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On two functional equations connected with distributivity of fuzzy implications

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Summary. The distributivity law for a fuzzy implication $I\colon [0,1]^2 \to [0,1]$ with respect to a fuzzy disjunction $S\colon [0,1]^2 \to [0,1]$ states that the functional equation $I(x, S(y,z)) = S(I(x,y), I(x,z))$ is satisfied for all pairs (x,y) from the unit square. To compare some results obtained while solving this equation in various classes of fuzzy implications, Wanda Niemyska has reduced the problem to the study of the following two functional equations: $h(\min(xg(y),1)) = \min(h(x) + h(xy),1)$, $x \in (0,1)$, $y \in (0,1]$, and $h(xg(y)) = h(x) + h(xy)$, $x, y \in (0,\infty)$, in the class of increasing bijections $h\colon [0,1] \to [0,1]$ with an increasing function $g\colon (0,1] \to [1,\infty)$ and in the class of monotonic bijections $h\colon (0,\infty) \to (0,\infty)$ with a function $g\colon (0,\infty) \to (0,\infty)$, respectively. A description of solutions in more general classes of functions (including nonmeasurable ones) is presented.

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fuzzy implication;
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Dedicated to Professor Henryk Hudzik
on the occasion of his 70th birthday.

The following tautology of the classical sentential calculus:

$$[x \to (y \vee z)] \equiv [(x \to y) \vee (x \to z)]$$

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may be viewed as distributivity of implication with respect to logical sum (disjunction). Replacing the logical connectives $\to$ and $\vee$ by a fuzzy implication $I\colon[0,1]^2 \to [0,1]$ and a fuzzy disjunction $S\colon[0,1]^2 \to [0,1]$ (a t-conorm), respectively, leads to a functional equation

$$I\big(x, S(y,z)\big) = S\big(I(x,y), I(x,z)\big),$$

assumed to be valid for all pairs (x,y) from the unit square. This equation plays an important role in fuzzy control systems and was solved in various classes of fuzzy implications (see [1, 2] and [4]). To compare these results, Wanda Niemyska (in her doctoral dissertation [4]) has reduced the problem to the study of the following two functional equations:

$$h\big(\min(xg(y),1)\big) = \min\big(h(x) + h(xy), 1\big), \quad x \in (0,1),\ y \in (0,1],$$

and

$$h\big(xg(y)\big) = h(x) + h(xy), \quad x, y \in (0,\infty),$$

in the class of increasing bijections $h\colon[0,1] \to [0,1]$ with an increasing function $g\colon(0,1] \to [1,\infty)$, and in the class of monotonic bijections $h\colon(0,\infty) \to (0,\infty)$ with a function $g\colon(0,\infty) \to (0,\infty)$, respectively.

In what follows, our attention will be focused upon the latter functional equation. We proceed with the following folklore lemma whose proof will be omitted.

1. Lemma. *Given a real number x, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x and let $\mu(x) = x - \lfloor x \rfloor$. Then for every irrational number δ the set $\{\mu(n\delta) : n \in \mathbb{N}\}$ is dense in $[0,1]$.*

2. Theorem. *Assume that the functions $g, h\colon(0,\infty) \to (0,\infty)$ satisfy the functional equation*

$$h\big(xg(y)\big) = h(x) + h(xy), \quad x, y \in (0,\infty). \tag{1}$$

If h is continuous, then there exist a real constant $M > 0$ and a nonzero real p such that

$$h(x) = Mx^p,\ x \in (0,\infty), \quad \text{and} \quad g(x) = (1 + x^p)^{\frac{1}{p}},\ x \in (0,\infty).$$

Conversely, each pair (h,g) of such functions yields a solution to equation (1).

Proof. Let the pair (h,g) be a suitable solution to equation (1). Obviously, the positivity of h forces the inequality $g(1) \neq 1$. Put $p := [\log_2 g(1)]^{-1}$ and $k(x) := x^{-p}h(x), x \in (0,\infty)$; then (1) assumes the form

$$g(y)^p k\big(xg(y)\big) = k(x) + y^p k(xy), \quad x, y \in (0,\infty). \tag{2}$$

Let $f\colon\mathbb{R} \to (0,\infty)$ be defined by the formula

$$f(t) := k\big(2^{\frac{t}{p}}\big), \quad t \in \mathbb{R}.$$

On setting $y = 1$ in (2), in view of the equality $g(1) = 2^{\frac{1}{p}}$ resulting from the definition of p, we easily derive the periodicity of f:

$$f(t+1) = f(t), \quad t \in \mathbb{R}.$$

Moreover, a simple calculation shows that

$$g(y)^p f\big(t + p\log_2 g(y)\big) = f(t) + y^p f(t + p\log_2 y), \quad t \in \mathbb{R},\ y \in (0,\infty). \tag{3}$$

Fix arbitrarily a positive irrational number r and put $y_0 := g(1)^r = 2^{\frac{r}{p}}$ and $q := p\log_2 g(y_0)$; then $r = p\log_2 y_0$. Put $y = y_0$ in (3) to obtain

$$2^q f(t+q) = f(t) + 2^r f(t+r), \quad t \in \mathbb{R}. \tag{4}$$

Due to continuity, periodicity and positivity of f, the symbols

$$M := \max\big\{f(t) : t \in \mathbb{R}\big\} \quad \text{and} \quad m := \min\big\{f(t) : t \in \mathbb{R}\big\}$$

denote well-defined positive real numbers attained as values of f at some points a, b from the unit interval, i.e. $M = f(a)$ and $m = f(b)$. Set $t = a - q$ and $t = b - q$ in (4) to get the estimates

$$2^q M = f(a-q) + 2^r f(a-q+r) \leqslant M + 2^r M \tag{5}$$

and

$$2^q m = f(b-q) + 2^r f(b-q+r) \geqslant m + 2^r m, \tag{6}$$

forcing $2^q \leqslant 1 + 2^r \leqslant 2^q$, which shows that inequalities (5) and (6) are in fact equalities. Therefore, we have, in particular,

$$f(a-q) + 2^r f(a-q+r) = M + 2^r M,$$

which by the definition of M implies that

$$f(a-q) = f(a-q+r) = M.$$

Plainly, at least one of the numbers $-q, -q+r$ must be irrational (since r is irrational). Thus we have $f(a) = M = f(a+\delta)$ for some irrational $\delta \in \mathbb{R}$, which shows that the role of a may be assumed by $a+\delta$. By induction, we infer that

$$f(a+n\delta) = M \quad \text{for every } n \in \mathbb{N}.$$

In view of the 1-periodicity of f, we obtain

$$M = f(a+n\delta) = f\big(\lfloor a+n\delta\rfloor + \mu(a+n\delta)\big) = f\big(\mu(a+n\delta)\big) = f\big(a+\mu(n\delta)\big)$$

for all $n \in \mathbb{N}$. In virtue of the lemma, the set $D := \{\mu(n\delta) : n \in \mathbb{N}\}$ is dense in $[0,1]$, whence its shift $D+a$ is dense in the interval $[a, a+1]$. Since $f|_{D+a} = M$ and f is continuous, we get the equality $f|_{[a,a+1]} = M$, which by the 1-periodicity of f implies that $f(x) = M$ for all $x \in \mathbb{R}$. This, in turn, easily implies that $h(x) = Mx^p$ for all $x \in (0,\infty)$ and, by equation (1), we obtain $g(x) = (1 + x^p)^{\frac{1}{p}}$, $x \in (0,\infty)$, as claimed. Since the reverse implication is trivial, the proof is complete. □

3. Remark. As pointed out in [4], when dealing with the functional equation

$$I\big(x, S(y,z)\big) = S\big((I(x,y), I(x,z)\big) \tag{7}$$

one basically looks for solutions of (1) that are homeomorphisms of the positive half-line $(0,\infty)$ onto itself. More specifically, one needs to find solutions to the equation

$$h\big(\min(xg(y),1)\big) = \min\big(h(x) + h(xy),1\big), \quad x \in (0,1),\ y \in (0,1], \tag{8}$$

in the class of increasing bijections $h\colon [0,1] \to [0,1]$ with an increasing function $g\colon (0,1] \to [1,\infty)$. Imitating the method of proof of Theorem 2, it has been shown in [4] that the only solutions to equation (8) (in that class) are the functions $h(x) = x^p$, $x \in [0,1]$, and $g(x) = (1 + x^p)^{\frac{1}{p}}$, $x \in [0,1]$, where p stands for a positive real constant.

4. Remark. There exist discontinuous solutions to equation (1) with a bijective function h that fail to be Lebesgue measurable. In fact, take a discontinuous additive bijection a of the real line onto itself (to convince yourself that such monsters do exist, cf., e.g., M. Kuczma's monograph [3, Chapter XII, Section 5, Theorem 1]) and put $h := \exp \circ a \circ \log$. Then h is a discontinuous (actually, nonmeasurable) bijection of the half-line $(0,\infty)$ onto itself and

$$h(xy) = h(x)h(y) \quad \text{for all} \quad x, y \in (0,\infty).$$

Nevertheless, the pair (h, g) with g given by the formula

$$g(x) = h^{-1}\big(1 + h(x)\big), \quad x \in (0,\infty),$$

yields a solution to equation (1).

5. Theorem. *Assume that the functions $g, h\colon (0,\infty) \to (0,\infty)$ satisfy the functional equation*

$$h\big(xg(y)\big) = h(x) + h(xy), \quad x, y \in (0,\infty). \tag{1}$$

If h is bijective, then there exist a positive real constant M and a bijection c of the half-line $(0,\infty)$ onto itself such that

$$c(xy) = c(x)c(y) \quad \textit{for all } x, y \in (0,\infty),$$

and

$$h(x) = Mc(x),\ g(x) = c^{-1}\big(1 + c(x)\big), \quad x \in (0, \infty).$$

Conversely, each pair of such functions h and g yields a solution to equation (1)*. In particular, if h is a continuous and bijective solution to equation* (1)*, then*

$$h(x) = Mx^p, \quad \text{and} \quad g(x) = (1 + x^p)^{\frac{1}{p}}, \qquad x \in (0, \infty),$$

where p stands for a real nonzero constant.

Proof. We have

$$xg(y) = h^{-1}\big(h(x) + h(xy)\big) \quad \text{for all } x, y \in (0, \infty),$$

or, equivalently,

$$xg\left(\frac{y}{x}\right) = h^{-1}\big(h(x) + h(y)\big) \quad \text{for all } x, y \in (0, \infty).$$

Consequently, for every $\lambda \in (0, \infty)$, we get

$$\lambda h^{-1}\big(h(x) + h(y)\big) = \lambda xg\left(\frac{y}{x}\right) = h^{-1}\big(h(\lambda x) + h(\lambda y)\big) \quad \text{for all } x, y \in (0, \infty).$$

Hence,

$$h\big(\lambda h^{-1}(h(x) + h(y))\big) = h(\lambda x) + h(\lambda y) \quad \text{for all } x, y \in (0, \infty),$$

and, setting here $h^{-1}(x)$ in place of x and $h^{-1}(y)$ in place of y, we arrive at the equation

$$H_\lambda(x + y) = H_\lambda(x) + H_\lambda(y)$$

valid for all $x, y \in (0, \infty)$, where $H_\lambda(x) := h(\lambda h^{-1}(x))$, $x \in (0, \infty)$. Since h is positive, so are the additive functions H_λ and, *a fortiori*, they are continuous (Bernstein–Doetsch theorem, see, e.g., M. Kuczma [3]), which implies that

$$h\big(\lambda h^{-1}(x)\big) = H_\lambda(x) = c(\lambda)x \quad \text{for all } \lambda, x \in (0, \infty),$$

whence

$$h(\lambda x) = c(\lambda)h(x) \quad \text{for all } \lambda, x \in (0, \infty).$$

Now, on setting $M := h(1) > 0$, we infer that

$$h(\lambda) = Mc(\lambda),\ \lambda \in (0, \infty), \quad \text{and} \quad c(\lambda)c(x) = c(\lambda x) \qquad \text{for all } \lambda, x \in (0, \infty).$$

Obviously, the function c is bijective because so is the function h. The form of the function h just obtained, in combination with equation (1) (with $x = 1$), forces that the function g is of the form

$$g(y) = c^{-1}\big(1 + c(y)\big), \quad y \in (0, \infty).$$

With the aid of Theorem 6 from M. Kuczma' s monograph [3, Chapter XIII, Section 1], the proof of the remaining assertions is straightforward. □

Some concluding remarks:

- any continuous solution $h\colon (0,\infty) \to (0,\infty)$ of equation (1), normalized by $h(1) = 1$, is bijective and multiplicative (see Theorem 2);
- normalized bijective solutions have to be multiplicative but need not be continuous (see Theorem 5);
- each solution $h\colon (0,\infty) \to (0,\infty)$ of equation (1) admits a representation of the form
$$h(x) = x^p f\left(\log_2 x^p\right),\ x \in (0,\infty),$$
where $f\colon \mathbb{R} \to \mathbb{R}$ is a 1-periodic function and $p := [\log_2 g(1)]^{-1}$ (see the first part of the proof of Theorem 2);
- any multiplicative solution h of Abel's functional equation
$$h\big(g(y)\big) = h(y) + 1$$
yields a solution to equation (1) (direct calculation);
- for any solution $h\colon (0,\infty) \to (0,\infty)$ of equation (1) one has
$$\sup h\big((0,\infty)\big) = \infty \quad \text{and} \quad \inf h\big((0,\infty)\big) = 0.$$
Indeed, put $y = 1$ in equation (1); then simple induction with $\alpha := g(1)$ gives the equalities
$$h(\alpha^n x) = 2^n h(x) \quad \text{for all } n \in \mathbb{N} \text{ and all } x \in (0,\infty),$$
whence
$$h\left(\frac{1}{\alpha^n}x\right) = \frac{1}{2^n}h(x) \quad \text{for all } n \in \mathbb{N} \text{ and all } x \in (0,\infty);$$
- any solution $h\colon (0,\infty) \to (0,\infty)$ to equation (1) with the Darboux property is surjective (an immediate consequence of the preceding remark).

We close this paper with more detailed information concerning the occurrence of equation (8) (see Remark 3 above). The following result has been proved in [4] (one of the main results of the dissertation [4]):

6. Theorem. *Let S be an Abelian binary operation (not necessarily associative) in the unit interval with increasing sections $S(x,\cdot)$ and $S(\cdot,y)$, $x,y \in [0,1]$, and with 0 as a neutral element. Let, further, $I\colon [0,1]^2 \to [0,1]$ be a fuzzy implication of the form*
$$I(x,y) := \begin{cases} 1, & \text{if } x \leqslant y, \\ \varphi^{-1}\left(\frac{\varphi(y)}{\varphi(x)}\right), & \text{if } x > y, \end{cases} \qquad x,y \in (0,\infty),$$
where φ stands for an increasing bijection of the unit interval onto itself. If the pair (S,I) yields a solution to the distributivity equation (7), then either
$$S(x,y) := \begin{cases} \max(x,y), & \text{if } x = 0 \text{ or } y = 0, \\ 1, & \text{otherwise}, \end{cases} \qquad x,y \in [0,1],\ (\textit{the so-called drastic t-conorm}),$$

or there exists an increasing function $g\colon (0,1] \to [1,\infty)$ *such that for all* $x, y \in (0,1)$, $x \geqslant y$, *one has*

$$S(x,y) = \varphi^{-1}\Big(\min\Big(g\Big(\frac{\varphi(y)}{\varphi(x)}\Big) \cdot \varphi(x), 1\Big)\Big).$$

Conversely, all such pairs (S, I) *are solutions to the distributivity equation* (7).

M. Baczyński and B. Jayaram [1] have solved the equation (7) in a less general family of functions S, i.e. continuous and Archimedean t-conorms, which, in particular, are associative; associativity is not assumed in Theorem 6. Through a description of solutions to equation (8) one may show (cf. [4]) that the Baczyński–Jayaram result is a particular case of Theorem 6.

7. Remark. In a paper of J. Balasubramaniam and C. J. M. Rao [2], the authors write that they *have a strong feeling* the max function is the only t-conorm satisfying equation (7). Theorem 6 shows, among other things, that their conjecture was inaccurate; numerous other solutions exist.

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