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Local structure of generalized Orlicz–Lorentz function spaces

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Summary. We study the local structure of a separated point x in the generalized Orlicz–Lorentz space Λ^{φ} which is a symmetrization of the respective Musielak–Orlicz space L^{φ}. We present criteria for an LM point and a UM point, and sufficient conditions for a point of order continuity and an $LLUM$ point, in the space Λ^{φ}. We prove also a characterization of strict monotonicity of the space Λ^{φ}.

Keywords
symmetric spaces; symmetrization of the Banach function space; generalized Orlicz–Lorentz space; Musielak–Orlicz space; monotonicity properties; order continuity; local structure of a separated point

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Dedicated to Professor Henryk Hudzik
on his 70th birthday in friendship and esteem.

1. Introduction

Geometry of Banach spaces has been deeply investigated over the recent decades. Rotundity and uniform rotundity are fundamental properties in "global" geometry of Banach spaces. Strict and uniform monotonicity play an analogous role in the "global" geometry of Banach lattices. Note that the study of global properties is not always sufficient. If a Banach space (Banach lattice) does not have a global property, then it is natural to ask about the structure of separated points. This leads, among others, to the notion of an extreme

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point, a point of lower (upper) monotonicity, etc. Such a local structure is currently being intensely investigated (see [3, 4, 9, 11, 12, 15, 18, 20–23]). It is known that the global (local) monotonicity structure can be applied to (local) dominated approximation problems in Banach lattices (see [3, 4, 10, 25]). Clearly, the role of rotundity (uniform rotundity) is similar in dominated approximation problems for Banach spaces.

Recall that the symmetrization $E^{(*)}$ of a Banach function space E has been intensely studied recently ([16–18] and [19]). Marcinkiewicz and Lorentz spaces are basic particular cases of this construction. Generalized Orlicz–Lorentz spaces are another important case of $E^{(*)}$ (see [5, 6] and [7]). Criteria for an *LM* point, a *UM* point, a point of order continuity, and a point of lower local uniform monotonicity in the symmetrizations $E^{(*)}$ have been given in [18]. We apply them to characterize the local structure of the generalized Orlicz–Lorentz space Λ^{φ}. Note that the space Λ^{φ} is a symmetrization of the Musielak–Orlicz space L^{φ}. Moreover, criteria for the local structure of $E^{(*)}$ involve the structure of E (see [18]). Consequently, we need to study some properties of the Musielak–Orlicz space L^{φ}.

We also present a description of strict monotonicity of the generalized Orlicz–Lorentz spaces.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real numbers and positive integers, respectively. Denote by $S(X)$ (resp. $B(X)$) the unit sphere (resp. the closed unit ball) of a quasi-Banach space $(X, \|\cdot\|_X)$.

The symbol L^0 stands for the set of all (equivalence classes of) extended real valued Lebesgue measurable functions on $I = [0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$. Let m be the Lebesgue measure on $[0, \alpha)$.

A quasi-Banach lattice $(E, \|\cdot\|_E)$ is called a *quasi-Banach function space* (or a *quasi-Köthe space*) if it is a linear subspace of L^0 satisfying the following conditions:

- If $x \in L^0$, $y \in E$, and $|x| \leqslant |y|$ m-a.e., then $x \in E$ and $\|x\|_E \leqslant \|y\|_E$.
- There exists a strictly positive $x \in E$.

Let E_+ be the positive cone of E, that is, $E_+ = \{x \in E : x \geqslant 0\}$. For $x \in L^0$ set

$$\mathcal{S}_x = \{t \in I : x(t) \neq 0\}.$$

Recall that the weighted quasi-Banach function space $E(w)$ is defined by

$$E(w) = \{x \in L^0 : xw \in E\} \text{ with the norm } \|x\|_{E(w)} = \|xw\|_E,$$

where $w \geqslant 0$.

A point $x \in E$ is said to have an *order continuous norm* (x is an OC point) if for any sequence (x_n) in E such that $0 \leqslant x_n \leqslant |x|$ and $x_n \to 0$ m-a.e. we have $\|x_n\|_E \to 0$.

A quasi-Banach function space E is called *order continuous* ($E \in (OC)$) if every element of E has an order continuous norm (see [13, 26]). As usual, E_a stands for the subspace of order continuous elements of E.

A point $x \in E_+ \setminus \{0\}$ is said to be a *point of upper monotonicity* if for any $y \in E_+$ such that $x \leqslant y$ and $y \neq x$, we have $\|x\|_E < \|y\|_E$. A point $x \in E_+ \setminus \{0\}$ is said to be a *point of lower monotonicity* if for any $y \in E_+$ such that $y \leqslant x$ and $y \neq x$, we have $\|y\|_E < \|x\|_E$. A point $x \in E_+ \setminus \{0\}$ is called a *point of lower local uniform monotonicity* if $\|x_n - x\|_E \to 0$ for any sequence (x_n) in E such that $x \geqslant x_n \geqslant 0$ and $\|x_n\|_E \to \|x\|_E$. We will write briefly that x is a *UM point,* an *LM* point and an *LLUM point,* respectively. The space E is called *strictly monotone* ($E \in (SM)$) provided each point of $E_+ \setminus \{0\}$ is a *UM* point (see [2, 8]). Moreover, $E \in (SM)$ if and only if each point of $E_+ \setminus \{0\}$ is an *LM* point (see [8]). Similarly, if each point of $E_+ \setminus \{0\}$ is an *LLUM* point, then we say that E is *lower locally uniformly monotone* ($E \in (LLUM)$).

Given $x \in L^0$, its *decreasing rearrangement* x^* is defined by

$$x^*(t) = \inf\{\lambda > 0 : d_x(\lambda) \leqslant t\}, \quad t \geqslant 0,$$

where d_x is the *distribution function,* that is,

$$d_x(\lambda) = m\{s \in [0, \alpha) : |x(s)| > \lambda\}, \quad \lambda \geqslant 0 \text{ (see [1, 24]).}$$

Set $x^*(\infty) = \lim_{t\to\infty} x^*(t)$ if $I = [0, \infty)$ and $x^*(\infty) = 0$ if $I = [0, 1)$. Note also that the function x^* is right-continuous.

Two functions $x, y \in L^0$ are called *equimeasurable* ($x \sim y$ for short) if $d_x = d_y$. We say that a quasi-normed function space $(E, \|\cdot\|_E)$ is *rearrangement invariant* (r.i. for short) or symmetric if, whenever $x \in L^0$ and $y \in E$ with $x \sim y$, then $x \in E$ and $\|x\|_E = \|y\|_E$. For more details, the reader is referred to [1, 24].

3. Symmetrizations of Banach function spaces

For a Banach function space E on I, define a symmetrization of E, denoted by $E^{(*)}$, by the formula

$$E^{(*)} = \left\{x \in L^0(I) : x^* \in E\right\},$$

with the functional

$$\|x\|_{E^{(*)}} = \|x^*\|_E.$$

Of course, the non-trivial case of the space $E^{(*)}$ arises for non-symmetric E.

3.1. Example. Lorentz and Marcinkiewicz spaces are examples of symmetrizations. Recall that for any quasi-concave function ϕ on I (that is $\phi(0) = 0, \phi(t)$ is positive, nondecreasing, and $\phi(t)/t$ is non-increasing for $t \in (0, m(I))$), the *Marcinkiewicz function space* M_ϕ is defined by the norm

$$\|x\|_{M_\phi} = \sup_{t\in I} \phi(t)\, x^{**}(t), \quad x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)ds.$$

The *Lorentz function space* Λ_ϕ is defined by the norm

$$\|x\|_{\Lambda_\phi} = \int_I x^*(t)d\phi(t) = \phi(0^+)\|x\|_{L^\infty(I)} + \int_I x^*(t)\phi'(t)dt,$$

where ϕ is a concave function on I. Recall also that the *fundamental function* f_E of a symmetric function space E on I is defined by the formula $f_E(t) = \|\chi_{[0,\,t]}\|_E$, $t \in I$ (see [1]). For a symmetric Banach function space E with the concave fundamental function f_E, there is a largest and a smallest symmetric Banach space with the same fundamental function. Namely,

$$\Lambda_{f_E} \overset{1}{\hookrightarrow} E \overset{1}{\hookrightarrow} M_{f_E}.$$

There is also a Marcinkiewicz space $M_\phi^{(*)}$ different than M_ϕ, defined by

$$M_\phi^{(*)} = M_\phi^{(*)}(I) = \left\{x \in L^0(I) : \|x\|_{M_\phi^*} = \sup_{t\in I} \phi(t)x^*(t) < \infty\right\}.$$

The Marcinkiewicz space $M_\phi^{(*)}$ is a quasi-Banach space and we always have $M_\phi \overset{1}{\hookrightarrow} M_\phi^{(*)}$. Moreover, $M_\phi^{(*)} \overset{C}{\hookrightarrow} M_\phi$ if and only if (see [19])

$$\int_0^t \frac{1}{\phi(s)}ds \leqslant C\frac{t}{\phi(t)} \quad \text{for all } t \in I.$$

Notice that $M_\phi^{(*)} = (L^\infty(\phi))^{(*)}$. Moreover, $\Lambda_\phi = \left(L^1(\phi')\right)^{(*)}$ provided $\phi(0+) = 0$. The local structure of spaces $M_\phi^{(*)}$ and Λ_ϕ has been discussed in [18].

The *dilation operator* D_s, $s > 0$, defined by $D_s x(t) = x(t/s)\chi_I(t/s)$, $t \in I$, is bounded in any symmetric space E on I and $\|D_s\|_{E\to E} \leqslant \max(1, s)$ (see [27, Lemma 1] for $I = (0, 1)$, [24, pp. 96–98] for $I = (0, \infty)$ and [26, p. 130] for both cases). A. Kamińska and Y. Raynaud proved

3.2. Theorem. *The functional* $\|\cdot\|_{E^{(*)}}$ *is a quasi-norm if and only if there is a constant* $1 \leqslant C < \infty$ *such that* (*see* [17, Lemma 1.4])

$$\|D_2 x^*\|_E \leqslant C\,\|x^*\|_E \quad \textit{for all } x^* \in E. \tag{1}$$

3.3. Remark.

(i) $E^{(*)} \neq \{0\}$ if and only if $\chi_{(0,t)} \in E$ for some $t > 0$.

(ii) If $E^{(*)} \neq \{0\}$ and condition (1) is satisfied, then $\chi_{(0,t)} \in E$ for each $t > 0$. Consequently, $E^{(*)}$ has a weak unit $x_0 = \sum_{i=1}^{\infty} x_n$ with $x_n = \frac{\chi_{[n-1,n)}}{b_n \|\chi_{[n-1,n)}\|_E}$, where b_n is chosen such that the sequence $\left\{b_n \left\|\chi_{[n-1,n)}\right\|_E\right\}$ is increasing and $\sum_{n=1}^{\infty} 1/b_n < \infty$.

3.4. Remark. Let C_E be the smallest constant satisfying (1). Recall that the Hardy operator H is defined by

$$Hx(t) = \frac{1}{t} \int_0^t x(s)\, ds, \quad \text{with } t \in I \setminus \{0\}.$$

If E is a Banach function space on I and the operator H is bounded in E, then (1) holds with $C_E \leqslant 2\|H\|_{E \to E}$. Indeed, we have

$$\|Hx^*\|_E = \| \int_0^1 x^*(st)\, ds\|_E \geqslant \| \int_0^{1/2} x^*(st)\, ds\|_E \geqslant \frac{1}{2}\|x^*(t/2)\|_E.$$

The spaces $E^{(*)}$ have been studied, among others, in the papers [16–18] and [19]. Kamińska and Raynaud studied the connections between the structure of $E^{(*)}$ and the structure of E (see [17]). The local structure of a separated point in $E^{(*)}$ with respect to the properties of its nonincreasing rearrangement x^* in the space E has been studied in [18]. In a natural way the following new notions appear (see [18]). Let P be a local property of a point $x \in E$ (an LM point, a UM point, a point of order continuity, etc.). We say that $x = x^*$ is a P^* point provided that it is a P point but restricted in the definition to nonnegative and nonincreasing elements. Namely,

3.5. Definition. A point $x = x^*$ is said to be an LM^* point of E whenever, for any $y \in E_+$ such that $y = y^* \leqslant x$ and $y \neq x$, we have $\|y\|_E < \|x\|_E$.

The notion of a UM^* point of E and the notion of an OC^* point of E are to be understood analogously. The following characterizations have been proved in [18, Theorems 3.6, 3.8, 3.9]. In general, the notion of a P^* point is essentially weaker than the notion of a respective P point.

3.6. Theorem.

(i) *A point* $0 \leqslant x \in E^{(*)}$ *is an LM point of* $E^{(*)}$ *if and only if* $m\left\{t \in I : 0 < x(t) \leqslant x^*(\infty)\right\} = 0$ *and* x^* *is an* LM^* *point of* E.

(ii) *A point* $0 \leqslant x \in E^{(*)}$ *is a UM point of* $E^{(*)}$ *if and only if* $m\left\{t \in I : x(t) < x^*(\infty)\right\} = 0$ *and* x^* *is a* UM^* *point of* E.

(iii) *A point* $x \in E^{(*)}$ *is an OC point of* $E^{(*)}$ *if and only if* $x^*(\infty) = 0$ *and* x^* *is an* OC^* *point of* E.

We will apply these results in the context of the generalized Orlicz–Lorentz spaces.

4. The generalized Orlicz–Lorentz spaces

A function Φ is called an *Orlicz function* whenever $\Phi\colon [0,\infty) \to [0,\infty]$, Φ is convex, vanishing and continuous at zero, not identically equal to zero (or infinity), and left-continuous on $(0, b_\Phi)$ if $\Phi(b_\Phi) = \infty$ or on $(0, b_\Phi]$ if $\Phi(b_\Phi) < \infty$, where

$$a_\Phi = \sup\{u \geqslant 0 : \Phi(u) = 0\} \quad \text{and} \quad b_\Phi = \sup\{u \geqslant 0 : \Phi(u) < \infty\}.$$

We write $\Phi > 0$ when $a_\Phi = 0$ and $\Phi < \infty$ when $b_\Phi = \infty$.

A function $\varphi\colon I \times [0,\infty) \longrightarrow [0,\infty)$ is said to be a *Musielak–Orlicz function* if $\varphi(\cdot, u)$ is measurable for each $u \in \mathbb{R}_+$ and $\varphi(t,\cdot)$ is an Orlicz function for m-a.e. $t \in I$. We define on L^0 a convex modular I_φ by

$$I_\varphi(x) = \int_I \varphi(t, |x(t)|)\, dt$$

for every $x \in L^0$. By the *Musielak–Orlicz space* L^φ we mean

$$L^\varphi = \{x \in L^0 : I_\varphi(cx) < \infty \text{ for some } c > 0\}$$

equipped with the so-called *Luxemburg–Nakano norm* defined as follows

$$\|x\|_\varphi = \inf\left\{\epsilon > 0 : I_\varphi\left(\frac{x}{\epsilon}\right) \leqslant 1\right\}.$$

Set

$$\begin{aligned}
\theta_\varphi(x) &= \inf\{\lambda > 0 : I_\varphi(x/\lambda) < \infty\},\\
a_\varphi(t) &= \sup\{u \geqslant 0 : \varphi(t,u) = 0\},\\
b_\varphi(t) &= \sup\{u \geqslant 0 : \varphi(t,u) < \infty\}.
\end{aligned}$$

The generalized Orlicz–Lorentz space Λ^φ is a symmetrization of the respective Musielak–Orlicz space L^φ, that is, $\Lambda^\varphi = (L^\varphi)^{(*)}$. Thus

$$\Lambda^\varphi = \{x \in L^0 : x^* \in L^\varphi\} \quad \text{and} \quad \|x\|_{\Lambda^\varphi} = \|x^*\|_\varphi.$$

4.1. Remark.

(i) In the paper we assume that $\chi_{(0,t)} \in L^\varphi$ for some $t > 0$ and condition (1) is satisfied for $E = L^\varphi$, so that $(\Lambda^\varphi, \|\cdot\|_{\Lambda^\varphi})$ is a nontrivial quasi-Banach function space with a weak unit (see Remark 3.3). Clearly, $(\Lambda^\varphi, \|\cdot\|_{\Lambda^\varphi})$ is symmetric.

(ii) From (i) we conclude that $\chi_{(0,t)} \in L^\varphi$ for each $t > 0$ (see Remark 3.3), which is equivalent to the following condition:

$$\text{for each } t \in (0, m(I)) \text{ there is } u > 0 \text{ such that } \int_0^t \varphi(s,u)\, ds < \infty.$$

For $t \in (0, m(I))$ denote

$$u_t = \sup\left\{u > 0 : \int_0^t \varphi(s,u)\,ds < \infty\right\}. \tag{2}$$

The structure of generalized Orlicz–Lorentz spaces has been extensively investigated recently under some stronger assumptions which make the space Λ^φ a Banach space (see [5–7]). The spaces Λ^φ are generalizations of the two-weighted Orlicz–Lorentz spaces $\Lambda_{\Phi,w,v}(I)$ studied, as quasi-Banach spaces, by A. Kamińska and Y. Raynaud in [17], which in turn include the classical Orlicz–Lorentz spaces $\Lambda_{\Phi,w}(I)$ and the Orlicz–Marcinkiewicz spaces $M_{\Phi,w}(I)$.

A natural problem is to establish sufficient conditions for boundedness of the dilation operator D_2 in the cone of nonnegative and nonincreasing elements of $\left(L^\varphi, \|\cdot\|_\varphi\right)$.

4.2. Proposition. *Assume there is a constant $C > 0$ such that*

$$\int_0^{2w} \varphi(t,u)\,dt \leqslant \int_0^{w} \varphi(t,Cu)\,dt \tag{3}$$

for all $u > 0$ and each $w > 0$ with $2w \in I$. Then $\|\cdot\|_{\Lambda^\varphi}$ is a quasi-norm on Λ^φ.

Proof. By Theorem 3.2, it is enough to show that (3) implies the dilation operator D_2 is bounded in the cone of nonnegative and nonincreasing elements of the space $\left(L^\varphi, \|\cdot\|_\varphi\right)$. The proof runs similarly as the proof of Proposition 4.5 in [17]. In our case, $\|\cdot\|_\varphi$ is a norm on L^φ and $\varphi(t,\cdot)$ is convex for m-a.e. $t \in I$. Consequently, we have $\varphi(t,u) \leqslant u\varphi'(t,u) \leqslant \varphi(t,2u)$ for m-a.e. $t \in I$, where φ' is the right derivative of φ with respect to the second variable.

Finally, notice that inequality (3) gives condition (4.3) in Proposition 4.5 from [17], for $\varphi_0(t,u) = \varphi(uv(t))\,w(t)$, where φ is an Orlicz function. □

Criteria for an LM and a UM point in Musielak–Orlicz spaces $\left(L^\varphi, \|\cdot\|_\varphi\right)$ have been proved in [11, Theorem 1 and 2]. We will need the respective criteria for LM^* and UM^* points in $\left(L^\varphi, \|\cdot\|_\varphi\right)$, which require quite different proofs.

4.3. Theorem. *Let $x = x^* \in S(L^\varphi)$. Then x is an LM^* point of L^φ if and only if:*

(i) $\theta_\varphi\left(x\chi_{(0,\alpha)}\right) < 1$ *for each* $\alpha \in (0, m(S_x))$.

(ii) *If* $I_\varphi(x) = 1$, *then* $m\left\{t \in (a, m(S_x)) : x(t) > a_\varphi(t)\right\} > 0$ *for each* $a \in (0, m(S_x))$.

(iii) *Let* $0 < a < b < m(S_x)$. *If x is not constant in (a,b) or x is not continuous at $t = b$, then* $m\left\{t \in (a,b) : x(t) > a_\varphi(t)\right\} > 0$ *and* $\theta_\varphi\left(x\chi_{(b,m(S_x))}\right) < 1$.

Proof. Necessity. (i) Suppose $\theta_\varphi\left(x\chi_{(0,\alpha)}\right) = 1$ for some $\alpha \in (0, m(S_x))$. Setting $y = x\chi_{(0,\alpha)}$, we have $y = y^*$, $0 \leqslant y \leqslant x$, and $y \neq x$. Moreover, $\|y\|_\varphi = 1$, hence x is not an LM^* point.

(ii) Suppose that $I_\varphi(x) = 1$ and $m\left\{t \in (a, m(S_x)) : x(t) > a_\varphi(t)\right\} = 0$ for some $a \in (0, m(S_x))$. Taking $y = x\chi_{(0,a)}$, we obtain $y = y^*$, $y \leqslant x$, and $y \neq x$. Moreover, $I_\varphi(y) = I_\varphi(x) = 1$, hence $\|y\|_\varphi = 1$.

(iii) Assume that there are numbers $0 < a < b < m(S_x)$, such that $x(a) > x(b^-)$ and $x(t) \leqslant a_\varphi(t)$ for m–a.e. $t \in (a, b)$. Let

$$y = x\chi_{I\backslash(a,b)} + x(b^-)\chi_{(a,b)}.$$

Then $y = y^*$, $y \leqslant x$, and $y \neq x$. We have $I_\varphi(y) = I_\varphi(x)$. It is enough to show that $\|y\|_\varphi = \|x\|_\varphi$. If $I_\varphi(x) = 1$, then $\|y\|_\varphi = 1 = \|x\|_\varphi$.

Suppose that $I_\varphi(x) < 1$. We claim that $\theta_\varphi\left(x\chi_{(b,m(I))}\right) = 1$. Otherwise, by (i), we get $\theta_\varphi(x) < 1$. Consequently, there is $\lambda_0 < 1$ with $I_\varphi\left(\frac{x}{\lambda_0}\right) < \infty$ and, by continuity of the function $f(\lambda) = I_\varphi(\lambda x)$ on the interval $(0, 1/\lambda_0)$, we conclude that $I_\varphi\left(\frac{x}{\lambda_1}\right) < 1$ for some $\lambda_1 < 1$. Hence, $\|x\|_\varphi \leqslant \lambda_1 < 1$, a contradiction. This proves the claim. Therefore $\theta_\varphi\left(y\chi_{(b,m(I))}\right) = 1$ and $\|y\|_\varphi = 1$.

Assume that x is constant in (a, b), $x(b^-) > x(b)$ and $m\left\{t \in (a, b) : x(t) > a_\varphi(t)\right\} = 0$. Let $y = x\chi_{I\backslash(a,b)} + x(b)\chi_{(a,b)}$. Then $y = y^*$, $y \leqslant x$ and $y \neq x$. We have $I_\varphi(y) = I_\varphi(x)$, and we proceed as above.

Suppose x is not constant in (a, b) and $\theta_\varphi\left(x\chi_{(b,m(S_x))}\right) = 1$. It is enough to take $y = x\chi_{I\backslash(a,b)} + x(b^-)\chi_{(a,b)}$. Finally, if x is constant in (a, b), $x(b^-) > x(b)$, and $\theta_\varphi\left(x\chi_{(b,m(S_x))}\right) = 1$, then we set $y = x(b)\chi_{(0,b)} + x\chi_{(b,m(S_x))}$.

Sufficiency. Let $y = y^*$, $y \leqslant x$, and $y \neq x$. Setting $A = \{t : y(t) < x(t)\} \subset S_x$, we can find an interval $(a, b) \subset A$, because the nonincreasing rearrangement is right-continuous. We split the proof in two parts.

1. Assume x is not constant in (a, b) or x is not continuous at $t = b$. By (iii), we have $m\left\{t \in (a, b) : x(t) > a_\varphi(t)\right\} > 0$, hence $I_\varphi(y) < I_\varphi(x)$. By (i) and (iii), we have $\theta_\varphi(y) \leqslant \theta_\varphi(x) < 1$, so there is $\lambda < 1$ with $I_\varphi\left(\frac{y}{\lambda}\right) < \infty$, and consequently $I_\varphi\left(\frac{y}{\lambda_0}\right) < 1$ for some $\lambda_0 < 1$. Thus $\|y\|_\varphi < 1$.
2. Assume that x is constant in (a, b) and x is continuous at $t = b$. If x is not constant in $(b, m(S_x))$, then we may go back to case 1 because $y = y^*$. Thus it is enough to consider the case $x(t) = c > 0$ for $t \in (a, m(S_x))$. Then $y(t) \leqslant y(a) < x(a)$ for $t \in (a, m(S_x))$. We consider two subcases.
   A. Suppose that $I_\varphi(x) = 1$. Then, by (ii), $m\left\{t \in (a, m(S_x)) : x(t) > a_\varphi(t)\right\} > 0$. Hence $I_\varphi(y) < 1$. Moreover, there is $\lambda_1 < 1$ with $I_\varphi\left(\frac{y}{\lambda_1}\chi_{(0,a)}\right) < \infty$, by (i). Next, $I_\varphi\left(\frac{y}{\lambda_2}\chi_{(a,m(S_x))}\right) < \infty$ for $\lambda_2 < 1$, so that $\frac{1}{\lambda_2}y(a) < x(a)$. For $\lambda = \max\{\lambda_1, \lambda_2\}$ we have $I_\varphi\left(\frac{y}{\lambda}\right) < \infty$ and, as above, $I_\varphi\left(\frac{y}{\lambda_0}\right) < 1$ for some $\lambda_0 \in (\lambda, 1)$. Thus $\|y\|_\varphi < 1$.
   B. Suppose $I_\varphi(x) < 1$. Then $I_\varphi(y) < 1$ and, as above, we conclude that $\|y\|_\varphi < 1$.

□

4.4. Theorem. *Let $x = x^* \in S(L^\varphi)$. Then x is a UM* point of L^φ if and only if the following statements are satisfied:*

(i) *Let $0 < a < b$ with $x(a) > x(b^-)$. Then*

$$m\{t \in (a,b) : x(t) + 1/n \geqslant a_\varphi(t)\} > 0 \quad \text{for each } n \in \mathbb{N}.$$

Moreover, if $I_\varphi(x) < 1$, then

$$m\{t \in (a,b) : x(t) + 1/n \geqslant b_\varphi(t)\} > 0 \quad \text{for each } n \in \mathbb{N}.$$

(ii) *Let $0 < a \leqslant m(S_x)$ with $x(a^-) > x(a)$. Then*

$$m\{t \in (a,b) : x(t) + 1/n \geqslant a_\varphi(t)\} > 0 \quad \text{for all } b > a \text{ and } n \in \mathbb{N}.$$

Moreover, if $I_\varphi(x) < 1$, then

$$m\{t \in (a,b) : x(t) + 1/n \geqslant b_\varphi(t)\} > 0 \quad \text{for all } b > a \text{ and } n \in \mathbb{N}.$$

(iii) *Let $0 < a \leqslant m(S_x)$. If x is constant in $(0,a)$, then*

$$m\{t \in (0,b) : x(t) + 1/n \geqslant a_\varphi(t)\} > 0 \quad \text{for all } b < a \text{ and } n \in \mathbb{N}.$$

Furthermore, if $I_\varphi(x) < 1$, then

$$m\{t \in (0,b) : x(t) + 1/n \geqslant b_\varphi(t)\} > 0 \quad \text{for all } b < a \text{ and } n \in \mathbb{N}.$$

Proof. Necessity. We divide the proof into several parts.

(i.1) Assume that there are numbers $0 < a < b$ such that $x(a) > x(b^-)$ and $x(t) + 1/n < a_\varphi(t)$ for some $n \in \mathbb{N}$ and for m-a.e. $t \in (a,b)$.

(a) If there is a point of discontinuity $t_0 \in (a,b)$ of x, we set

$$y = x\chi_{I\setminus(t_0,b)} + (x+c)\chi_{(t_0,b)},$$

where $c = \min\{(x(t_0 - 0) - x(t_0))/2, 1/n\}$. Then $y = y^*$, $y \geqslant x$, and $y \neq x$. Applying $I_\varphi(x) = I_\varphi(y)$, we get $\|y\|_\varphi = 1$.

(b) If x is continuous on the interval (a,b), we can find $t_0 \in (a,b)$ such that $x(t_0) > x(b^-)$ and $x(t_0) - x(b^-) \leqslant 1/n$. Then it is enough to take

$$y = x\chi_{I\setminus(t_0,b)} + x(t_0)\chi_{(t_0,b)}.$$

(i.2) Suppose there are numbers $0 < a < b$ such that $x(a) > x(b^-)$, $I_\varphi(x) < 1$, and $x(t) + 1/n_0 < b_\varphi(t)$ for m-a.e. $t \in (a,b)$ and some n_0. Without loss of generality we may assume that $x(b^-) > 0$.

(a) If x is continuous on the interval (a, b), there are $t_0, t_1 \in (a, b)$ with $t_0 < t_1$, $0 < x(t_0) - x(t_1) < 1/n_0$, and

$$I_\varphi \left(x\chi_{I\setminus(t_0,t_1)} + x(t_0)\,\chi_{(t_0,t_1)} \right) \leqslant 1.$$

Let

$$y = x\chi_{I\setminus(t_0,t_1)} + x(t_0)\,\chi_{(t_0,t_1)}.$$

Then $y = y^*$, $y \geqslant x$, and $y \neq x$. Moreover, $I_\varphi(y) \leqslant 1$. By $\|x\|_\varphi = 1$, we get $\|y\|_\varphi = 1$.

(b) Suppose there is a point of discontinuity $t_0 \in (a, b)$ of x. We find $t_1 \in (t_0, b)$ such that

$$I_\varphi \left(x\chi_{I\setminus(t_0,t_1)} + (x + c)\,\chi_{(t_0,t_1)} \right) \leqslant 1 \quad \text{where } c = \min\left\{ \left(x(t_0 - 0) - x(t_0) \right)/2, 1/n_0 \right\}.$$

Taking $y = x\chi_{I\setminus(t_0,t_1)} + (x + c)\,\chi_{(t_0,t_1)}$, we finish as above.

(ii.1) Assume $0 < a \leqslant m(S_x)$, $x(a^-) > x(a)$, and $m\left\{ t \in (a, b) : x(t) + 1/n_0 > a_\varphi(t) \right\} = 0$ for some $n_0 \in \mathbb{N}$ and $b > a$. Define

$$y = x\chi_{I\setminus(a,b)} + \left(x + \min\left\{ 1/n_0, x(a^-) - x(a) \right\} \right) \chi_{(a,b)}.$$

Then $y = y^*$, $y \geqslant x$, and $y \neq x$. Moreover, $I_\varphi(x) = I_\varphi(y)$, so we get $\|y\|_\varphi = 1$.

(ii.2) Suppose $0 < a \leqslant m(S_x)$, $x(a^-) > x(a)$, $I_\varphi(x) < 1$, and

$$m\left\{ t \in (a, b) : x(t) + 1/n_0 \geqslant b_\varphi(t) \right\} = 0$$

for some $n_0 \in \mathbb{N}$, $b > a$. There exist $t_0 \in (a, b)$ and $\delta \in \left(0, \min\left\{ 1/n_0, x(a^-) - x(a) \right\}\right)$ with $I_\varphi \left(x\chi_{I\setminus(a,t_0)} + (x + \delta)\,\chi_{(a,t_0)} \right) \leqslant 1$. Taking

$$y = x\chi_{I\setminus(a,t_0)} + (x + \delta)\,\chi_{(a,t_0)},$$

we finish as in case (i).

(iii.1) Assume that $0 < a \leqslant m(S_x)$, x is constant in $(0, a)$, and

$$m\left\{ t \in (0, b) : x(t) + 1/n_0 > a_\varphi(t) \right\} = 0$$

for some $n_0 \in \mathbb{N}$ and $b < a$. Let

$$y = \left(x + \frac{1}{2n_0} \right) \chi_{(0,b)} + x\chi_{I\setminus(0,b)}.$$

Clearly, $y = y^*$, $y \geqslant x$, and $y \neq x$. By $I_\varphi(x) = I_\varphi(y)$, we get $\|y\|_\varphi = 1$.

(iii.2) Assume that $0 < a \leqslant m(S_x)$, $I_\varphi(x) < 1$, and $m\{t \in (0,b) : x(t) + 1/n_0 \geqslant b_\varphi(t)\} = 0$ for some $n_0 \in \mathbb{N}$ and $b < a$. There are $t_0 \in (0,b)$ and $0 < \delta < 1/n_0$ with

$$I_\varphi\left(x\chi_{I\setminus(0,t_0)} + (x+\delta)\chi_{(0,t_0)}\right) \leqslant 1.$$

Taking

$$y = x\chi_{I\setminus(0,t_0)} + (x+\delta)\chi_{(0,t_0)},$$

we finish as in case (i)

Sufficiency. Let $y = y^*$, $y \geqslant x$, and $y \neq x$. Setting $A = \{t : y(t) > x(t)\}$, we can find an interval $(a,b) \subset A$ and $n_0 \in \mathbb{N}$ such that $a \leqslant m(S_x)$ and $y(t) > x(t) + 1/n_0$ for $t \in (a,b)$. We split the proof in two parts.

a. Assume that $x(a) > x(b^-)$. By (i), $m\{t \in (a,b) : x(t) + 1/n_0 > a_\varphi(t)\} > 0$ and, consequently, $I_\varphi(y) > I_\varphi(x)$. If $I_\varphi(x) = 1$, then $\|y\|_\varphi > 1$. Otherwise, by (i) we have $m\{t \in (a,b) : x(t) + 1/n_0 \geqslant b_\varphi(t)\} > 0$ and, consequently, $I_\varphi(y) = \infty$. Thus $\|y\|_\varphi > 1$.
b. Suppose x is constant in (a,b). If $x(a^-) > x(a)$, then by (ii) we get $I_\varphi(y) > I_\varphi(x)$ and we finish as above.

Now assume that $x(a^-) = x(a)$. If there is $t_0 \in (0,a)$ with $x(t_0) > x(a)$, we proceed as in case 1 or 2 because $y = y^*$. Otherwise, x is constant in $(0,b)$. Since $y = y^*$, $y(t) > x(t) + 1/n_0$ for $t \in (0,b)$. By (iii), it follows that $I_\varphi(y) > I_\varphi(x)$. Thus, applying again (iii), we conclude that $\|y\|_\varphi > 1$ as in case 1. □

4.5. Example. A UM^*point in L^φ need not be a UM point. Let $I = (0,\infty)$. Consider the following Musielak–Orlicz function

$$\varphi(t,u) = \begin{cases} \max\{0, u-(t+2)\} & \text{if } 0 \leqslant t \leqslant 1/2 \text{ and } u \geqslant 0, \\ u & \text{if } 1/2 < t < 1, \\ \max\{0, u-(t-1)\} & \text{if } t \geqslant 1 \text{ and } u \geqslant 0. \end{cases}$$

Note that

$$a_\varphi(t) = \begin{cases} t+2 & \text{if } 0 \leqslant t \leqslant 1/2, \\ 0 & \text{if } 1/2 < t < 1, \\ t-1 & \text{if } t \geqslant 1. \end{cases}$$

Let $x = 2\chi_{(0,1)}$. Then $x = x^*$ and $I_\varphi(x) = \int_{1/2}^1 \varphi(t,2)\,dt = 1$, hence $\|x\|_\varphi = 1$. By Theorem 4.4, x is an UM^* point of L^φ. On the other hand, x is not an UM point of L^φ by Theorem 1 in [11]. Note also that x is an LM^* point of L^φ and x is not an LM point of L^φ (see

Theorem 4.3 above and Theorem 2 in [11]). Moreover, this example shows that condition (ii) in Theorem 4.4 cannot replaced by the following simpler one:

Let $0 < a \leqslant m(S_x)$ with $x(a^-) > x(a)$.

Then $m\left\{t \in (a,b) : x(t) \geqslant a_\varphi(t)\right\} > 0$ for all $b > a$.

Indeed, we have $m\left\{t \in (1,b) : x(t) \geqslant a_\varphi(t)\right\} = 0$ for all $b > 1$. Similarly, considering condition (iii) in Theorem 4.4, we have $m\left\{t \in (0,b) : x(t) + 1/n \geqslant a_\varphi(t)\right\} > 0$ for all $b < 1$ and $n \in \mathbb{N}$, but $m\left\{t \in (0,b) : x(t) \geqslant a_\varphi(t)\right\} = 0$ for all $b < 1/2$.

Applying Theorems 3.6, 4.3 and 4.4, we get

4.6. Corollary. *Let $0 \leqslant x \in S(\Lambda^\varphi)$. Then x is an LM point of Λ^φ if and only if:*
(i) $m\{t \in I : 0 < x(t) \leqslant x^*(\infty)\} = 0$.
(ii) $\theta_\varphi\left(x^*\chi_{(0,\alpha)}\right) < 1$ *for each* $\alpha \in (0, m(S_x))$.
(iii) *If* $I_\varphi(x^*) = 1$ *then* $m\left\{t \in (a, m(S_x)) : x^*(t) > a_\varphi(t)\right\} > 0$ *for each* $a \in (0, m(S_x))$.
(iv) *Let* $0 < a < b < m(S_x)$. *If* x^* *is not constant in* (a,b) *or* x^* *is not continuous at* $t = b$, *then* $m\left\{t \in (a,b) : x^*(t) > a_\varphi(t)\right\} > 0$ *and* $\theta_\varphi\left(x^*\chi_{(b,m(S_x))}\right) < 1$.

4.7. Corollary. *Let $0 \leqslant x \in S(\Lambda^\varphi)$. Then x is a UM point of Λ^φ if and only if:*
(i) $m\{t \in I : x(t) < x^*(\infty)\} = 0$.
(ii) *Let* $0 < a < b$ *with* $x^*(a) > x^*(b^-)$. *Then*

$$m\left\{t \in (a,b) : x^*(t) + 1/n \geqslant a_\varphi(t)\right\} > 0 \quad \text{for each } n \in \mathbb{N}.$$

Moreover, if $I_\varphi(x^*) < 1$, *then*

$$m\left\{t \in (a,b) : x^*(t) + 1/n \geqslant b_\varphi(t)\right\} > 0 \quad \text{for each } n \in \mathbb{N}.$$

(iii) *Let* $0 < a \leqslant m(S_x)$ *with* $x^*(a^-) > x^*(a)$. *Then*

$$m\left\{t \in (a,b) : x^*(t) + 1/n \geqslant a_\varphi(t)\right\} > 0 \quad \text{for all } b > a \text{ and } n \in \mathbb{N}.$$

Moreover, if $I_\varphi(x^*) < 1$, *then*

$$m\left\{t \in (a,b) : x^*(t) + 1/n \geqslant b_\varphi(t)\right\} > 0 \text{ for all } b > a \text{ and } n \in \mathbb{N}.$$

(iv) *Let* $0 < a \leqslant m(S_x)$. *If* x^* *is constant in* $(0,a)$, *then*

$$m\left\{t \in (0,b) : x^*(t) + 1/n \geqslant a_\varphi(t)\right\} > 0 \quad \text{for all } b < a \text{ and } n \in \mathbb{N}.$$

If additionally $I_\varphi(x^*) < 1$, *then*

$$m\left\{t \in (0,b) : x^*(t) + 1/n \geqslant b_\varphi(t)\right\} > 0 \quad \text{for all } b < a \text{ and } n \in \mathbb{N}.$$

We say that Λ^{φ} satisfies the *norm-modular* condition (briefly, $\Lambda^{\varphi} \in (n-m)$) provided $\|x\|_{\varphi} = 1$ implies $I_{\varphi}(x^*) = 1$ for all $x \in \Lambda^{\varphi}$.

Recall that a Banach function space E is *strictly monotone* if and only if each point of $E_+ \setminus \{0\}$ is a *UM* point (equivalently, each point of $E_+ \setminus \{0\}$ is an *LM* point). Applying Corollaries 4.6 and 4.7, we can prove the following

4.8. Theorem. *The generalized Orlicz–Lorentz space Λ^{φ} is strictly monotone if and only if:*

(i) *The space Λ^{φ} satisfies the norm-modular condition.*

(ii) *Set $A_n = \left\{t : a_{\varphi}(t) < 1/n\right\}$. Then $m((a,b) \cap A_n) > 0$ for all $n \in \mathbb{N}$ and $0 < a < b$.*

(iii) *$x^*(\infty) = 0$ holds for each $x \in \Lambda^{\varphi}$.*

Proof. Necessity. (i) Suppose, to the contrary, that Λ^{φ} is strictly monotone and there is $x \in S(\Lambda^{\varphi})$ with $I_{\varphi}(x^*) < 1$. Then $\theta_{\varphi}(x^*) = 1$. Choose $0 < a < m(S_x)$. Applying Corollary 4.6 (ii), we conclude that $\theta_{\varphi}\left(x^*\chi_{(0,a)}\right) < 1$, that is, $I_{\varphi}\left((1+\delta)x^*\chi_{(0,a)}\right) < \infty$ for some $\delta > 0$. Consequently, $\theta_{\varphi}\left(x^*\chi_{(a,m(S_x))}\right) = 1$. Thus, by Corollary 4.6 (iv), x^* is constant in $(0, m(S_x))$. By Corollary 4.7 (iv), $m\left\{t \in (0,a) : x^*(t) + 1/n \geqslant b_{\varphi}(t)\right\} > 0$ for all $n \in \mathbb{N}$. Taking $n_0 \in \mathbb{N}$ such that $n_0 x^*(a)\delta > 1$, we get $(1+\delta)x^*(t) > x^*(t) + 1/n_0$ for $t \in (0,a)$, hence $I_{\varphi}\left((1+\delta)x^*\chi_{(0,a)}\right) = \infty$, a contradiction.

(ii) Assume that $m((a,b) \cap A_{n_0}) = 0$ for some $n_0 \in \mathbb{N}$ and $0 < a < b$. Setting $B_{n_0} = (a,b) \cap A_{n_0}^c$, where $A_{n_0}^c = (a,b) \setminus A_{n_0}$, we have $m(B_{n_0}) = b - a$. Denote by u_a the number from condition (2).

If $u_a = \infty$ or $\int_0^a \varphi(s,u_a)\,ds > 1$ with $u_a < \infty$, then we claim that

$$\int_0^a \varphi(s,\alpha)\,ds = 1 \text{ for some number } \alpha < u_a.$$

Define $F(t) = \int_0^a \varphi(s,t)\,ds$. The claim follows from the following facts:

(a) If $u_a = \infty$, the function F is continuous in $(0,\infty)$ (since $L^1 \in (OC)$), $F(0) = 0$, and $F(\infty) = \infty$.

(b) If $u_a < \infty$ and $\int_0^a \varphi(s,u_a)\,ds > 1$, the function F is continuous in $(0,u_a)$, $F(0) = 0$, and $F(u_a) > 1$.

Take $n_1 \geqslant n_0$ with $1/n_1 \leqslant \alpha$. Setting

$$x = \alpha\chi_{(0,a)} + (1/n_1)\,\chi_{(a,b)}$$

we conclude that $x^* = x$ and x is not an *LM* point, by Corollary 4.6 (iii).

If $\int_0^a \varphi(s,u_a)\,ds \leqslant 1$, then we set $x = u_a\chi_{(0,a)} + (1/n_1)\,\chi_{(a,b)}$ with $n_1 \geqslant n_0$ satisfying $1/n_1 \leqslant u_a$. Then $\|x\|_{\Lambda^{\varphi}} = 1$ and $\theta_{\varphi}\left(x^*\chi_{(0,u_a)}\right) = 1$, hence x is not an *LM* point, by Corollary 4.6 (ii).

The condition (iii) follows from Corollary 4.6 (i) (see also Corollary 3.13 in [18]).

Sufficiency. Take $0 \leqslant x \leqslant y \in S(\Lambda^\varphi)$ and $x \neq y$. By (i), $I_\varphi(y^*) = 1$. Clearly, $x^* \leqslant y^*$. By (iii), we have $x^* \neq y^*$ (see Lemma 3.2 in [14] or Lemma 2.1 in [3] for a more general case). Moreover, by the right-continuity of the decreasing rearrangement, there is an interval (a, b) with $x^*(t) < y^*(t)$ for $t \in (a, b)$. We can find a number n and an interval $(c, d) \subset (a, b)$ such that $y^*(t) > 1/n$ for $t \in (c, d)$. Thus $I_\varphi(x^*) < 1$ by (ii). Finally, $\|x\|_{\Lambda^\varphi} < 1$ by (i). □

4.9. Remark. Theorem 4.8 is a generalization of Theorem 5.1 in [5]. First, note that if φ satisfies condition Δ_2^Λ (see Definition 2.3 in [5]), then $\Lambda^\varphi \in (n-m)$ by Proposition 2.10 in [5]. Moreover, the author of [5] assumes in Theorem 5.1 the so-called conditions $(L1)$ and $(L2)$. Condition $(L1)$ guarantees that $\|\cdot\|_{\Lambda^\varphi}$ is a norm in Λ^φ (see Theorem 1.2 in [5]), however, for monotonicity properties it is natural to consider also quasi-normed spaces. Moreover, condition $(L2)$ implies (iii) in Theorem 4.8 automatically (see Proposition 1.6 in [5]). Finally, condition (i) from Theorem 5.1 in [5] is

- essentially stronger than condition (ii) in the above theorem and
- not necessary in general; it is necessary when we assume conditions $(L1)$ and $(L2)$.

Now we discuss sufficient conditions for a point x to be a point of order continuity or of lower local uniform monotonicity in Λ^φ. Applying Definition 1 from [20] for $E = L^1$, we get

4.10. Definition. Let $x \in L^\varphi$. We say φ satisfies a local $\Delta_2^{L^1}(x)$ condition with respect to x ($\varphi \in \Delta_2^{L^1}(x)$, for short) if for each $l > 1$ we have

$$I_\varphi\left(lx\chi_{A_k^l}\right) \to 0 \text{ as } k \to \infty,$$

where

$$A_k^l = \left\{t \in S_x : l|x(t)| < b_\varphi(t) \text{ and } \varphi(t, lx(t)) > k\varphi(t, x(t))\right\}.$$

Clearly, if $b_\varphi \equiv \infty$, $x \in B(L^\varphi)$ and $\varphi \in \Delta_2^{L^1}(x)$, then $\theta_\varphi(x) = 0$. Applying Theorem 11 from [20] for $E = L^1$, we obtain

4.11. Corollary. *Let $x \in B(L^\varphi)$. Then $x \in (L^\varphi)_a$ if and only if:*

(i) $\varphi \in \Delta_2^{L^1}(x\chi_C)$, *where* $C = \left\{t \in S_x : a_\varphi(t) < |x(t)|\right\}$.
(ii) $\varphi \circ (ma_\varphi)\chi_{C_m} \in L^1$ *for every* $m \in N$, *where* $C_m = \left\{t \in S_x : \frac{1}{m}a_\varphi(t) \leqslant |x(t)| \leqslant a_\varphi(t)\right\}$.
(iii) $m(S_x \cap D) = 0$, *where* $D = \left\{t \in I : b_\varphi(t) < \infty\right\}$.

Now taking into account Theorem 3.6, we have

4.12. Corollary. *Let $x \in B(\Lambda^\varphi)$. Assume the following conditions are satisfied.*

(i) $\varphi \in \Delta_2^{L^1}(x^*\chi_C)$, *where* $C = \left\{t \in S_{x^*} : a_\varphi(t) < x^*(t)\right\}$.
(ii) $\varphi \circ (ma_\varphi)\chi_{C_m} \in L^1$ *for every* $m \in N$, *where* $C_m = \left\{t \in S_{x^*} : \frac{1}{m}a_\varphi(t) \leqslant x^*(t) \leqslant a_\varphi(t)\right\}$.

(iii) $m(S_{x^*} \cap D) = 0$, *where* $D = \{t \in I : b_\varphi(t) < \infty\}$.
(iv) $x^*(\infty) = 0$.
Then $x \in (\Lambda^\varphi)_a$.

Recall that if E is a symmetric Banach function space then $x \in E_+$ is an *LLUM* point if and only if x is an *LM* point and an $x \in E_a$ (see [3, Theorem 2.1]). Note that under additional assumption that $E \hookrightarrow L^1 + L^\infty$ almost the same proof works for a symmetric quasi-Banach function space. Notice also that if $E \hookrightarrow L^1 + L^\infty$ then $E^{(*)} \hookrightarrow L^1 + L^\infty$. Consequently, applying Corollaries 4.12 and 4.6, we get

4.13. Corollary. *Let* $L^\varphi \hookrightarrow L^1 + L^\infty$ *and* $x \in S(\Lambda^\varphi)$. *Assume the following conditions are satisfied:*
(i) $\varphi \in \Delta_2^{L^1}(x^*\chi_C)$, *where* $C = \{t \in S_{x^*} : a_\varphi(t) < x^*(t)\}$.
(ii) $\varphi \circ (ma_\varphi)\chi_{C_m} \in L^1$ *for every* $m \in N$, *where* $C_m = \{t \in S_{x^*} : \frac{1}{m}a_\varphi(t) \leqslant x^*(t) \leqslant a_\varphi(t)\}$.
(iii) $m(S_{x^*} \cap D) = 0$, *where* $D = \{t \in I : b_\varphi(t) < \infty\}$.
(iv) $x^*(\infty) = 0$.
(v) $\theta_\varphi\left(x^*\chi_{(0,\alpha)}\right) < 1$ *for each* $\alpha \in (0, m(S_x))$.
(vi) *If* $I_\varphi(x^*) = 1$, *then* $m\{t \in (a, m(S_x)) : x^*(t) > a_\varphi(t)\} > 0$ *for each* $a \in (0, m(S_x))$.
(vii) *Let* $0 < a < b < m(S_x)$. *If* x^* *is not constant in* (a, b) *or* x^* *is not continuous at* $t = b$, *then* $m\{t \in (a, b) : x^*(t) > a_\varphi(t)\} > 0$ *and* $\theta_\varphi\left(x^*\chi_{(b,m(S_x))}\right) < 1$.
Then x *is an LLUM point of* Λ^φ.

4.14. Problem. Find necessary and sufficient conditions for boundedness of the dilation operator D_2 in the cone of nonnegative and nonincreasing elements of the space $(L^\varphi, \|\cdot\|_\varphi)$ and compare them with conditions $(L1)$ and $(L2)$ from [5]. Note that in the case of two-weighted Orlicz–Lorentz spaces $\Lambda_{\Phi,w,v}(I)$ the respective condition (3) is necessary and sufficient for boundedness of the operator D_2 under some additional assumptions (see Corollary 4.8 in [17]).

4.15. Problem. Find the full criteria for the point of order continuity in Λ^φ. By Theorem 3.6, it is enough to prove a characterization for $x = x^* \in L^\varphi$ to be an OC^* point in L^φ. It seems that the respective conditions can be essentially weaker than in Corollary 4.11.

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