obtained in the last fifteen years (and quite often in the last five years). We give a self-contained presentation of the basic properties of Lipschitz-free Banach spaces and investigate some specific topics: non-linear transfer of asymptotic smoothness, approximation properties, norm-attainment. Section 5 consists mainly of unpublished results. A list of open problems with comentary is provided.

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1. Introduction

Let M be a metric space equipped, for convenience, with a distinguished point 0. The space $\mathrm{Lip}_0(M)$ of real-valued Lipschitz functions defined on M which vanish at 0 is a Banach space whose unit ball is compact in the topology of pointwise convergence on M. Therefore, $\mathrm{Lip}_0(M)$ is isometric to a dual space. The corresponding predual is the norm-closed subspace of $\mathrm{Lip}_0(M)^*$ generated by the Dirac measures. This space, which has been investigated for many years in the important book [60], and in some subsequent works (see, for instance, the early works of J. Johnson [38–40]), is called the Arens–Eels space over M. Today it is usually called, following [28], the Lipschitz-free space over M and denoted $\mathcal{F}(M)$.

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A simple diagram-chasing argument shows that Lipschitz-free spaces enjoy a canonical (and useful) linearization property: every Lipschitz map between metric spaces extends to a linear continuous operator between the corresponding free spaces (see section 2 below). We can expect that this universal property yields to an intricate structure of these spaces, and, indeed, the free spaces constitute a nice collection of Banach spaces which are very easy to define but difficult to analyse. Moreover, the arrow $M \to \mathcal{F}(M)$ carries the diversity of (separable) metric spaces over to (separable) Banach spaces, and although this map is not one-to-one (see [18]), we can expect to meet all kinds of free spaces. Understanding these spaces is a fascinating research program and we hope that this survey will motivate and help those who feel like contributing to this topic.

We refer to [5] as an authoritative book on non-linear geometry of Banach spaces as it was in 2000. The second edition of the book [1] contains an updated chapter on this topic, and we refer also to [31] for a recent survey which focuses mainly on the contribution of Nigel Kalton (1946–2010). The present survey is by no means exhaustive and significant articles are not mentioned in the bibliography. However, I still hope that this bibliography contains most names of the recent contributors and thus the readers should be able to gather updated knowledge of the frontline research by using it.

2. Lipschitz-free spaces: definitions and basic properties

We recall in this section the basic properties of Lipschitz-free Banach spaces and the notation which is used in this paper and in most recent articles on this topic. Our main reference is the article [28].

Lipschitz-free spaces. Let M be a pointed metric space, that is, a metric space equipped with a distinguished point denoted 0. The space $\mathrm{Lip}_0(M)$ is the space of real-valued Lipschitz functions on M which vanish at 0. When equipped with the Lipschitz norm defined by

$$\|f\|_L = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} \ : \ x \neq y \in M\right\}$$

the space $\mathrm{Lip}_0(M)$ becomes a Banach space whose dual contains in particular the Dirac measure $\delta(x)$ at any point $x \in M$.

The Dirac map $\delta\colon M \to \mathrm{Lip}_0(M)^*$ (denoted δ_M when necessary) defined by the formula $\langle g, \delta(x)\rangle = g(x)$ is an isometric embedding from M to a subset of $\mathrm{Lip}_0(M)^*$. Indeed, it follows from the definition of the dual norm that δ is 1-Lipschitz. On the other hand, if we define

$$f_x(z) = d(z,x) - d(x,0)$$

then $f_x \in \mathrm{Lip}_0(M)$, f_x is 1-Lipschitz, and $f_x(y) - f_x(x) = d(x, y)$. Hence δ is indeed an isometry.

We denote by $\mathcal{F}(M)$ the norm-closed linear span of $\delta(M)$ in the dual space $\mathrm{Lip}_0(M)^*$. This space is called in [28] the *Lipschitz-free space over M* and we will keep this notation in this survey. The definition of the Lipschitz norm shows that $\mathcal{F}(M)$ is a norming subspace of $\mathrm{Lip}_0(M)^*$. But actually more is true: if Ψ is a continuous linear form on $\mathcal{F}(M)$, then the function $g = \Psi \circ \delta : M \to \mathbb{R}$ is Lipschitz and satisfies $\langle \mu, g \rangle = \Psi(\mu)$ for every $\mu \in \mathcal{F}(M)$. Therefore, the Lipschitz-free space $\mathcal{F}(M)$ is an isometric predual of the space $\mathrm{Lip}_0(M)$ whose w^*-topology coincides on the unit ball of $\mathrm{Lip}_0(M)$ with the pointwise convergence on M.

It is clear that when M is separable, $\mathcal{F}(M)$ is separable as well since $\delta(M)$ spans a dense subspace of that space. We should mention that although Lipschitz-free spaces over separable metric spaces constitute a class of separable Banach spaces which are easy to define, the structure of these spaces is very poorly understood to this day. Improving our understanding of this class is a fascinating research program.

The free spaces provide a canonical linearization process: if we identify (through the Dirac map) a metric space M with a subset of $\mathcal{F}(M)$, any Lipschitz map F from M to a metric space N which maps 0 to 0 extends to a continuous linear map from $\mathcal{F}(M)$ to $\mathcal{F}(N)$. Indeed the composition map

$$C_F(g) = g \circ F$$

maps $\mathrm{Lip}_0(N)$ to $\mathrm{Lip}_0(M)$, therefore $(C_F)^* : \mathrm{Lip}_0(M)^* \to \mathrm{Lip}_0(N)^*$; and if we call $\widehat{F}$ the restriction of $(C_F)^*$ to $\mathcal{F}(M)$, then $\widehat{F}$ maps $\mathcal{F}(M)$ to $\mathcal{F}(N)$ and satisfies $\widehat{F} \circ \delta_M = \delta_N \circ F$ (where δ_E denotes the Dirac map associated with the metric space E). This linearization procedure applies, in particular, to canonical injections: if A is a metric space and B is a non-empty subset of A, real-valued Lipschitz functions on B can be extended to Lipschitz functions on A with the same Lipschitz constant by an inf-convolution formula. Namely, if $f : B \to \mathbb{R}$ is L-Lipschitz, then the formula

$$\overline{f}(a) = \inf\{f(b) + Ld(a, b) \,:\, b \in B\},$$

which goes back to Mac Shane [54], defines an L-Lipschitz function $\overline{f}$ on A which extends f. It follows that if $j : B \to A$ is the canonical injection, then $\widehat{j} : \mathcal{F}(B) \to \mathcal{F}(A)$ is a linear isometric injection. We can, therefore, identify $\mathcal{F}(B)$ with a subspace of $\mathcal{F}(A)$.

So if we move up to the free spaces, the Lipschitz maps become linear but of course the structure is shifted from the map to the space: when the arrows become simple, the sets on which they act bear the complexity. This may explain why the structure of Lipschitz-free spaces is not easy to analyze.

2.1. Example. The first example is provided by the real line whose free space is isometric to L_1. To check this, observe that every Lipschitz function f from $\mathbb{R}$ to $\mathbb{R}$ is differentiable almost everywhere and, moreover,

$$f(x) - f(0) = \int_0^x f'(t)dt.$$

Hence the map $D(f) = f'$ induces an isometry from $\mathrm{Lip}_0(\mathbb{R})$ onto $L_\infty(\mathbb{R})$, and so their (unique) isometric preduals $\mathcal{F}(\mathbb{R})$ and $L_1(\mathbb{R})$ are isometric as well. More precisely, the map $J(\delta(x)) = \mathbb{1}_{[0,x]}$ extends to an isometry from $\mathcal{F}(\mathbb{R})$ onto $L_1(\mathbb{R})$.

Actually, metric spaces M whose free spaces are isometric to a subspace of L_1 are characterized in [24] as subsets of metric trees equipped with the shortest path metric. On the other hand, the free space of the plane $\mathbb{R}^2$ does not embed isomorphically into L_1 [55]. Note that the Lipschitz-free spaces are often called Arens–Eells spaces and that they can be interpreted in terms of the "transportation cost" (see [60]).

The free space of a Banach space. Banach spaces are, in particular, pointed metric spaces (pick the origin as the distinguished point) and we can apply the previous construction. Note that the isometric embedding $\delta: X \to \mathcal{F}(X)$ is, of course, non-linear since there exist Lipschitz functions on X which are not affine.

This Dirac map has a linear left inverse $\beta: \mathcal{F}(X) \to X$ (denoted β_X when necessary) which is the quotient map such that $x^*(\beta(\mu)) = \langle x^*, \mu \rangle$ for all $x^* \in X^*$; that is, $\beta(\mu)$ is the restriction of μ to the subspace X^* of $\mathrm{Lip}_0(X)$ and this restriction belongs to X since it trivially does for the dense subspace of $\mathcal{F}(X)$ spanned by $\delta(X)$. In other words, β is the extension to $\mathcal{F}(X)$ of the barycenter map.

The barycenter map β provides an extension result which explains the terminology "free space" by analogy to free groups. Indeed, if $F: M \to X$ is a Lipschitz map from a metric space M to a Banach space X such that $F(0) = 0$, then the linear map $\overline{F} = \beta_X \circ \widehat{F}$ has norm equal to the Lipschitz constant of F and satisfies $\overline{F} \circ \delta_M = F$. That is, Banach space-valued Lipschitz maps defined on a metric space M extend canonically to bounded linear maps on $\mathcal{F}(M)$.

Following [28], let us say that a Banach space X has the *lifting property* if there is a continuous linear map $R: X \to \mathcal{F}(X)$ such that $\beta R = \mathrm{Id}_X$, or, equivalently, if for Banach spaces Y and Z and continuous linear maps $S: Z \to Y$ and $T: X \to Y$, the existence of a Lipschitz map $\mathcal{L}$ such that $T = S\mathcal{L}$ implies the existence of a continuous linear operator L such that $T = SL$. Indeed, the operator $L = \beta_Z \widehat{\mathcal{L}} R$ does the job in this case. We note that $\|L\| \leqslant \|R\| \|\mathcal{L}\|_L$. In all examples where a linear section $R: X \to \mathcal{F}(X)$ has been shown to exist, we have moreover that $\|R\| = 1$. Hence the *isometric lifting property* holds true, where this notation means that a linear operator can be found with $\|L\| = \|\mathcal{L}\|_L$.

This setting provides canonical examples of Lipschitz-isomorphic spaces. Indeed, if we let $Z_X = Ker(\beta_X)$, it follows easily from $\beta_X \delta_X = \mathrm{Id}_X$ that the space $Z_X \oplus X = \mathcal{G}(X)$ is Lipschitz-isomorphic to $\mathcal{F}(X)$. The linear map

$$\widehat{\delta_X}: \mathcal{F}(X) \to \mathcal{F}(\mathcal{F}(X))$$

is a linear lifting of the quotient map $\beta_{\mathcal{F}(X)}$ and thus, for any Banach space X, the free space $\mathcal{F}(X)$ over X has the lifting property. Since the lifting property passes over to complemented subspaces and X is complemented in $\mathcal{G}(X)$, it follows that $\mathcal{G}(X)$ is linearly isomorphic to $\mathcal{F}(X)$ if and only if X has the lifting property [28]. Therefore, any Banach space X which fails to have the lifting property provides a couple of spaces (namely $\mathcal{F}(X)$ and $\mathcal{G}(X)$) which are Lipschitz-isomorphic but not linearly isomorphic. It turns out (see [28]) that all non-separable reflexive spaces, including the spaces $\ell_\infty(\mathbb{N})$ and $c_0(\Gamma)$ when Γ is uncountable, fail to have the lifting property and this provides canonical examples of pairs of Lipschitz-isomorphic but not linearly isomorphic spaces. However, we will see below that this technique fails to provide a pair of separable examples.

We should mention at this stage that the first couple of Lipschitz- but not linearly isomorphic Banach spaces was discovered in [2]. It was shown later ([15], see [16, Theorem VI.8.9]) that if K is a scattered compact space with a finite Cantor derivation index, then $C(K)$ is Lipschitz-isomorphic to the space $c_0(\Gamma)$ of the same density character. The proof relies on the existence of Lipschitz (but not linear!) liftings from quotient spaces isomorphic to $c_0(\Gamma)$ spaces. If K is separable and not metrizable, the space $C(K)$ is not isomorphic to a $c_0(\Gamma)$ space.

The lifting property for separable spaces. The following result from [28] shows a useful regularity property of separable spaces.

2.2. Theorem. *Every separable Banach space X has the isometric lifting property.*

Proof. We will actually give two proofs. In the first proof, since X is separable, there exists a Gaussian measure γ whose support is dense in X. Then we can use the result (see [6]) that if $\mathcal{L}$ is a Banach-space valued Lipschitz map defined on X, then the convolution

$$(\mathcal{L} * \gamma)(x) = \int_X \mathcal{L}(x-t)\,d\gamma(t)$$

is Gâteaux-differentiable. We apply this to the map $\mathcal{L} = \delta_X$ and then, in the above notation, $T = (\delta_X * \gamma)'(0)$ satisfies $\beta_X T = \mathrm{Id}_X$.

The second proof is essentially self-contained. It consists in replacing the Gaussian measure by a cube measure and this will be useful later. It underlines the simple fact that being separable is equivalent to being "compact-generated".

Let $(x_i)_{i\geqslant 1}$ be a linearly independent sequence of vectors in X such that

$$\overline{\mathrm{vect}}[(x_i)_{i\geqslant 1}] = X$$

and $\|x_i\| = 2^{-i}$ for all i. Let $H = [0,1]^{\mathbb{N}}$ be the Hilbert cube and $H_n = [0,1]^{\mathbb{N}_n}$ be the copy of the Hilbert cube where the factor of rank n is omitted, that is, $\mathbb{N}_n = \mathbb{N}\backslash\{n\}$. We denote by λ (resp.λ_n) the natural probability measure on H (resp. H_n) obtained by taking the product of the Lebesgue measure on each factor.

Let $E = \mathrm{span}[(x_i)_{i\geqslant 1}]$ and denote by $R\colon E \to \mathcal{F}(X)$ the unique linear map which for all $n \geqslant 1$ and all $f \in \mathrm{Lip}_0(X)$ satisfies

$$R(x_n)(f) = \int_{H_n}\Big[f\Big(x_n + \sum_{\substack{j=1\\ j\neq n}}^{\infty} t_j x_j\Big) - f\Big(\sum_{\substack{j=1\\ j\neq n}}^{\infty} t_j x_j\Big)\Big]\, d\lambda_n(t).$$

Pick $f \in \mathrm{Lip}_0(X)$. If the function f is Gâteaux-differentiable, Fubini's theorem shows that for all $x \in E$

$$R(x)(f) = \int_H \Big\langle \{\nabla f\}\Big(\sum_{j=1}^{\infty} t_j x_j\Big), x\Big\rangle\, d\lambda(t).$$

Thus $|R(x)(f)| \leqslant \|x\|\|f\|_L$ in this case. But since X is separable, any $f \in \mathrm{Lip}_0(X)$ is a uniform limit of a sequence f_j of Gâteaux-differentiable functions such that $\|f_j\|_L \leqslant \|f\|_L$. It follows that

$$\|R\| \leqslant 1.$$

We may now extend R to a linear map $\overline{R}\colon X \to \mathcal{F}(X)$ such that $\|\overline{R}\| = 1$ and it is clear that $\overline{R}(x)(x^*) = x^*(x)$ for all $x \in X$ and all $x^* \in X^*$. □

The above proof follows [28]. We refer to [25] for an elementary approach along the lines of the second argument which uses only finite-dimensional arguments and is accessible at the undergraduate level.

Figiel's theorem. We will apply the lifting property to the construction of *linear* isometries from separable Banach spaces to Banach spaces when existence of isometries is assumed. We first show an improvement due to Figiel of the classical Mazur–Ulam theorem. For this purpose we need the following lemma.

2.3. Lemma. *Let E be a finite-dimensional normed space, with norm $\|\cdot\|$. Pick $x \in E$ a point of differentiability of the norm $\|\cdot\|$ with $\|x\| = 1$. Then $\{\nabla\|\cdot\|\}(x)$ is the only 1-Lipschitz map $\varphi\colon E \to \mathbb{R}$ such that $\varphi(tx) = t$ for all $t \in \mathbb{R}$.*

Proof. Let $\varphi\colon E \to \mathbb{R}$ a 1-Lipschitz map such that $\varphi(tx) = t$ for all $t \in \mathbb{R}$. Pick $y \in E$.

For all $t \neq 0$, one has

$$1 = \left|t\varphi(y) - t\varphi\big((\varphi(y) + 1/t)x\big)\right| \leqslant \|x - t(y - \varphi(y)x)\|.$$

Therefore, the right-hand -side function attains its minimum at $t = 0$. Differentiation gives

$$\big\langle \{\nabla\|\cdot\|\}(x), y - \varphi(y)x \big\rangle = 0$$

and thus $\{\nabla\|\cdot\|\}(x) = \varphi$. □

2.4. Lemma. *Let E be a normed space of finite dimension n, let F be a normed space and let $\phi\colon E \to F$ be an isometry such that $\phi(0) = 0$. We assume that $\overline{\mathrm{vect}}[\phi(E)] = F$. Then there exists a unique continuous linear map $T\colon F \to E$ such that $T \circ \phi = \mathrm{Id}_E$ and, moreover, $\|T\| = 1$.*

Proof. We first consider the one-dimensional case. Let $j\colon \mathbb{R} \to F$ be an isometry such that $j(0) = 0$. For all $k \in \mathbb{N}$ there exists $x_k^* \in F^*$ with norm 1 such that $\langle x_k^*, j(k) - j(-k)\rangle = 2k$. It is easily seen that $\langle x_k^*, j(t)\rangle = t$ for all $t \in [-k, k]$. It follows by weak* compactness that there exists $x^* \in F^*$ with norm 1 such that $\langle x^*, j(t)\rangle = t$ for all $t \in \mathbb{R}$, and this linear form x^* does the job.

Take now $\phi\colon E \to F$ as above. Pick any $x \in E$ where the norm $\|\cdot\|$ is differentiable. By the one-dimensional case, there exists $f_x^* \in F^*$ with norm 1 such that $\langle f_x^*, \phi(tx)\rangle = t$ for all $t \in \mathbb{R}$. Lemma 2.3 shows that $f_x^* \circ \phi = \{\nabla\|.\|\}(x)$.

We recall now that any norm on a finite-dimensional space is differentiable at every point of a dense subset. It follows that for any $z \in E\backslash\{0\}$, there is a point x' of differentiability of the norm such that $\{\nabla\|\cdot\|\}(x')(z) \neq 0$. It follows that we can find points of differentiability $x_1, x_2, \ldots, x_n$ such that the set of linear forms $\big(\{\nabla\|\cdot\|\}(x_i)\big)_{1\leqslant i\leqslant n}$ is a basis of E^*.

We denote by $(z_j)_{1\leqslant j\leqslant n}$ the dual basis in E such that

$$\{\nabla\|\cdot\|\}(x_i)(z_j) = \delta_{i,j}.$$

For all $1 \leqslant i \leqslant n$, there exists $f_{x_i}^* \in F^*$ with norm 1 such that

$$\{\nabla\|\cdot\|\}(x_i) = f_{x_i}^* \circ \phi.$$

We define $T\colon F \to E$ by

$$T(y) = \sum_{i=1}^{n} f_{x_i}^*(y) z_i.$$

The map T is linear and continuous and $T \circ \phi = \mathrm{Id}_E$.

Uniqueness of such a map T follows immediately from $\overline{\mathrm{vect}}[\phi(E)] = F$.

Moreover, for all $x' \in \Omega_{\|\cdot\|}$, one has

$$f^*_{x'} = \{\nabla\|.\|\}(x') \circ T, \tag{1}$$

since these continuous linear forms coincide on the dense set $\mathrm{vect}[\phi(E)]$. We pick now any $y \in F$ and let $z = T(y)$. Since $\|z\| = \sup\{\langle\{\nabla\|\cdot\|\}(x'), z\rangle\}$, where the supremum is taken over the points x' of differentiability of the norm, it follows from (1) that $\|z\| \leqslant \|y\|$ and thus $\|T\| = 1$. □

It is now easy to deduce Figiel's theorem [20] from this lemma. Note that this theorem immediately implies Mazur–Ulam's theorem: every onto isometry $\Phi: X \to Y$ between Banach spaces such that $\Phi(0) = 0$ is linear.

2.5. Theorem. *Let X be a separable infinite-dimensional Banach space. Let F be a normed space and let $\phi: X \to F$ be an isometry such that $\phi(0) = 0$. We assume that $\overline{\mathrm{vect}}[\phi(X)] = F$. Then there exists a unique continuous linear map $T: F \to X$ such that $T \circ \phi = \mathrm{Id}_X$ and, moreover, $\|T\| = 1$.*

Proof. We complete the proof in the case when X is separable. Easy modifications provide the general case. We write

$$X = \overline{\bigcup_{k\geqslant 1} E_k}$$

where $(E_k)_{k\geqslant 1}$ is an increasing sequence of finite-dimensional subspaces. We let $F_k = \mathrm{vect}[\Phi(E_k)]$. By Lemma 2.4, there exists a unique continuous linear map $T_k: F_k \to E_k$ such that $T_k(\Phi(x)) = x$ for all $x \in E_k$ and, moreover, $\|T_k\| = 1$.

Uniqueness implies that we can consistently define $T: \bigcup_{k\geqslant 1} F_k \to X$ by $T(y) = T_k(y)$ if $y \in F_k$, with $\|T\| = 1$ since $\|T_k\| = 1$ for all k. Finally, our assumption implies that $F = \overline{\bigcup_{k\geqslant 1} F_k}$ and T can be extended to F since it takes values in the complete space X. □

Some applications of the lifting property. Figiel's theorem and the lifting property of separable Banach spaces provide the following result from [28]. It should be underlined that the conclusion fails to hold in general if X is not assumed to be separable.

2.6. Theorem. *Let X be a separable Banach space. If there exists an isometry Φ from X into a Banach space Y, then Y contains a closed linear subspace which is linearly isometric to X.*

Proof. We may and do assume that $\Phi(0) = 0$ and that $\overline{\mathrm{vect}}[\Phi(X)] = Y$. By Theorem 2.5, there is a quotient map $Q: Y \to X$ of norm 1 such that $Q \circ \Phi = \mathrm{Id}_X$. We can, therefore, apply Theorem 2.2 with $\mathcal{L} = \Phi$, and this shows the existence of $S: X \to Y$ with $\|S\| = 1$ and $Q \circ S = \mathrm{Id}_X$. It is now clear that S is a linear isometry from X into Y. □

Another application of the lifting property concerns quasi-linear projections. We recall that if Y is a closed linear subspace of a Banach space X, a map $p: X \to Y$ is called a *quasi-linear* projection if $p(0) = 0$ and $p(x + y) = p(x) + y$ for every $x \in X$ and $y \in Y$. In other words, p commutes with the translations which are parallel to Y. It is easy to check that there exists a quasi-linear Lispchitz projection from X onto Y if and only if the quotient map $Q: X \to X/Y$ admits a Lipschitz lifting. Therefore, the lifting property translates into the following

2.7. Theorem. *Let X be a Banach space and Y a closed linear subspace of X such that X/Y is separable. If there exists a Lipschitz quasi-linear projection $p: X \to Y$, then Y is complemented in X.*

For instance, if a separable space X is reflexive and strictly convex and Y is a closed subspace of X, then every $x \in X$ has a unique nearest point $p_Y(x)$ in Y and the map p_Y is a quasi-linear projection. It follows from Theorem 2.7 that such maps p_Y will not be Lipschitz unless Y is linearly complemented in X.

3. Free spaces and the approximation property

A Banach space X has the approximation property (in short, A.P.) if for any compact subset K of X and any $\epsilon > 0$, there exists a bounded finite rank operator R such that $\|x - R(x)\| < \epsilon$ for every $x \in K$. If this property holds with the extra condition $\|R\| \leqslant \lambda$, we say that X has the bounded approximation property (in short, B.A.P) or, more precisely, the λ-B.A.P. The 1-B.A.P. is called the metric approximation property (in short, M.A.P.). The uniform boundedness principle shows that a separable Banach space X has the bounded approximation property if there exists a sequence of finite rank operators T_n such that $\lim \|T_n(x) - x\| = 0$ for every $x \in X$.

It is natural to investigate for which metric spaces M the free space $\mathcal{F}(M)$ has the approximation property or one of its quantitative versions. We will see below that this question is tightly related to the extension properties of Lipschitz functions. We can observe first that if X is a separable Banach space which fails to have the A.P. then $\mathcal{F}(X)$ fails to have the A.P. as well since X is isomorphic to a complemented subspace of $\mathcal{F}(X)$. On the other hand, $\mathcal{F}(\mathbb{R}) = L_1$ has the M.A.P., and it is shown in [28] that, more generally, $\mathcal{F}(\mathbb{R}^n)$ has the M.A.P. for every $n \in \mathbb{N}$ and every norm on $\mathbb{R}^n$.

Free spaces over compact spaces. Free spaces over Banach spaces can have or fail to have the approximation property. A problem that goes back to [38] asks whether the free space

$\mathcal{F}(K)$ has the A.P. or even the M.A.P. for every metric compact space K. This question has been answered in [32] as follows.

3.1. Theorem. *Let X be a separable Banach space and let C be a closed convex set containing 0 such that $\overline{\text{span}}[C] = X$. Then X is isomorphic to a complement subspace of $\mathcal{F}(C)$.*

Note that for any separable space X there are compact sets C which satisfy the assumptions of this theorem and if X fails to have the A.P. then $\mathcal{F}(C)$ fails to have the A.P. as well.

Proof. We use the second proof of Theorem 2.2. In the notation of this proof, we may assume that the vectors $(x_i)_{i\geqslant 1}$ are contained in $C/2$. Then the proof shows that $\overline{R}(X) \subset \mathcal{F}(C) \subset \mathcal{F}(X)$. Therefore $\overline{R}(X)$ is a 1-complemented subspace of $\mathcal{F}(C)$ and is isometric to X. □

Thus some free spaces $\mathcal{F}(K)$ over compact spaces fail to have the approximation property. Otherwise, it is frequent: for instance, if C is a closed convex subset of an Euclidean space H, then, since $\mathcal{F}(H)$ has the M.A.P. ([28]) and C is a 1-Lipschitz retract of H, $\mathcal{F}(C)$ is 1-complemented in $\mathcal{F}(H)$ and so it has the M.A.P. We will now relate the validity of the B.A.P for $\mathcal{F}(K)$ with the extension properties of Lipschitz functions defined on subsets of K.

In what follows, a metric compact set M is understood as the limit of a nested sequence of finite sets (M_n). A subset S of a metric space M is said to be ϵ-dense if for all $m \in M$, one has $\inf\{d(m,s) : s \in S\} \leqslant \epsilon$. We denote by $\delta_n\colon M_n \to \mathcal{F}(M_n)$ the Dirac map relative to M_n. If X is a Banach space, $\text{Lip}(M,X)$ denotes the space of Lipschitz functions $F\colon M \to X$. If K is a compact metric space and $T\colon \text{Lip}(K) \to \text{Lip}(K)$ is a continuous linear operator, we denote by $\|T\|_L$ its operator norm when $\text{Lip}(K)$ is equipped with the Lipschitz norm, and by $\|T\|_{L,\infty}$ its norm when the domain space is equipped with the Lipschitz norm and the range space with the uniform norm, alternatively: $\|T\|_{L,\infty}$ is the norm of T from $\text{Lip}(K)$ to $C(K)$ when these spaces are equipped with their canonical norms. We use the same notation for X-valued Lipschitz functions. It should be noted that if M is a metric compact space, then the uniform norm induces on the unit ball of $\text{Lip}(M)$ the weak* topology associated with the free space $\mathcal{F}(M)$.

With this notation, the following holds ([27]).

3.2. Theorem. *Let M be a compact metric space. Let $(M_n)_n$ be a sequence of finite ϵ_n-dense subsets of M with $\lim(\epsilon_n) = 0$. We denote by $R_n(f)$ the restriction to M_n of a function f defined on M. Let $\lambda \geqslant 1$. The following assertions are equivalent:*

(i) *The free space $\mathcal{F}(M)$ over M has the λ-B.A.P.*

(ii) *There exist* $\alpha_n \geqslant 0$ *with* $\lim \alpha_n = 0$ *such that for every Banach space* X *there exist linear operators* $E_n\colon Lip(M_n, X) \to Lip(M, X)$ *with* $\|E_n\|_L \leqslant \lambda$ *and*

$$\|R_n E_n - I\|_{L,\infty} \leqslant \alpha_n.$$

(iii) *There exist linear operators* $G_n\colon Lip(M_n) \to Lip(M)$ *with* $\|G_n\|_L \leqslant \lambda$ *and* $\lim \|R_n G_n - I\|_{L,\infty} = 0$.

(iv) *For every Banach space* X *there exist* $\beta_n \geqslant 0$ *with* $\lim \beta_n = 0$ *such that for every* 1*-Lipschitz function* $F\colon M_n \to X$ *there exists a* λ*- Lipschitz function* $H\colon M \to X$ *such that* $\|R_n(H) - F\|_{l_\infty(M_n, X)} \leqslant \beta_n$.

Proof. (i)⇒(ii): Let $Z = c((\mathcal{F}(M_n))$ be the Banach space of sequences (μ_n), with $\mu_n \in \mathcal{F}(M_n)$ for all n, such that (μ_n) is norm convergent in the Banach space $\mathcal{F}(M)$. We equip Z with the supremum norm and denote by $Q\colon Z \to \mathcal{F}(M)$ the canonical quotient operator which maps every sequence in Z to its limit.

The kernel $Z_0 = c_0((\mathcal{F}(M_n))$ of Q is an M-ideal in Z and the quotient space Z/Z_0 is isometric to $\mathcal{F}(M)$. It follows from (1) and the Ando–Choi–Effros theorem (see [36, Theorem II.2.1]) that there exists a linear map $L\colon \mathcal{F}(M) \to Z$ such that $QL = \mathrm{Id}_{\mathcal{F}(M)}$ and $\|L\| \leqslant \lambda$.

We let π_n be the canonical projection from Z onto $\mathcal{F}(M_n)$ and we define

$$g_n = \pi_n L\delta\colon M \to \mathcal{F}(M_n).$$

The maps g_n are λ-Lipschitz and for every $m \in M$ we have

$$\lim \|g_n(m) - \delta(m)\|_{\mathcal{F}(M)} = 0.$$

Since M is compact, this implies by an equicontinuity argument that if we let

$$\alpha_n = \sup_{m\in M} \|g_n(m) - \delta(m)\|_{\mathcal{F}(M)},$$

then $\lim \alpha_n = 0$. Let now X be a Banach space and $F\colon M_n \to X$ a Lipschitz map. There exists a unique continuous linear map $\overline{F}\colon \mathcal{F}(M_n) \to X$ such that $\overline{F} \circ \delta_n = F$ and its norm is equal to the Lipschitz constant of F. In the notation of [28], one has $\overline{F} = \beta_X \circ \widehat{F}$ and, in particular, $\overline{F}$ depends linearly upon F. We now let

$$E_n(F) = \overline{F} \circ g_n$$

and it is easy to check that the sequence (E_n) satisfies the requirements of (ii).

(ii)⇒(iii): it suffices to take $X = \mathbb{R}$ in (ii).

(ii)⇒(iv): it suffices to take $H = E_n(F)$ and (iv) follows with $\beta_n = \alpha_n$ (independent of X).

(iii)⇒(i): We let $\|R_nG_n - I\|_{L,\infty} = \gamma_n$, with $\lim \gamma_n = 0$. If $H \in \mathrm{Lip}(M)$, then

$$\|R_nG_nR_n(H) - R_n(H)\|_{l_\infty(M_n)} \leqslant \gamma_n\|H\|_L.$$

In other words,

$$\|R_n[G_nR_n(H) - H]\|_{l_\infty(M_n)} \leqslant \gamma_n\|H\|_L.$$

If we now let $T_n = G_nR_n\colon \mathrm{Lip}(M) \to \mathrm{Lip}(M)$, we have $\|T_n\|_L \leqslant \lambda$ and, since M_n is ϵ_n-dense in M with $\lim \epsilon_n = 0$, it follows from the above that for every $H \in \mathrm{Lip}(M)$ one has

$$\lim \|T_n(H) - H\|_{l_\infty(M)} = 0.$$

The operator R_n is a finite rank operator which is weak-star to norm continuous and so is T_n, since $T_n = G_nR_n$. In particular, there exists $A_n\colon \mathcal{F}(M) \to \mathcal{F}(M)$ such that $A_n^* = T_n$. It is clear that $\|A_n\|_{\mathcal{F}(M)} \leqslant \lambda$ and that the sequence (A_n) converges to the identity for the weak operator topology, and this shows (i).

(iv)⇒(i): It will be sufficient to apply condition (iv) to a very natural sequence of 1-Lipschitz maps. We let $X = l_\infty(\mathcal{F}(M_n))$ and $j_n \circ \delta_n = \tilde{\delta}_n\colon M_n \to X$, where $j_n = \mathcal{F}(M_n) \to X$ is the obvious injection such that $(j_n(\mu))_k = 0$ if $k \neq n$ and $(j_n(\mu))_n = \mu$. The map $\tilde{\delta}_n$ is an isometric injection from M_n into X.

By (iv), there exist λ-Lipschitz maps $H_n\colon M \to X$ such that

$$\|R_n(H_n) - \tilde{\delta}_n\|_{l_\infty(M_n,X)} \leqslant \beta_n.$$

We let $V_n = P_nH_n$, where $P_n\colon X \to \mathcal{F}(M_n)$ is the canonical projection. The maps V_n are λ-Lipschitz and for every $m \in M_n$ one has, since $P_n\tilde{\delta}_n = \delta_n$, that

$$\|V_n(m) - \delta_n(m)\|_{\mathcal{F}(M_n)} \leqslant \beta_n.$$

The Lipschitz map $V_n\colon M \to \mathcal{F}(M_n)$ extends to a linear map $\overline{V_n}\colon \mathcal{F}(M) \to \mathcal{F}(M_n)$ with $\|\overline{V_n}\| \leqslant \lambda$. By the above, the sequence $C_n = J_n\overline{V_n}$, where $J_n\colon \mathcal{F}(M_n) \to \mathcal{F}(M)$ is the canonical injection, converges to the identity of $\mathcal{F}(M)$ in the strong operator topology. This concludes the proof. □

In what follows, we will restrict our attention to actual extension operators, in other words, to the case $\alpha_n = \beta_n = \gamma_n = 0$. Note that Mac Shane's formula

$$\overline{f}(a) = \inf\{f(b) + Ld(a,b) \,:\, b \in B\},$$

used for extending real-valued Lipschitz functions from a subset B to a metric space A, preserves the Lipschitz constant, but does not work for Banach-space valued Lipschitz functions (since it uses the order structure on $\mathbb{R}$) and is not linear in f for real-valued functions, hence does not provide a bounded linear extension operator for Lipschitz functions. The above result states that such improved extensions are possible uniformly on a tower of finite subsets of a compact set K exactly when $\mathcal{F}(K)$ has the B.A.P. Conversely, compact sets K such that $\mathcal{F}(K)$ fails to have the B.A.P. provide natural examples where such extensions do not exist (see [53], [9] for previous examples). This is explained in the following remarks.

3.3. Remark.

1) Let M be a compact metric space with distinguished point 0_M such that $\mathcal{F}(M)$ fails to have the B.A.P. We denote by M^∞ the Cartesian product of countably many copies of M equipped with $d^\infty(x_n, y_n) = \sup d(x_n, y_n)$ and by $P_n : M^\infty \to M$ the corresponding sequence of projections. We use the notation of the proof of (iv)⇒(i) and, in particular, we let $X = l_\infty(\mathcal{F}(M_n))$. We define a map Δ from the subset $L = \Pi_{n \geqslant 1} M_n$ of M^∞ to X by the formula

$$\Delta((m_n)) = (\tilde{\delta}_n(m_n))_n.$$

The map Δ is 1-Lipschitz. We denote by $i_n : M \to M^\infty$ the natural injection defined by $(i_n(m))_k = m$ if $k = n$ and 0_M otherwise. Assume that Δ admits a λ-Lipschitz extension $H : M^\infty \to X$. Then for every n the map $H_n = P_n H i_n$ is a λ-Lipschitz extension of $\tilde{\delta}_n$. But then the proof of (iv)⇒(i) shows that $\mathcal{F}(M)$ has the λ-B.A.P., contrarily to our assumption. Hence Δ cannot be extended to a Lipschitz map from M^∞ to X.

2) In the notation of Remark 1), assume that there exists a linear extension operator $E : \mathrm{Lip}(L) \to \mathrm{Lip}(M^\infty)$ with $\|E\|_L = \lambda < \infty$. If π_n denotes the canonical projection from L onto $i_n(M_n)$, then π_n is 1-Lipschitz and thus the map $E_n : \mathrm{Lip}(i_n(M_n)) \to \mathrm{Lip}(M^\infty)$ defined by $E_n(F) = E(F \circ \pi_n)$ satisfies $\|E_n\|_L \leqslant \lambda$. Composing E_n with the restriction to $i_n(M)$ shows the existence of a linear extension operator from $\mathrm{Lip}(M_n)$ to $\mathrm{Lip}(M)$ with norm at most λ for all n, and by (iii)⇒(i) this cannot be if $\mathcal{F}(M)$ fails to have the B.A.P.

The lifting property and the B.A.P. We conclude this section with another application ([7]) of the lifting property which shows that the B.A.P. is identical to its natural Lipschitz counterpart. Note that the condition (i) is obviously independent of the choice of the approximating sequence (E_n) and hence so are conditions (ii) and (iii).

3.4. Theorem. *Let X be a separable Banach space. Let $(E_n)_{n \geqslant 1}$ be an increasing sequence of finite dimensional subspaces of X whose union is dense in X. The following assertions are equivalent:*

(i) *The space X has the λ-B.A.P.*

(ii) *There is a sequence of linear operators $R_n : X \to E_n$ such that $\sup \|R_n\| \leqslant \lambda$ and for every $x \in X$*

$$\lim \|R_n(x) - x\| = 0.$$

(iii) *There is a sequence of Lipschitz maps* $F_n\colon X \to E_n$ *such that* $\sup \|F_n\|_L \leqslant \lambda$ *and for every* $x \in X$

$$\lim \|F_n(x) - x\| = 0.$$

Proof. We first show that (iii) implies (ii). The argument is similar to the proof of (i) ⇒ (ii) in Theorem 3.2 above. Let $Z = c((E_n))$ be the Banach space of sequences (x_n), with $x_n \in E_n$ for all n, such that (x_n) is norm convergent in the Banach space X. We equip Z with the supremum norm and denote by $Q\colon Z \to X$ the canonical quotient operator which maps every sequence in Z to its limit. The map $\Sigma\colon X \to Z$ defined by $\Sigma(x) = (F_n(x))_{n\geqslant 1}$ is a λ-Lipschitz right inverse to the quotient map Q. By the isometric lifting property (Theorem 2.2) of separable spaces, there exists a linear map $L\colon X \to Z$ with $\|L\| \leqslant \lambda$ such that $QL = \mathrm{Id}_X$. It suffices to let $R_n = P_n L$, where P_n is the canonical projection from Z to its nth coordinate E_n.

It is obvious that (ii) implies (i) and (iii). Finally, assume that (i) holds. Then, with the above notation, the space $Z_0 = Ker(Q)$ of sequences in Z which converge to 0 is an M-ideal in Z and the quotient Z/Z_0 is isometric to X. Since X has the λ-B.A.P., the Ando––Choi–Effros theorem provides a linear map $L\colon X \to Z$ with $\|L\| \leqslant \lambda$ such that $QL = \mathrm{Id}_X$, and we deduce (ii) as above. □

A Banach space X has the λ-Lipschitz B.A.P if for every compact subset K of X and every $\epsilon > 0$ there is a Lipschitz map $F\colon X \to V$, where V is some finite-dimensional subspace of X, such that $\|F(x) - x\| < \epsilon$ for every $x \in K$. It is shown in [28] that the λ-Lipschitz B.A.P. is actually equivalent to the λ-B.A.P. and this can be deduced quite easily from the above theorem. Using the result that the free spaces over abitrary finite-dimensional Banach spaces have the M.A.P., it is shown in [28] that X has λ-B.A.P. if and only if $\mathcal{F}(X)$ has the λ-B.A.P. It follows, in particular, that the bounded approximation property is stable under Lipschitz-isomorphisms.

Some recent progress. Nigel Kalton studied in [45] and [46] the approximation properties of free spaces in relation to non-linear approximation procedures that would be available for *every* separable Banach space. Some of the most important results (and problems) in the area stem from these works (see Problem 6.2 below).

Following [32], work was done to decide if free spaces over "small" compact spaces could fail to have the A.P. It has been shown in [12] that if K is a countable compact, then $\mathcal{F}(K)$ has the M.A.P., and in [13] that, more generally, if M is a countable proper metric space (where "proper" means that the closed balls are compact), then $\mathcal{F}(M)$ is isometric to a dual space with the M.A.P. Using results of A. Godard [24] characterizing the spaces M whose free spaces embed isometrically into L_1, it was shown in [13] that if M is proper and ultrametric, then $\mathcal{F}(M)$ is isometric to the dual of an isomorphic copy of c_0. On the other hand, it was shown in [10] that if M is ultrametric and separable, then $\mathcal{F}(M)$ is isomorphic to l_1 and has a monotone Schauder basis. We refer to [14] for the characterization of free spaces which are isometric to l_1.

Thus, when K is too small then $\mathcal{F}(K)$ enjoys the M.A.P. On the other hand, it was shown in [34] that there exists a totally discontinuous compact metric space K such that $\mathcal{F}(K)$ fails to have the A.P. In this same paper [34], it was shown that if K is countable, then $\mathcal{F}(K)$ has the Schur property. This should be compared with the recent result [11] that the free space over $\mathbb{R}^n$ is weakly sequentially complete (although, if $n > 1$, this space is not isomorphic to a subspace of L_1 [55]). Note that it was shown in [21] that the free space over the Urysohn space has the M.A.P.

Several results have been shown on stronger forms of the B.A.P. It has been shown in [7] that free spaces over $\mathbb{R}^n$ have F.D.D., and in [51] that these F.D.D. can be taken monotone in the case of l_1^n (and even l_1). A subsequent article [35] shows the existence of a Schauder basis in the free space over $\mathbb{R}^n$ and over l_1. In [56], it is shown in particular that if $\mathbb{R}^n$ is equipped with any norm and C is a compact convex subset, then the free space over C has the M.A.P. Finally, the article [51] contains the remarkable result that if M is a doubling metric space, then $\mathcal{F}(M)$ has the B.A.P. Hence, when M is somewhat "finite-dimensional", then $\mathcal{F}(M)$ has the B.A.P. It is not clear at this moment what kind of uniform control on the approximation constants is available (see Problem 4). Note that it is shown in [49], among other results, that if F is a subset of $\mathbb{R}^n$ which contains a subset which is Lipschitz-isomorphic to the unit ball of $\mathbb{R}^n$, then $\mathcal{F}(F)$ is isomorphic to $\mathcal{F}(\mathbb{R}^n)$.

4. The quotient norm of the linear extension of a Lipschitz isomorphism

Let $f\colon X \to Y$ be a Lipschitz-isomorphism from a Banach space X onto a Banach space Y, where there word "isomorphism" means of course that both f and its inverse f^{-1} are Lipschitz. As seen above, there is a linear continuous map $\overline{f}\colon \mathcal{F}(X) \to Y$ such that $\overline{f} \circ \delta_X = f$. Since f is onto, $\overline{f}$ is, in particular, a quotient map.

It turns out that the corresponding quotient norm on Y enjoys natural smoothness properties when they hold for the original norm on X. These properties are asymptotic in the sense that they do not change too much when we decide to ignore at each point finitely many dimensions. We now recall the definition of the modulus of asymptotic smoothness.

4.1. Definition. Let X be a Banach space equipped with the norm $\|.\|$. If $\|x\| = 1$, $\tau > 0$ and Y is a closed finite-codimensional subspace of X, we let

$$\rho(\tau, x, Y) = \sup_{y \in S_Y} \|x + \tau y\| - 1,$$

where S_Y denotes the unit sphere of Y. Then we let

$$\rho(\tau, x) = \inf_{Y} \rho(\tau, x, Y),$$

where the infimum is taken over all closed finite-codimensional subspaces. Finally, we let

$$\rho(\tau) = \sup_{x \in S_X} \rho(\tau, x).$$

This function ρ (or ρ_X if the space X needs to be specified) is called the modulus of asymptotic uniform smoothness of X. It is sometimes denoted $\overline{\rho}$ to distinguish it from the modulus of uniform smoothness, but this latter notion is not used in this note. A Banach space X is said to be asymptotically uniformly smooth (in short, a.u.s) if

$$\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0.$$

The space $X = c_0$ is a.u.s., and $\rho_{c_0}(\tau) = 0$ for all $\tau \in (0,1]$. More generally, a Banach space E is called asymptotically uniformly flat if there exists $\tau_0 > 0$ such that $\rho_E(\tau_0) = 0$. It is shown in [29] that a separable space has an equivalent asymptotically uniformly flat norm if and only if it is isomorphic to a subspace of c_0.

Here is a practical way of computing the modulus ρ.

4.2. Lemma. *Let X be a Banach space with a separable dual, $\tau \in (0,1]$ and $x \in S_X$. We let*

$$\eta(\tau, x) = \sup\Big[\overline{\lim}_{n\to\infty} \|x + x_n\| - 1\Big],$$

where the supremum is taken over all sequences (x_n) which converge weakly to 0 and such that $\|x_n\| \leqslant \tau$ for all n. Let $\eta(\tau) = \sup_{x\in S_X} \eta(\tau, x)$. Then $\eta(\tau, x) = \rho(\tau, x)$ and $\eta(\tau) = \rho(\tau)$ for every $\tau \in (0,1]$.

Proof. Let (x_n) be a sequence which converges weakly to 0 and such that $\|x_n\| \leqslant \tau$ for all n. Let $Y \subset X$ be a closed subspace of finite codimension. The distance $d(x_n, Y)$ from x_n to Y tends to 0, so, given $\epsilon > 0$, for n large enough there exists $y_n \in Y$ with $\|x_n - y_n\| < \epsilon$. Then $\|y_n\| < \tau + \epsilon$ and

$$\|x + x_n\| - 1 \leqslant \|x + y_n\| - 1 + \|x_n - y_n\| \leqslant \rho(\tau + \epsilon, x, Y) + \epsilon.$$

Since Y of finite codimension is arbitrary, for n large enough we have

$$\|x_n + x\| - 1 \leqslant \rho(\tau + \epsilon, x) + \epsilon$$

and, since $\epsilon > 0$ is arbitrary, it follows that $\eta(\tau, x) \leqslant \rho(\tau, x)$.

Conversely, we have $\eta(\tau, x) \geqslant \rho(\tau, x)$. Indeed, let (x_j^*) be a dense sequence in X^*, and let

$$Y_n = \bigcap_{j=0}^{n} Ker(x_j^*).$$

Given $\epsilon > 0$, there is $x_n \in Y_n$ with $\|x_n\| \leqslant \tau$ such that

$$\|x + x_n\| - 1 + \epsilon \geqslant \rho(\tau, x, Y_n) \geqslant \rho(\tau, x).$$

It is easy to check that the sequence (x_n) weakly converges to 0. Since $\epsilon > 0$ is arbitrary, it follows that $\eta(\tau, x) \geqslant \rho(\tau, x)$. Hence these two quantities are equal and the last assertion follows immediately by taking the supremum over $x \in S_X$. □

Non-linear transfer of asymptotic smoothness. Up to the notation, the following result is Theorem 5.4 in [30] with an actual estimate of the constants. The computations were left to the reader in [30] since the result is the Lipschitz case of the previous Theorem 5.3 from [30] which concerns uniform homeomorphisms. But it turns out that these computations are non-trivial even in this special case. For the convenience of the reader, we provide a complete proof below.

4.3. Theorem. *Let X and Y be two separable Banach spaces. We assume that X is asymptotically uniformly smooth and that there exists a Lipschitz-isomorphism f from X onto Y. Let ρ_X denote the modulus of asymptotic uniform smoothness of X and let $M = \mathrm{Lip}(f^{-1})$. Then $\overline{f}: \mathcal{F}(X) \to Y$ is a quotient map whose quotient norm $|\cdot|$ on Y is asymptotically uniformly smooth with modulus ρ_Y satisfying for every $\tau \in (0,1]$*

$$\rho_Y(\tau/4M) \leqslant 2\rho_X(\tau).$$

Proof. Since X is asymptotically uniformly smooth, X^* is separable and then it follows from [4] that Y^* is separable as well. We may and do assume that $\mathrm{Lip}(f) = 1$ and $\mathrm{Lip}(f^{-1}) = M$. Since $\mathcal{F}(X)^* = \mathrm{Lip}_0(X)$, the unit ball of $\mathcal{F}(X)$ is the norm-closed convex hull of the set consisting of all $(\delta_X(x) - \delta_X(y))/\|x - y\|$, when (x, y) runs through all pairs of distinct points of X.

Therefore, the unit ball of the norm $|\cdot|$ is the norm-closed convex hull of the vectors $(f((x) - f(y))/\|x - y\|$, where (x, y) runs over all pairs of distinct elements of X. This means, incidentally, that this norm $|\cdot|$ is the largest norm on Y for which the map f is 1-Lipschitz.

The dual norm $|\cdot|^*$ on Y^* is defined by the formula

$$|y^*|^* = \sup\left\{ \frac{|\langle y^*, f(x) - f(x')\rangle|}{\|x - x'\|} \;:\; x \neq x' \right\},$$

where the above supremum is taken over all pairs (x, x') of distinct points in X.

We claim that this norm satisfies the requested conditions. By Lemma 4.2, we need to show that $\eta_Y(\tau/4M) \leqslant 2\rho_X(\tau) = 2\rho(\tau)$, where $\eta_Y = \eta$ is obtained from $|\cdot|$ along the lines of this Lemma. Let $y \in Y$ with $|y| = 1$ and let (y_n) be a sequence in Y which converges weakly to 0 and such that $|y_n| \leqslant \tau/4M$ for all n. We have to show that

$$\overline{\lim}_{n\to\infty} |y + y_n| - 1 \leqslant 2\rho(\tau).$$

For all n, we pick $y_n^* \in Y^*$ with $|y_n^*|^* = 1$ such that $\langle y_n^*, y + y_n\rangle = |y + y_n|$. We may and do assume that the sequence (y_n^*) is weak* convergent to y^* with $|y^*|^* \leqslant 1$ and that $\lim |y^* - y_n^*|^* = l$ exists. Pick $\epsilon > 0$ and $x \neq x'$ in X such that

$$\langle y^*, f(x) - f(x')\rangle \geqslant (1-\epsilon)|y^*|^* \|x - x'\|.$$

We may and do assume that $x' = -x$ (hence $x \neq 0$) and $f(x') = -f(x)$, and thus

$$\langle y^*, f(x)\rangle \geqslant (1-\epsilon)|y^*|^* \|x\|.$$

Pick any $\beta > \rho(\tau)$. By definition of $\rho(\tau)$, there exists a subspace X_0 of finite codimension in X such that if $z \in X_0$ and $\|z\| \leqslant \tau\|x\|$, then

$$\|x + z\| \leqslant (1+\beta)\|x\|.$$

Pick $b < \tau\|x\|/2M$ and let $d = \tau\|x\|/2$. Since f^{-1} is M-Lipschitz (for the original norm, and thus for the larger norm $|\cdot|$), we can apply Gorelik's principle ([29, Prop. 2.7]) for these values of b and d and the finite-codimensional subspace X_0 to conclude that there exists a compact set K such that $bB_{|\cdot|} \subset K + f(2dB_{X_0})$.

We now observe that the sequence $(y_n^* - y^*)$ converges to 0 uniformly on the compact set K. It follows that there exists a sequence (z_n) in X_0 such that $\|z_n\| \leqslant 2d = \tau\|x\|$ and $\lim\langle y_n^* - y^*, f(z_n)\rangle = -bl$.

We set $A_n = \langle y_n^*, f(x) - f(z_n)\rangle$. We have $A_n \leqslant |y_n^*|^* \|x - z_n\| \leqslant (1+\beta)\|x\|$. Moreover,

$$A_n = \langle y^*, f(x) - f(z_n)\rangle + \langle y_n^* - y^*, f(x)\rangle - \langle y_n^* - y^*, f(z_n)\rangle$$

and, since $(y_n^* - y^*)$ weak* converges to 0 and $f(-x) = -f(x)$, one has

$$A_n = 2\langle y^*, f(x)\rangle - \langle y^*, f(z_n) - f(-x)\rangle + bl + \epsilon(n)$$

with $\lim \epsilon(n) = 0$. Since we have

$$\langle y^*, f(z_n) - f(-x)\rangle \leqslant |y^*|^* \|z_n + x\| \leqslant |y^*|^* (1+\beta)\|x\|$$

il follows that

$$A_n \geqslant 2(1-\epsilon)|y^*|^* \|x\| - |y^*|^* (1+\beta)\|x\| + bl + \epsilon(n).$$

We can now combine the two inequalities for A_n and let n increase to infinity to obtain

$$(1+\beta)\|x\| \geqslant (1-\beta-2\epsilon)|y^*|^* \|x\| + bl.$$

Playing on β and b leads to

$$(1+\rho(\tau))\|x\| \geqslant (1-\rho(\tau)-2\epsilon)|y^*|^* \|x\| + l\tau\|x\|/2M$$

and since we can divide by $\|x\| \neq 0$ and $\epsilon > 0$ is arbitrary, it follows that

$$|y^*|^* \leqslant 1 + \frac{2\rho(\tau)}{1-\rho(\tau)} - \frac{l\tau}{2M(1-\rho(\tau))}.$$

We have

$$|y + y_n| = \langle y_n^*, y + y_n\rangle = \langle y_n^* - y^*, y\rangle + \langle y_n^* - y^*, y_n\rangle + \langle y^*, y + y_n\rangle$$

and thus

$$\overline{\lim}|y + y_n| \leqslant (\tau/4M)\lim|y_n^* - y^*|^* + |y^*|^* = \frac{l\tau}{4M} + |y^*|^*.$$

If $\frac{l\tau}{4M} \leqslant 2\rho(\tau)$, then, since $|y^*|^* \leqslant 1$, it follows that $\overline{\lim}|y + y_n| - 1 \leqslant 2\rho(\tau)$. If $\frac{l\tau}{4M} > 2\rho(\tau)$, then

$$|y^*|^* \leqslant 1 - \frac{l\tau}{4M(1-\rho(\tau))} \leqslant 1 - \frac{l\tau}{4M}$$

and thus $\overline{\lim}|y + y_n| \leqslant 1$. Hence, in both cases we have

$$\overline{\lim}_{n\to\infty}|y + y_n| - 1 \leqslant 2\rho(\tau)$$

and this concludes the proof. □

It is natural to wonder which special properties of norms, beyond asymptotic uniform smoothness, could be preserved by the transfer method used in Theorem 4.3. The first applications of this result are the following corollaries ([29]):

4.4. Corollary. *The class of linear subspaces of $c_0(\mathbb{N})$ is stable under Lipschitz isomorphisms.*

Proof. Indeed, a separable Banach space X is isomorphic to a subspace of c_0 if and only if it admits an equivalent asymptotically uniformly flat norm (that is, a norm such that $\rho_X(\tau_0) = 0$ for some $\tau_0 > 0$) ([29, Theorem 2.4]). Theorem 4.3 shows immediately that the existence of such a norm is a Lipschitz invariant. □

4.5. Corollary. *A Banach space X which is Lipschitz-isomorphic to $c_0(\mathbb{N})$ is linearly isomorphic to $c_0(\mathbb{N})$.*

Proof. Indeed, we know by the above that X is isomorphic to a subspace of $c_0(\mathbb{N})$. On the other hand, X is a $\mathcal{L}^\infty$ space since the class of $\mathcal{L}^\infty$ spaces is stable under Lipschitz isomorphisms [37]. Finally, it is shown in [41] that a $\mathcal{L}^\infty$ subspace of $c_0(\mathbb{N})$ is isomorphic to $c_0(\mathbb{N})$. □

Theorem 4.3 provides an example of an asymptotic property which happens to be Lipschitz-invariant. This result and its proof, which relies, in particular, on the Gorelik principle, suggest that asymptotic properties are good candidates for Lipschitz or uniform invariants. This intuition is essentially correct, as shown in the fundamental articles due to Nigel Kalton ([47, 48]), where such ideas are explored in considerable depth.

Some recent progress. The property (M) is defined in [43] as follows: a Banach space X has (M) if for every $(u,v) \in X^2$ with $\|u\| = \|v\|$ and every weakly-null sequence (x_n), one has

$$\lim \|u + x_n\| = \lim \|v + x_n\|,$$

provided these limits exist. It is shown in particular in [43] that Orlicz sequence spaces h_F can be renormed to have property (M). This asymptotic homogeneity property implies, in particular, that norms with (M) are asymptotically uniformly smooth and optimally so among all equivalent norms [19]. This is an operative tool for computing estimates on the Szlenk index of Orlicz sequence spaces [8] and using these estimates to show that two uniformly homeomorphic Orlicz sequence spaces contain the same l_p spaces or, in other words, have the same Matuszewska–Orlicz indices. Coarse and uniform embeddings between Orlicz sequence spaces have been further investigated in [50]. In the spirit of Corollary 4.4, it is shown in [17] that the class of separable quotients of c_0 (which is a subclass of the collection of all subspaces of c_0) is stable under Lipschitz isomorphisms, provided that we restrict ourselves to the quotient spaces whose dual spaces have the A.P. It is not known whether one can dispense with this technical restriction.

5. Norm attainment

When a Lipschitz map f defined on a Banach space X attains its norm on a couple of points (x, y), this can provide useful information on the behavior of f at the point x or in the direction $(y - x)$. We refer to [57], where this technique is used for obtaining smooth points of real-alued Lipschitz maps on Asplund spaces. However, it turns out that it is not easy to approximate Lipschitz maps by norm-attaining ones (in this strong sense). We now show some results on this largely unexplored topic.

Our first result is an application [26] of Theorem 4.3. We use the notation from the above Section 12. Let us say that a Lipschitz map from a metric space M to a Banach space Y attains its norm *in the strong sense* (or strongly attains its norm) if there exists a pair of distinct points $(m,s) \in M^2$ such that $\|f(m) - f(s)\| = \|f\|_L d(m,s)$. We say that f attains its norm on $\mathcal{F}(M)$ if the corresponding linear operator $\overline{f}\colon \mathcal{F}(M) \to Y$ attains its operator norm. We say that f attains its norm in the direction $y \in Y$ if $\|y\| = \|f\|_L$ and there exist couples of distinct points (m_n, s_n) of M such that

$$\lim_n (f(m_n) - f(s_n))/d(m_n, s_n) = y.$$

It is clear that f attains its norm in the strong sense on the couple (m, s) if and only if it attains its Lipschitz norm on $\mathcal{F}(M)$ on the corresponding molecule $(\delta(m)-\delta(s))/d(m,s)$, and in that case, it also attains its Lipschitz norm in the direction $(f(m)-f(s))/d(m,s)$. With this notation, the following holds.

5.1. Theorem. *Let X and Y be separable Banach spaces. We assume that X is asymptotically uniformly smooth and that there exists a Lipschitz isomorphism from X onto Y which attains its norm in some direction $y \in Y$. Then there is a constant $C > 0$ such that $\rho_Y(y, \tau/C) \leqslant 2\rho_X(\tau)$ for all $\tau \in (0,1]$.*

Proof. We may and do assume that $\mathrm{Lip}(f) = 1$. We denote by $\|\cdot\|$ the original norm on the space Y. Then $1 = \|y\| \leqslant |y|$, where $|\cdot|$ denotes the equivalent norm on Y constructed in Theorem 4.3. Moreover $|y| \leqslant 1$ since $y = \lim_n (f(u_n) - f(v_n))/\|u_n - v_n\|$. Hence $\|y\| = |y| = 1$. Since $\|z\| \leqslant |z|$ for all $z \in Y$, Theorem 4.3 implies that $\rho_Y(y, \tau/4M) \leqslant 2\rho_X(\tau)$ for all $\tau \in (0,1]$, where $M = \mathrm{Lip}(f^{-1})$. □

It follows easily from this result (see [26]) that if X is asymptotically uniformly flat and Y has the Kadec–Klee property (that is, the weak and norm topologies agree on the unit sphere of Y), then no Lipschitz isomorphism between X and Y can attain its norm in a direction $y \in Y$. Therefore, Theorem 5.1 provides couples of Banach spaces (X, Y) such that the set of norm-attaining Lipschitz maps (in some direction $y \in Y$) is not dense in the space $\mathrm{Lip}(X, Y)$ equipped with the Lipschitz norm.

If we replace the norm-attainment in a direction $y \in Y$ by the strong norm-attainment, then examples are easier to find: actually, it is shown in [42] that the set of strongly norm-attaining Lipschitz maps from the real line to itself is not dense in the Lipschitz norm in $\mathrm{Lip}_0(\mathbb{R})$. Indeed, if we identify this space with $L_\infty(\mathbb{R})$ and thus its predual $\mathcal{F}(\mathbb{R})$ with $L_1(\mathbb{R})$, a function $f \in L_\infty(\mathbb{R})$ attains its norm on a couple (m, s) if and only if

$$\left|\int_m^s f(u)\,du\right| = |s-m|\,\|f\|_\infty$$

and thus if A is a measurable subset of $\mathbb{R}$ such that $0 < m(A \cap I) < m(I)$ for every open interval I and $f = \mathbb{1}_A - \mathbb{1}_{\mathbb{R}\setminus A}$ then $\|f\| = 1 \leqslant \|f - g\|$ for every strongly norm-attaining function g. Note that this function f attains its norm on the free space $\mathcal{F}(\mathbb{R}) = L_1$ but not on a "molecule" $(\delta_s - \delta_m)/|m-s|$.

Clearly, we may replace the real line by the compact set $[0,1]$ in the above argument and reach the same conclusion, i.e. the non-denseness of the set of strongly norm-attaining Lipschitz functions on $[0,1]$ and the existence of Lipschitz functions which attain their norm on $\mathcal{F}([0,1])$ but not on a molecule $(\delta_s - \delta_m)/|m-s|$. However, we now investigate compact spaces which behave quite differently in this respect.

We recall that there are compact metric spaces K such that the free space $\mathcal{F}(K)$ is isometric to the dual space of the little Lipschitz space $\mathrm{lip}_0(K)$ consisting of all Lipschitz functions f with $f(0) = 0$ such that for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $|f(x) - f(y)| \leqslant \epsilon d(x, y)$. This happens exactly when $\mathrm{lip}_0(K)$ strongly separates K, that is, when there exists $a > 0$ such that for all $(s, t) \in K^2$ there exists $f \in \mathrm{lip}_0(K)$ such that $f(s) - f(t) = d(s, t)$ and $\|f\|_L \leqslant a$ (see Chapter 3 in [60]). This condition is satisfied when K is the usual middle-thirds Cantor set, when $K = L^\alpha$ is a compact Holder metric space obtained from a compact space L by snowflaking (see Proposition 3.2.2 in [60]), or when K is countable or ultrametric ([13]).

We note that for any compact metric space K, the space $\mathrm{lip}_0(K)$ is $(1+\epsilon)$-isomorphic to a subspace of c_0 for every $\epsilon > 0$ (Theorem 6.6 in [44]). It follows in particular that when $\mathrm{lip}_0(K)$ strongly separates $\mathcal{F}(K)$, then it is an M-ideal in its bidual $\mathrm{Lip}_0(K)$ (see [36] for this notion). In order to exploit the powerful theory of M-ideals, we need the following lemma.

5.2. Lemma. *Let X be a Banach space which is an M-ideal in its bidual. If $x^{**} \in X^{**}$ attains its norm on B_{X^*}, then it attains its norm on some $x^* \in B_{X^*} \cap Ext(B_{X^{***}})$. Therefore, the set of all $x^{**} \in X^{**}$ which attain their norm on some $x^* \in B_{X^*} \cap Ext(B_{X^{***}})$ is norm dense in X^{**}.*

Proof. We may and do assume that $\|x^{**}\| = 1$. Let $F = B_{X^{***}} \cap (x^{**})^{-1}(1)$. We clearly have $Ext(F) = F \cap Ext(B_{X^{***}})$, and thus $Ext(F) \subset X^* \cup X^\perp$, since X is an M-ideal in its bidual. We cannot have $Ext(F) \subset X^\perp$ since this would imply by the Krein–Milman theorem that $F \subset X^\perp$ and, by assumption, $F \cap X^* \neq \varnothing$. Therefore $\varnothing \neq Ext(F) \cap X^* = F \cap X^* \cap Ext(B_{X^{***}})$.

The second assertion follows from the first since the Bishop–Phelps theorem asserts that the set of all $x^{**} \in X^{**}$ which attain their norm on some $x^* \in B_{X^*}$ is norm dense in X^{**}. □

We are now ready to show the following fact.

5.3. Proposition. *Let K be a metric compact space. We assume that $\mathrm{lip}_0(K)$ strongly separates K. Then every Lipschitz function $f \in \mathrm{Lip}_0(K)$ which attains its norm on $\mathcal{F}(K)$ strongly attains it, that is, there exist distinct points $(k, l) \in K^2$ such that $|f(k) - f(l)| = \|f\|_L d(k, l)$. Therefore, the set of functions which strongly attain their norm is norm dense in $\mathrm{Lip}_0(K)$.*

Proof. We can apply Lemma 5.2, which shows that any norm-attaining function f attains its norm at some point $x^* \in \mathcal{F}(K) \cap Ext(B_{\mathrm{Lip}_0(K)^*})$. But then Corollary 2.5.4 in [60] shows that $x^* = u[(\delta_k - \delta_l)/d(k, l)]$, for some scalar u with $|u| = 1$. The result follows. The last assertion follows again from the Bishop–Phelps theorem. □

We now turn to vector-valued Lipschitz functions. Let us note first that the canonical identification $\mathrm{Lip}_0(K,Y) = L(\mathcal{F}(K),Y)$ could be seen as a version of the classical Radon–Nikodym representation of operators. Indeed, if $K = [0,1]$, this equation boils down to

$$\mathrm{Lip}_0([0,1],Y) = L(L_1([0,1]),Y)$$

hence any operator $T\colon L_1 \to Y$ is represented by a Lipschitz function $F_T\colon [0,1] \to Y$ and, when the Banach space Y has the Radon–Nikodym property, this Lipschitz function is differentiable almost everywhere and its derivative provides a vector-valued measure which represents T.

We now come back to compact spaces K which are strongly separated by the little Lipschitz functions. Our goal is to extend Proposition 5.3 to vector-valued Lipschitz functions. We recall that if Y is an arbitrary Banach space, the canonical equation $\mathrm{Lip}_0(K,Y) = L(\mathcal{F}(K),Y)$ identifies a Lipschitz map f with an operator $\overline{f}$. We can now extend the first part of Proposition 5.3.

5.4. Proposition. *Let K be a metric compact space such that* $\mathrm{lip}_0(K)$ *strongly separates K. Let Y be a Banach space. Let $f\colon K \to Y$ be a Lipschitz function. Then the following assertions are equivalent:*

(i) *The function f strongly attains its norm.*

(ii) *The operator $\overline{f}\colon \mathcal{F}(K) \to Y$ attains its norm.*

Proof. The implication (i) ⇒ (ii) is clear and does not request the strong separation assumption. Conversely, assume that there is $\mu \in \mathcal{F}(K)$ with $\|\mu\| = 1$ and $\|\overline{f}(\mu)\| = \|\overline{f}\| = \|f\|_L$. We pick $y^* \in S_{Y^*}$ such that

$$\langle y^*, \overline{f}(\mu)\rangle = \|f\|_L$$

Then the function $(y^* \circ f)$ is a real-valued Lipschitz function which attains its norm on $\mathcal{F}(K)$ (at μ). Proposition 5.3 concludes the proof. □

It would be tempting to use Proposition 5.4 to conclude that the set of vector-valued Lipschitz functions which strongly attain their norm is dense, but the problem is that Bishop–Phelps theorem fails in general for operators. In other words, operators, in general, cannot be approximated by norm-attaining ones (see [52] for the first investigation of this phenomenon).

However, this obstruction can be lifted when the range space is finite-dimensional. In this case we can extend the second assertion of Proposition 5.3 as follows.

5.5. Proposition. *Let K be a compact metric space such that* lip_0 *strongly separates K. Let E be a finite-dimensional normed space. Then the set of operators $T\colon \mathcal{F}(K) \to E$ which attain their norm on $\mathcal{F}(K)$ is norm dense in $L(\mathcal{F}(K),E)$. Equivalently, the set of Lipschitz functions $f\colon K \to E$ which strongly attain their norm is $\|.\|_L$-dense in* $\mathrm{Lip}_0(K,E)$.

Proof. Since the space E is finite-dimensional, the following isometric identifications hold:

$$\mathrm{Lip}_0(K,E) = L(\mathcal{F}(K),E) = K(\mathcal{F}(K),E) = \mathrm{Lip}_0(K)\otimes_\epsilon E.$$

Moreover, by our assumption on K, we have

$$\mathrm{Lip}_0(K)\otimes_\epsilon E = (lip_0(K))^{**}\otimes_\epsilon E = (lip_0(K)\otimes_\epsilon E)^{**}.$$

We now claim that an operator $T\in L(\mathcal{F}(K),E) = \mathrm{Lip}_0(K,E)$ attains its norm on $\mathcal{F}(K)$ if and only if it attains its norm as a linear form on the predual $lip_0(K,E)^* = \mathcal{F}(K)\otimes_\pi E^*$. Once this is shown, the conclusion follows by the usual Bishop–Phelps theorem and, for the last sentence, from Proposition 5.4. The equivalence follows from the following lemma.

5.6. Lemma. *Let $X = Y^*$ be a separable dual and let E be a finite-dimensional normed space. Then $T\in L(X,E) = (X\otimes_\pi E^*)^*$ attains its norm as an operator on X if and only if it attains its norm as a linear form on $X\otimes_\pi E^*$.*

Indeed, if $\|T\| = \|T(x)\|_E = \langle e^*, T(x)\rangle$ with $x\in S_X$ and $e^*\in S_{E^*}$, then T attains its norm on $x\otimes e^*$. Conversely, if T attains its supremum $\|T\|$ on the unit ball of $X\otimes_\pi E^*$, then, since $X\otimes_\pi E^*$ has the Radon–Nikodym property, T attains its norm at some extreme point of this unit ball. But by [58] (see also [59]), we have

$$Ext(B_{X\otimes_\pi E^*}) = Ext(B_{(Y\otimes_\epsilon E)^*}) = Ext(B_X)\otimes Ext(B_{E^*})$$

and if $\|T\| = \langle T, x\otimes e^*\rangle$, then $\|T\| = \|T(x)\|_E$. □

5.7. Remark.

1) A slightly alternative proof of Proposition 5.5 runs as follows. By Proposition VI.3.1 in [36], the space $\mathrm{lip}_0(K,E) = \mathrm{lip}_0(K)\otimes_\epsilon E$ is an M-ideal in its bidual $\mathrm{lip}_0(K,E)$. Its dual is $\mathrm{Lip}_0(K,E)^* = \mathcal{F}(K)\otimes_\pi E^*$. By Lemma 5.2, if the norm of T is attained on $\mathcal{F}(K)\otimes_\pi E^*$, then it is attained on some point

$$z\in(\mathcal{F}(K)\otimes_\pi E^*)\cap Ext(B_{\mathrm{Lip}_0(K,E)^*})$$

By [58] or [59], we have

$$Ext(B_{\mathrm{Lip}_0(K,E)^*}) = Ext((B_{(\mathrm{Lip}_0(K)\otimes_\epsilon E)^*}) = Ext(B_{\mathrm{Lip}_0(K)^*})\otimes Ext(B_{E^*})$$

and this equation with Corollary 2.5.4 in [60] shows that there exist two distinct points $(x,y)\in K^2$ and $e^*\in Ext(B_{E^*})$ such that

$$z = (\delta_K(x)-\delta_K(y))/d(x,y)\otimes e^*$$

and then T attains its norm as an operator on $S_{\mathcal{F}(K)}$, more precisely, at the molecule $(\delta_K(x)-\delta_K(y))/d(x,y)$, and the corresponding Lipschitz function on K strongly attains its norm.

2) If X is an arbitrary Banach space and E is finite-dimensional, then an operator $T\colon X \to E$ which attains its norm on S_X attains its norm as a linear form on $(X \otimes_\pi E^*)$. But the converse fails in general, even when X is a free space. For instance, take $X = \mathcal{F}(\mathbb{T})$. We can identify X with $L_1(\mathbb{T})$, where the unit circle $\mathbb{T}$ is equipped with the Haar measure m. Let $E = l_2^2$ be the Euclidean plane. Any operator $T\colon L_1 \to E$ is represented by a function $F \in L_\infty(E)$ through the formula

$$T(g) = \int_{\mathbb{T}} F(t)g(t)dm(t)$$

and T attains its norm as a linear form if and only if there exists $h \in L_1(E)$ with $\|h\|_1 = 1$ such that

$$\|T\| = \int_{\mathbb{T}} \langle F(t), h(t)\rangle dm(t).$$

It follows, for instance, that if $F(t) = t$ for all $t \in \mathbb{T}$, the corresponding operator attains its norm as a linear form but not as operator on $L_1(\mathbb{T})$.

3) Let us summarize what has been shown in this section on various types of norm attainment: if K is a compact space, E is a finite-dimensional normed space, and $f \in \mathrm{Lip}_0(K, E)$ attains its norm in the strong sense, then the operator $\overline{f}\colon \mathcal{F}(K) \to E$ attains its norm, and thus this operator attains its norm as a linear form on $\mathcal{F}(K) \otimes_\pi E^*$. The converse implications both fail, but if K is strongly separated by the little Lipschitz space $\mathrm{lip}_0(K)$, then the three conditions are equivalent.

6. Open Problems

The canonical map $M \to \mathcal{F}(M)$ is a useful connection from the world of metric spaces to the world of Banach spaces and, since going to the free space allows linearization of Lipschitz maps between metric spaces, it is natural to believe that complexity does not evaporate and is carried from the arrows to the spaces. Hence, analyzing the free spaces should be a challenge.

What we provide now certainly is not an exhaustive list of the open problems in this field, but rather a gathering of those which appear naturally along the lines of the present work. The first two problems are classical and probably hard. I am more optimistic about the feasibility of the other ones.

6.1. Problem. Let X and Y be two separable Lipschitz-isomorphic Banach spaces. Are they linearly isomorphic?

This problem is open even if $X = l_1$ or $X = \mathcal{C}([0,1])$. The answer is negative if the assumptions are relaxed in various ways: couples of counterexamples exist in separable quasi-Banach spaces, or in non-separable Banach spaces, or for bi-uniform homeomorphisms between separable Banach spaces.

6.2. Problem. Let M be a uniformly discrete metric space, that is, there exists $\theta > 0$ such that $d(x, y) > \theta$ for all pairs (x, y) of distinct points. Does the free space $\mathcal{F}(M)$ have the Bounded Approximation Property?

This question was asked by Nigel Kalton in [45] (see the Remark after Prop. 4.4) and is related to an approximation property in [46] (see Problem 1). If the answer to Problem 6.2 is negative, it follows that there is an equivalent norm on l_1 which fails to have the metric approximation property (which would solve a 50-year-old problem). If the answer to Problem 6.2 is positive, it follows that *every* separable Banach space is approximable in the sense where the identity is the pointwise limit of a sequence of equi-uniformly continuous functions with relatively compact range.

6.3. Problem. Let X be a separable Banach space. Assume that there exists $\lambda \in \mathbb{R}$ such that for every compact subset $K \subset X$ and every $\epsilon > 0$, there is a λ-Lipschitz map $F: X \to X$ with relatively compact range such that $\|F(x) - x\| < \epsilon$ for all $x \in K$. Does it follow that X has the Bounded Approximation Property?

If we assume, moreover, that (in the above notation) $F(X)$ is contained in a finite dimensional space, then the positive answer follows from Theorem 5.3 in [28]. Note that if there exists a compact convex set K containing 0 which is a Lipschitz retract of X and such that $\bigcup_n (nK)$ is dense in X, then the assumptions of Problem 6.3 are satisfied. I do not know if such a set K exists for every separable Banach space. If yes, then the answer to Problem 6.3 is negative: take X failing to have the A.P.

I should mention that at the beginning of section 4 in [46] Nigel Kalton states that a Banach space is Lipschitz approximable if and only if it has the B.A.P. and this equivalence is the positive answer to Problem 6.3. For justification, Nigel simply refers to [28]. I do not know if he overlooked the finite-dimensional restriction in Definition 5.2 from [28] or he had in mind a complete argument. Sadly, [46] is a posthumous paper and there is no way to ask him...

6.4. Problem. Let M be a subset of $\mathbb{R}^n$. Does $\mathcal{F}(M)$ have the M.A.P.?

Note that since M is a doubling metric space, the space $\mathcal{F}(M)$ has the B.A.P. by [51]. Actually, if $M \subset l_2^n$ is equipped with the restriction of the Euclidean norm, then $\mathcal{F}(M)$ has the λ-B.A.P., where $\lambda \leqslant C\sqrt{n}$ with C a numerical constant (Proposition 2.3 in [51]).

It is not known if the B.A.P. constant can be bounded independently of the dimension, and $\lambda = 1$ would be a natural candidate for this bound.

6.5. Problem. Let K be a compact metric space and $lip_0(K)$ the space of little Lipschitz functions on K. Does the space $\mathrm{lip}_0(K)$ have the M.A.P.?

It frequently happens that the space $\mathrm{lip}_0(K)$ is trivial (e.g. when K is metrically convex) and, when it is so, the answer is obviously positive. Note that $\mathrm{lip}_0(K)$ is arbitrarily close to subspaces of c_0 (Theorem 6.6 in [45]), hence if $\mathrm{lip}_0(K)$ has the M.A.P. or merely the λ-commuting B.A.P. with $\lambda < 2$, then by ([33, Proposition 4.3]) its dual has the M.A.P. The case when $\mathrm{lip}_0(K)$ strongly separates K is of special interest. In this case, if $\mathrm{lip}_0(K)$ has the M.A.P., then $\mathcal{F}(K)$ has the M.A.P.

6.6. Problem. Let K be the Cantor compact space $\{0,1\}^{\mathbb{N}}$. If d is any metric on K which induces the natural (product) topology, we consider the free space $\mathcal{F}((K,d)) = X_d$. What is the topological nature of the set of all metrics d such that X_d fails to have the A.P.?

Any metric on K is a continuous function on K^2 and it is easy to check that when the set $\mathcal{M}$ of metrics which induce the usual topology is equipped with the topology of uniform convergence on K^2, then it is a countable intersection of open subsets of $\mathcal{C}(K^2)$. Hence $\mathcal{M}$ is a Polish space and, in particular, a Baire space. A more precise question is to know if the set $G = \{d \in \mathcal{M} : X_d \text{fails to have A.P.}\}$ is a residual subset of $\mathcal{M}$. By [34], this set is non empty. In the above we can replace the Cantor set by the infinite-dimensional compact convex subset of a Banach space, since by [32] the corresponding set G is non-empty. It would, of course, be great to have a smooth proof of the non-emptiness of G through a Baire category argument. Note that it is shown in [22] (resp. [23]) that the family of separable Banach spaces with the B.A.P (resp. with the (π)-property) is Borel.

6.7. Problem. What are the couples (K,E), where K is a compact metric space and E is a Banach space, such that the subset of $\mathrm{Lip}_0(K,E)$ consisting of functions which strongly attain their Lipschitz norm is dense in $\mathrm{Lip}_0(K,E)$ equipped with the Lipschitz norm?

By Proposition 5.5 above, the answer is positive when $\mathrm{lip}_0(K)$ strongly separates K and E is finite-dimensional. I do not know if denseness when $E = \mathbb{R}$ is 1-dimensional already implies that $\mathrm{lip}_0(K)$ strongly separates K. We recall that it is not known if, when X is an arbitrary Banach space and E is finite-dimensional (or even $E = l_2^2$ is the 2-dimensional Euclidean space), the set of norm-attaining operators is always norm dense in the space $L(X,E)$ of all linear operators (see [3]). The free spaces $\mathcal{F}(K) = X$ constitute a class where this problem can be tested, since then $L(X,E) = \mathrm{Lip}_0(K,E)$. By the above, denseness holds, in particular, when $\mathrm{lip}_0(K)$ strongly separates K.

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