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Stability of forward-backward finite difference schemes for certain problems in biology

Abstract. We present a discretization method for a generalized von Foerster-type equation in many spatial variables. Stability of finite difference schemes on regular meshes is studied. If characteristic curves are decreasing, there are forward difference quotients applied. Otherwise, the derivatives are replaced by backward difference quotients.

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1. Introduction. To describe an evolution of a population there were used various mathematical models, eg. Malthus [20], logistic [25], SIR [21], the Lotka-Volterra equations [19]. Although the above mentioned models enabled to investigate successfully a lot of phenomena, they do not yield any information concerning the age distribution of the population. Lotka [19] and von Foerster [9] proposed a model which takes into consideration an influence of the age structure on birth and death processes. These models were generalized in such a way that independent variables can describe not only the age structure, but also other parameters of the population like its size, psychological properties of households, nutrition state and others. In [2], [7, 8], [11], [23, 24] these models were applied for describing some phenomena in biology, ecology, epidemiology and medicine.

Many topics of the mathematical biology are presented in [3], [14], [19] and [22].

The existence, uniqueness, qualitative theory of the von Foerster-type equations were considered in [5, 6], [10], [15], [17], [23]. Results of [6] and [17] are based on transformations of given problems into systems of ordinary functional differential equations along bicharacteristics. Next, there are defined integral operators for which the existence of fixed-point equations is proved. Additional assumptions on given functions lead to uniqueness of solutions. In [10] existence and uniqueness of solutions is established by transformations of the main problem into a pair of coupled integral equations and studying its properties.

The papers [1, 2] are concerned with finite difference schemes for non-linear hyperbolic initial-boundary value problems in bounded domains with non-local boundary condition. In [18] there are studied stability and consistency for Euler schemes for the von Foerster-type equations in unbounded domains. The monograph [13] presents a systematic treatment of nonlinear functional differential problems, including difference methods for initial and mixed problems. Convergence results for unbounded solutions of first order differential-functional equations are studied in [16].

In our paper the following von Foerster model is studied. Suppose that $c_k\colon E \times \mathbb{R}_+ \to \mathbb{R}$, $k = 1, \ldots, n$, $\lambda\colon E \times \mathbb{R}_+^2 \to \mathbb{R}$, where $E = [0, a] \times \mathbb{R}_+^n$, $\mathbb{R}_+ = [0, +\infty)$, $a > 0$. Consider the initial value problem

$$\partial_t u(t,x) + \sum_{k=1}^{n} c_k\,(t,x,z(t))\,\partial_{x_k} u(t,x) = \lambda\bigl(t,x,u(t,x),z(t)\bigr), \tag{1}$$

where

$$z(t) = z[u(t,\cdot)] = \int_0^\infty \cdots \int_0^\infty u(t,x)\;dx_1 \ldots dx_n, \quad t \in [0,a] \tag{2}$$

with the initial condition

$$u(0,x) = v(x), \quad x \in \mathbb{R}_+^n, \tag{3}$$

where $v\colon \mathbb{R}_+^n \to \mathbb{R}_+$ is a given continuous and integrable function. The well-posedness of problem (1)–(3) demands the condition $c\,(t,0,q) \leq 0$, $q \in \mathbb{R}_+$, that is: the characteristics either go out of the set through the lateral boundary or meet the boundary and remain there.

According to [17] the above model can be generalized in the following ways: (i) including many species, (ii) taking into consideration past densities and past total sizes of species.

We are interested in a discretization of problem (1)–(3) using finite difference schemes on rectangular meshes. The main frame of our investigations is related to the Lax-Richtmyer equivalence theorem, which splits this task to stability and consistency. In this paper we focus on stability. Being inspired by [4] and [12, 13] we study a discretization method for problem (1)–(3) applying either forward or backward spatial difference quotients, depending on the flow of characteristics. The introduced finite difference scheme is explicit. We stress that even for two or three dimensional spaces the discretization involves very large number of arithmetic operations. We prove a stability theorem for this scheme with respect to: perturbations of the right hand side, initial conditions and cuts of the quadrature. The general theory is illustrated by some numerical experiments in $\mathbb{R}^3$.

2. Discretization of the differential problem. Let $\mathbb{N} = \{0, 1, \ldots\}$. For $x, y \in \mathbb{R}^n$ define $x * y = (x_1 y_1, \ldots, x_n y_n)$. We introduce in E a rectangular mesh as follows. For any given number $N_0 \in \mathbb{N}$ we define $h_0 = \dfrac{a}{N_0}$, and for given numbers $h_1, \dots, h_n \in (0, \infty)$, we denote by $h' = (h_1, \dots, h_n)$, $h = (h_0, h')$. The knots are denoted by $\left(t^{(i)}, x^{(j)}\right)$, where $t^{(i)} = ih_0$, $x^{(j)} = j * h'$. Let $E_h = \left\{\left(t^{(i)}, x^{(j)}\right) : i = 0, \dots, N_0,\ j \in \mathbb{N}^n\right\}$. The values of any discrete function $u\colon E_h \to \mathbb{R}_+$ at the knot $\left(t^{(i)}, x^{(j)}\right)$ will be denoted by $u^{(i,j)} = u\left(t^{(i)}, x^{(j)}\right)$.

Define the discrete operators $\delta_0, \delta_k^+, \delta_k^-, Q_h$:

$$\begin{aligned}
\delta_0 u^{(i,j)} &= \frac{u^{(i+1,j)} - u^{(i,j)}}{h_0}, \qquad \delta_k^+ u^{(i,j)} = \frac{u^{(i,j+e_k)} - u^{(i,j)}}{h_k}, \\
\delta_k^- u^{(i,j)} &= \frac{u^{(i,j)} - u^{(i,j-e_k)}}{h_k}, \qquad (Q_h u)_i = h_1 \dots h_n \sum_{j \in \mathbb{N}^n} u^{(i,j)},
\end{aligned}$$

where $e_k = (0, \dots, 1, \dots, 0)$ and Q_h is an infinite multidimensional quadrature. The number $\delta_0 u^{(i,j)}$ approximates the derivative $\partial_t u\left(t^{(i)}, x^{(j)}\right)$, whereas the derivatives $\partial_{x_k} u\left(t^{(i)}, x^{(j)}\right)$ can be approximated in two ways: either by progressive difference quotients $\delta_k^+ u^{(i,j)}$ or regressive difference quotients $\delta_k^- u^{(i,j)}$. The quadrature $(Q_h u)_i$ is a first-order approximation of the integral (2) at $t = t^{(i)}$. While performing practical computations we replace $(Q_h u)_i$ by the following finite sum

$$(Q_h^{N_h} u)_i = h_1 \dots h_n \sum_{j_1, \dots, j_n = 0}^{N_h} u^{(i, j_1, \dots, j_n)},$$

where N_h is a sufficiently large natural number, usually proportional to $\max\limits_{k=1,\dots,n} \left[(1/h_k) \log(1/h_k)\right]$. Notice that $\|h'\| N_h \to +\infty$ as $\|h'\| \to 0$, where $\|h'\| = \max\{h_1, \dots, h_n\}$.

In order to make the descriptions concise, denote

$$c_k^{(i,j)}[z] = c_k\left(t^{(i)}, x^{(j)}, z^{(i)}\right), \quad \lambda^{(i,j)}[u, z] = \lambda\left(t^{(i)}, x^{(j)}, u^{(i,j)}, z^{(i)}\right).$$

For $\left(t^{(i)}, x^{(j)}\right) \in E_h$, $q \in \mathbb{R}$ define the characteristic function

$$\chi_k^{(i,j)}[q] = \begin{cases} 1, & c_k^{(i,j)}[q] < 0, \\ 0, & c_k^{(i,j)}[q] \geq 0. \end{cases}$$

Consider a finite difference problem for (1)–(3)

$$\delta_0 u^{(i,j)} + \sum_{k=1}^{n} c_k^{(i,j)}[z] \left\{ \chi_k^{(i,j)}[z] \delta_k^+ u^{(i,j)} + \left(1 - \chi_k^{(i,j)}[z]\right) \delta_k^- u^{(i,j)} \right\} = \lambda^{(i,j)}[u, z] \tag{4}$$

on E_h, where $z^{(i)} = \left(Q_h u\right)_i$ with the initial condition

$$u^{(0,j)} = v^{(j)} \quad \text{for} \quad j \in \mathbb{N}^n. \tag{5}$$

Denote by $L^{\infty}\left(\mathbb{R}_{+}^{n}\right)$ and $L^{1}\left(\mathbb{R}_{+}^{n}\right)$ the classes of all essentially bounded measurable functions and Lebesgue integrable functions defined on $\mathbb{R}_{+}^{n}$. Denote by $C(X,\mathbb{R})$ the class of all continuous functions $u\colon X\to\mathbb{R}$. Define the following class of submonotone integrable functions: $f\in L^{1}_{\mathcal{M}}$ iff there exists a nonnegative and decreasing function $g\in L^{1}(\mathbb{R}_{+}^{n})$ such that $|f(x)|\le g(x)$ for $x\in\mathbb{R}_{+}^{n}$.

Introduce the following normed spaces. In the space l_{n}^{∞}, of all sequences $\psi=\left(\psi_{j}\right)_{j\in\mathbb{N}^{n}}$, we have the natural supremum norm

$$\|\psi\|_{\infty}=\sup_{j\in\mathbb{N}^{n}}|\psi_{j}|\quad\text{ for }(\psi_{j})\in l_{n}^{\infty}.$$

The space l_{n}^{1}, of all summable sequences $\psi=\left(\psi_{j}\right)_{j\in\mathbb{N}^{n}}$, is equipped with the norm

$$\|\psi\|_{1}=h_{1}\cdots h_{n}\sum_{j\in\mathbb{N}^{n}}|\psi_{j}|\quad\text{ for }(\psi_{j})\in l_{n}^{1}.$$

REMARK 2.1 Let $f\colon\mathbb{R}_{+}^{n}\to\mathbb{R}$, $f\in L^{1}_{\mathcal{M}}$ and $h'=(h_{1},\dots,h_{n})\in\mathbb{R}_{+}^{n}$. By f_{h} denote the restriction of the function f to the set $R_{h}=\left\{x^{(j)}\;:\;j\in\mathbb{N}^{n}\right\}$. Then $\|f_{h}\|_{1}<\infty$.

In the paper we assume that:

ASSUMPTION $[V]$. The initial function $v\colon\mathbb{R}_{+}^{n}\to\mathbb{R}_{+}$ is bounded, continuous and $v\in L^{1}_{\mathcal{M}}$.

ASSUMPTION $[C]$. The functions $c_{k}\colon E\times\mathbb{R}_{+}\to\mathbb{R}$, $k=1,\dots,n$, are continuous, bounded and there are: a constant $L_{c}>0$ and a bounded, nonnegative function $L_{c}^{*}\in L^{1}_{\mathcal{M}}$ such that

$$\left|c_{k}\left(t,x,q\right)-c_{k}\left(t,\bar{x},\bar{q}\right)\right|\le L_{c}\left\|x-\bar{x}\right\|+L_{c}^{*}\left(x+\bar{x}\right)\left|q-\bar{q}\right|$$

for $(t,x,q),(t,\bar{x},\bar{q})\in E\times\mathbb{R}_{+}$.

ASSUMPTION $[S]$. The functions $c_{k}\colon E\times\mathbb{R}_{+}\to\mathbb{R}$, $k=1,\dots,n$ and the steps h_{k}, $k=0,1,\dots,n$, satisfy the stability condition

$$1-\sum_{k=1}^{n}\frac{h_{0}}{h_{k}}\left|c_{k}(t,x,q)\right|\ge 0\quad\text{for}\quad(t,x,q)\in E\times\mathbb{R}_{+}.$$

ASSUMPTION $[\Lambda]$. The function $\lambda\colon E\times\mathbb{R}_{+}^{2}\to\mathbb{R}$ is bounded, continuous and there are: a positive constant L_{λ}, a bounded, nonnegative function L_{z} such that $L_{z}\in L^{1}_{\mathcal{M}}$ and

$$\left|\lambda(t,x,\bar{p},\bar{q})-\lambda(t,x,p,q)\right|\le L_{\lambda}\left|\bar{p}-p\right|+L_{z}(x)\left|\bar{q}-q\right|$$

for $(t,x)\in E,\ p,q,\bar{p},\bar{q}\in\mathbb{R}_{+}$.

ASSUMPTION $[L]$. The discrete function $u\colon E_{h}\to\mathbb{R}_{+}$ satisfies the discrete Lipschitz condition: $\left|u^{(i,j\pm e_{k})}-u^{(i,j)}\right|\le L_{u}h_{k}$ with some $L_{u}>0$ for $k=1,\dots,n$.

3. Stability of the scheme. To prove the stability of the finite difference scheme for problem (1)–(3) with respect to the right-hand side and the initial condition, we consider the perturbed scheme

$$\begin{aligned} \delta_0 u^{(i,j)} \quad &+ \quad \sum_{k=1}^{n} c_k^{(i,j)}[z] \left\{ \chi_k^{(i,j)}[z] \delta_k^+ u^{(i,j)} + \left(1 - \chi_k^{(i,j)}[z]\right) \delta_k^- u^{(i,j)} \right\} \\ &= \quad \lambda^{(i,j)}[u,z] + \xi^{(i,j)} \quad \text{on } E_h, \end{aligned} \tag{6}$$

with $z^{(i)} = (Q_h u)_i$ and the initial condition

$$u^{(0,j)} = v^{(j)} + \hat{\xi}^{(0,j)} \quad \text{for} \quad j \in \mathbb{N}^n. \tag{7}$$

The perturbations $\xi^{(i,j)}$ can be interpreted as local discretization errors for (1). The numbers $\hat{\xi}^{(0,j)}$ are perturbations of (3).

LEMMA 3.1 *Suppose that $u, \bar{u} \colon E_h \to \mathbb{R}_+$ are bounded, the discrete functions $u^{(i,\cdot)}$, $\bar{u}^{(i,\cdot)} \in l_n^1$ for $i = 1, \dots, N_0$ and*

(i) u is a solution of problem (4)–(5), satisfying Assumption [L],

(ii) $\bar{u}$ is a solution of problem (6)–(7) with perturbations satisfying the conditions

$$\left\| \xi^{(i)} \right\|_\infty \le C_h, \; \left\| \hat{\xi}^{(0)} \right\|_\infty \le C_{0,h}, \; \left\| \xi^{(i)} \right\|_1 \le \bar{C}_h, \; \left\| \hat{\xi}^{(0)} \right\|_1 \le \bar{C}_{0,h},$$

$i = 1, \dots, N_0$, where $C_{0,h}, C_h, \bar{C}_{0,h}, \bar{C}_h \to 0$ as $\|h\| \to 0$,

(iii) the functions $c_k \in C(E \times \mathbb{R}_+, \mathbb{R})$, $k = 1, \dots, n$, and the steps h_k, $k = 0, \dots, n$, satisfy Assumptions [C] and [S],

(iv) the function $\lambda \in C(E \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ satisfies Assumption [Λ].

Then $\bar{u}^{(i,j)} - u^{(i,j)}$ converges uniformly to 0 as $\|h\| \to 0$ in the supremum and l^1 norm.

REMARK 3.2 If the function c does not depend on the last variable, then Assumption [L] for the function u can be omitted.

LEMMA 3.3 *Suppose that $u, \bar{u} \colon E_h \to \mathbb{R}_+$ are bounded, the discrete functions $u^{(i,\cdot)}$, $\bar{u}^{(i,\cdot)} \in l_n^1$ for $i = 1, \dots, N_0$ and Assumptions (i)–(iii) of Lemma 3.1 are satisfied. Then*

$$\begin{aligned} &\sum_{k=1}^{n} \Bigg| \delta_k^+ u^{(i,j)} \left(c_k^{(i,j)}[z] \chi_k^{(i,j)}[z] - c_k^{(i,j)}[\bar{z}] \chi_k^{(i,j)}[\bar{z}] \right) \\ &+ \quad \delta_k^- u^{(i,j)} \left(c_k^{(i,j)}[z] \left(1 - \chi_k^{(i,j)}[z]\right) - c_k^{(i,j)}[\bar{z}] \left(1 - \chi_k^{(i,j)}[\bar{z}]\right) \right) \Bigg| \\ &\le n L_u \|L_c^*\|_\infty \left| z^{(i)} - \bar{z}^{(i)} \right|. \end{aligned} \tag{8}$$

Moreover, if we set $\omega^{(i,j)} = \bar{u}^{(i,j)} - u^{(i,j)}$ (the error of the scheme), then we have

$$\sum_{k=1}^{n} \sum_{j\in\mathbb{N}^n} \left\{ \frac{\left|c_k^{(i,j)}[\bar{z}]\right|}{h_k} \left(- \left|\omega^{(i,j)}\right| + \chi_k^{(i,j)}[\bar{z}] \left|\omega^{(i,j+e_k)}\right| \right.\right.$$

$$\left.\left. + \left(1 - \chi_k^{(i,j)}[\bar{z}]\right) \left|\omega^{(i,j-e_k)}\right| \right) \le 2nL_c \sum_{j\in\mathbb{N}^n} \left|\omega^{(i,j)}\right|. \tag{9}$$

Remark 3.4 If the function c does not depend on the last variable, then the left hand side of inequality (8) is equal to zero.

Proof (of Lemma 3.3) We analyze the left hand side of inequality (8) according to: if $c_k^{(i,j)}[\bar{z}] < 0$ and $c_k^{(i,j)}[z] < 0$, or if $c_k^{(i,j)}[\bar{z}] \ge 0$ and $c_k^{(i,j)}[z] \ge 0$, then $\left|\delta_k^+ u^{(i,j)} \left(c_k^{(i,j)}[z] - c_k^{(i,j)}[\bar{z}]\right)\right|$ and $\left|\delta_k^- u^{(i,j)} \left(c_k^{(i,j)}[z] - c_k^{(i,j)}[\bar{z}]\right)\right|$ are estimated by $L_u \left\|L_c^*\right\|_\infty \left|z^{(i)} - \bar{z}^{(i)}\right|$. If $c_k^{(i,j)}[\bar{z}] \ge 0$ and $c_k^{(i,j)}[z] < 0$, then we have $\left|\delta_k^+ u^{(i,j)} c_k^{(i,j)}[z] - \delta_k^- u^{(i,j)} c_k^{(i,j)}[\bar{z}]\right| \le \left|\left|\delta_k^- u^{(i,j)}\right| c_k^{(i,j)}[\bar{z}] - \left|\delta_k^+ u^{(i,j)}\right| c_k^{(i,j)}[z]\right| \le$ $L_u \left\|L_c^*\right\|_\infty \left|\bar{z}^{(i)} - z^{(i)}\right|$. The same estimate is obtained when $c_k^{(i,j)}[\bar{z}] < 0$ and $c_k^{(i,j)}[z] \ge 0$. Taking into consideration the above inequalities, we get (8).

Denote $J_k = (j_1, \ldots, j_{k-1}, 0, j_{k+1} \ldots, j_n) \in \mathbb{N}^n$. To shorten notation, assume that $c_l^{(i,J_k-e_k)}[\bar{z}] = 0$, $k, l = 1, \ldots, n$. Changing the order of summation in (9) and applying the condition $c_l^{(i,J_k)}[\bar{z}] \le 0$, $k, l = 1, \ldots, n$, we write the left hand side of (9) in the form

$$\sum_{k=1}^{n} \sum_{j\in\mathbb{N}^n} \frac{\left|\omega^{(i,j)}\right|}{h_k} \left\{ - \left|c_k^{(i,j)}[\bar{z}]\right| + \left|c_k^{(i,j-e_k)}[\bar{z}]\right| \chi_k^{(i,j-e_k)}[\bar{z}] \right.$$

$$\left. + \left|c_k^{(i,j+e_k)}[\bar{z}]\right| \left(1 - \chi_k^{(i,j+e_k)}[\bar{z}]\right) \right\}. \tag{10}$$

From Assumption $[C]$ it follows that

$$\left| \left|c_k^{(i,j)}[\bar{z}]\right| - \left|c_k^{(i,j\pm e_k)}[\bar{z}]\right| \right| \le \left|c_k^{(i,j)}[\bar{z}] - c_k^{(i,j\pm e_k)}[\bar{z}]\right| \le h_k L_c. \tag{11}$$

By $X_k^{(i,j)}$ denote

$$\left| \left|c_k^{(i,j-e_k)}[\bar{z}]\right| \chi_k^{(i,j-e_k)}[\bar{z}] + \left|c_k^{(i,j+e_k)}[\bar{z}]\right| \left(1 - \chi_k^{(i,j+e_k)}[\bar{z}]\right) - \left|c_k^{(i,j)}[\bar{z}]\right| \right|.$$

If $c_k^{(i,j-e_k)}[\bar{z}] < 0$ and $c_k^{(i,j+e_k)}[z] < 0$, or $c_k^{(i,j-e_k)}[\bar{z}] \ge 0$ and $c_k^{(i,j+e_k)}[\bar{z}] \ge 0$, then using (11), we have $X_k^{(i,j)} \le L_c h_k$. If $c_k^{(i,j-e_k)}[\bar{z}] < 0$ and $c_k^{(i,j+e_k)}[\bar{z}] \ge 0$, then

$$X_k^{(i,j)} \le \left|c_k^{(i,j+e_k)}[\bar{z}] - c_k^{(i,j-e_k)}[\bar{z}]\right| \le 2L_c h_k.$$

If $c_k^{(i,j-e_k)}[\bar{z}] \geq 0$ and $c_k^{(i,j+e_k)}[\bar{z}] < 0$, then we obtain $X_k^{(i,j)} \leq L_c h_k$. Hence (10) is estimated by $2nL_c \sum\limits_{j\in\mathbb{N}^n} \left|\omega^{(i,j)}\right|$. ■

Proof (of Lemma 3.1) Recall that the discrete function $\bar{u}$ is the solution of (6) with perturbations satisfying Assumption (ii) of Lemma 3.1. Subtracting (6) with the function $\bar{u}$ and (4) we obtain the explicit recurrence error equation

$$
\begin{aligned}
\omega^{(i+1,j)} &= \omega^{(i,j)}\left(1-\sum_{k=1}^{n}\frac{h_0}{h_k}\left|c_k^{(i,j)}[\bar{z}]\right|\right)\\
&+ h_0\sum_{k=1}^{n}\left\{\frac{\left|c_k^{(i,j)}[\bar{z}]\right|}{h_k}\left(\chi_k^{(i,j)}[\bar{z}]\omega^{(i,j+e_k)}+\left(1-\chi_k^{(i,j)}[\bar{z}]\right)\omega^{(i,j-e_k)}\right)\right.\\
&+ \delta_k^{+}u^{(i,j)}\left(c_k^{(i,j)}[z]\chi_k^{(i,j)}[z]-c_k^{(i,j)}[\bar{z}]\chi_k^{(i,j)}[\bar{z}]\right)\\
&+ \left.\delta_k^{-}u^{(i,j)}\left(c_k^{(i,j)}[z]\left(1-\chi_k^{(i,j)}[z]\right)-c_k^{(i,j)}[\bar{z}]\left(1-\chi_k^{(i,j)}[\bar{z}]\right)\right)\right\}\\
&+ h_0\left(\lambda^{(i,j)}[\bar{u},\bar{z}]-\lambda^{(i,j)}[u,z]\right)+h_0\xi^{(i,j)}.
\end{aligned}
\tag{12}
$$

Applying Assumptions $[S]$ and $[\Lambda]$, we obtain the inequality

$$
\begin{aligned}
\left|\omega^{(i+1,j)}\right| &\leq \left|\omega^{(i,j)}\right|\left(1-\sum_{k=1}^{n}\frac{h_0}{h_k}\left|c_k^{(i,j)}[\bar{z}]\right|\right)\\
&+ h_0\sum_{k=1}^{n}\left\{\frac{\left|c_k^{(i,j)}[\bar{z}]\right|}{h_k}\left(\chi_k^{(i,j)}[\bar{z}]\left|\omega^{(i,j+e_k)}\right|+\left(1-\chi_k^{(i,j)}[\bar{z}]\right)\left|\omega^{(i,j-e_k)}\right|\right)\right.\\
&+ \left|\delta_k^{+}u^{(i,j)}\left(c_k^{(i,j)}[z]\chi_k^{(i,j)}[z]-c_k^{(i,j)}[\bar{z}]\chi_k^{(i,j)}[\bar{z}]\right)\right.\\
&+ \left.\left.\delta_k^{-}u^{(i,j)}\left(c_k^{(i,j)}[z]\left(1-\chi_k^{(i,j)}[z]\right)-c_k^{(i,j)}[\bar{z}]\left(1-\chi_k^{(i,j)}[\bar{z}]\right)\right)\right|\right\}\\
&+ h_0L_\lambda\left|\omega^{(i,j)}\right|+h_0L_z\left(x^{(j)}\right)\left|\bar{z}^{(i)}-z^{(i)}\right|+h_0\left|\xi^{(i,j)}\right|.
\end{aligned}
\tag{13}
$$

Notice that

$$
\left|\bar{z}^{(i)}-z^{(i)}\right| = h_1\cdots h_n\left|\sum_{j\in\mathbb{N}^n}\left(\bar{u}^{(i,j)}-u^{(i,j)}\right)\right| \leq \left\|\omega^{(i)}\right\|_1. \tag{14}
$$

Using Lemma 3.3, Assumption $[\Lambda]$ and (14), we obtain the recurrence inequality

$$
\left\|\omega^{(i+1)}\right\|_\infty \leq (1+h_0L_\lambda)\left\|\omega^{(i)}\right\|_\infty + h_0L_1\left\|\omega^{(i)}\right\|_1 + h_0\left\|\xi^{(i)}\right\|_\infty, \tag{15}
$$

where $L_1 = 2nL_u \left\| L_c^* \right\|_\infty + \left\| L_z \right\|_\infty$. Summing all terms of (13) over $j \in \mathbb{N}^n$, we have the inequality

$$\begin{aligned}
\sum_{j\in\mathbb{N}^n} \left|\omega^{(i+1,j)}\right| \quad &\leq \quad \sum_{j\in\mathbb{N}^n} \left|\omega^{(i,j)}\right| + h_0 \sum_{k=1}^{n} \sum_{j\in\mathbb{N}^n} \frac{\left|c_k^{(i,j)}[\bar{z}]\right|}{h_k} \\
&\times\left\{ -\left|\omega^{(i,j)}\right| + \chi_k^{(i,j)}[\bar{z}] \left|\omega^{(i,j+e_k)}\right| + \left(1-\chi_k^{(i,j)}[\bar{z}]\right) \left|\omega^{(i,j-e_k)}\right| \right\} \\
&+h_0 \sum_{k=1}^{n} \sum_{j\in\mathbb{N}^n} \Big| \delta_k^+ u^{(i,j)} \left(c_k^{(i,j)}[z]\chi_k^{(i,j)}[z] - c_k^{(i,j)}[\bar{z}]\chi_k^{(i,j)}[\bar{z}] \right) \\
&\quad + \delta_k^- u^{(i,j)} \left(c_k^{(i,j)}[z] \left(1-\chi_k^{(i,j)}[z]\right) - c_k^{(i,j)}[\bar{z}] \left(1-\chi_k^{(i,j)}[\bar{z}]\right) \right) \Big| \\
&+h_0 L_\lambda \sum_{j\in\mathbb{N}^n} \left|\omega^{(i,j)}\right| + h_0 \left|\bar{z}^{(i)} - z^{(i)}\right| \sum_{j\in\mathbb{N}^n} L_z\left(x^{(j)}\right) + h_0 \sum_{j\in\mathbb{N}^n} \left|\xi^{(i,j)}\right|
\end{aligned} \tag{16}$$

Multiplying the both sides of (16) by $h_1 \cdot \ldots \cdot h_n$ and applying Lemma 3.3 to the second line of (16), Assumptions $[C]$ and $[L]$ to the third and fourth lines of (16), and Assumption $[\Lambda]$ to the last line of (16), we obtain the following recurrence inequality

$$\left\|\omega^{(i+1)}\right\|_1 \leq (1+h_0L_2) \left\|\omega^{(i)}\right\|_1 + h_0 \left\|\xi^{(i)}\right\|_1, \tag{17}$$

where $L_2 = L_\lambda + 2nL_c + 2L_u \left\|L_c^*\right\|_1 + \left\|L_z\right\|_1$. Let us consider the comparison recurrence equations with respect to (15) and (17):

$$\begin{aligned}
\eta^{(i+1)} &= \eta^{(i)}(1+h_0L_\lambda) + h_0L_1\tilde{\eta}^{(i)} + h_0 \left\|\xi^{(i)}\right\|_\infty, \\
\tilde{\eta}^{(i+1)} &= \tilde{\eta}^{(i)}(1+h_0L_2) + h_0 \left\|\xi^{(i)}\right\|_1.
\end{aligned} \tag{18}$$

Taking into consideration the initial conditions

$$\left\|\omega^{(0)}\right\|_1 \leq \tilde{\eta}^{(0)} = \bar{C}_{0,h} \to 0, \quad \left\|\omega^{(0)}\right\|_\infty \leq \eta^{(0)} = C_{0,h} \to 0,$$

we obtain the estimates $\left\|\omega^{(i)}\right\|_\infty \leq \eta^{(i)}$ and $\left\|\omega^{(i)}\right\|_1 \leq \tilde{\eta}^{(i)}$, hence the solutions of (15), (17) satisfy

$$\begin{aligned}
\left\|\omega^{(i)}\right\|_1 &\leq \tilde{\eta}^{(i)} \leq e^{L_2 a}\left(\bar{C}_{0,h} + \frac{\bar{C}_h}{L_2}\right) =: \hat{C}_h, \\
\left\|\omega^{(i)}\right\|_\infty &\leq \eta^{(i)} \leq e^{L_\lambda a}\left(C_{0,h} + \frac{L_1\hat{C}_h + C_h}{L_\lambda}\right).
\end{aligned}$$

The right-hand sides of these estimates are derived from the system of comparison recurrence equations (18), because $\eta^{(i)} \leq \eta^{(N_0)}$, $\tilde{\eta}^{(i)} \leq \tilde{\eta}^{(N_0)}$ and $(1+h_0L)^i \leq e^{h_0 iL} \leq e^{aL}$, $i = 0, \ldots, N_0$. ■

3.1. Stability - the case of finite quadrature. Since only a finite number of terms can be involved in practical computations, we shall prove a lemma on stability with respect to cut-offs of the quadrature for the forward-backward scheme. Denoting by u_h the solution of this scheme with the finite quadrature $Q_h^{N_h}$ instead of Q_h, we write it as follows:

(19)
$$\delta_0 u_h^{(i,j)} + \sum_{k=1}^{n} c_k^{(i,j)}[z_h] \left\{ \chi_k^{(i,j)}[z_h] \delta_k^+ u_h^{(i,j)} + \left(1 - \chi_k^{(i,j)}[z_h]\right) \delta_k^- u_h^{(i,j)} \right\} = \lambda^{(i,j)}[u_h, z_h]$$

with

$$z_h^{(i)} = \left(Q_h^{N_h} u_h\right)_i, \tag{20}$$

and the initial condition

$$u_h^{(0,j)} = v^{(j)} \quad \text{for} \quad j \in \mathbb{N}^n. \tag{21}$$

LEMMA 3.5 *Suppose that*

(*i*) *the numbers* N_h *satisfy the condition:* $\|h\| N_h \to +\infty$ *as* $\|h\| \to 0$,

(*ii*) *the functions* $c_k \in C(E \times \mathbb{R}_+, \mathbb{R})$, $k = 1, \ldots, n$ *and the steps* $h = (h_0, h_1, \ldots, h_n)$ *satisfy Assumptions* $[C]$ *and* $[S]$,

(*iii*) *the function* $\lambda \in C\left(E \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}\right)$ *satisfies Assumption* $[\Lambda]$.

Then the scheme (4)–(5) is stable with respect to cut-offs of the quadrature.

PROOF Suppose that a discrete function $u \colon E_h \to \mathbb{R}_+$ is a solution of problem (4)–(5) such that u is bounded, $u^{(i,\cdot)} \in l_n^1$, $i = 1, \ldots, N_0$ and satisfy Assumption $[L]$. Denote by u_h the only solution of (19)–(21), which clearly exists. Observe that u_h is also bounded and $u_h^{(i,\cdot)} \in l_n^1$, $i = 1, \ldots, N_0$. Denote $\varepsilon^{(i,j)} = u^{(i,j)} - u_h^{(i,j)}$. Similarly to the proof of Lemma 3.1, we subtract (4) and (19). We obtain the explicit recurrence error equation with zero initial condition. Applying the stability condition $[S]$, Assumption $[\Lambda]$, we have

$$\begin{aligned}
\left|\varepsilon^{(i+1,j)}\right| \quad &\leq \quad \left|\varepsilon^{(i,j)}\right| \left(1 - \sum_{k=1}^{n} \frac{h_0}{h_k} \left|c_k^{(i,j)}[z_h]\right|\right) \\
&+ h_0 \sum_{k=1}^{n} \left\{ \frac{\left|c_k^{(i,j)}[z_h]\right|}{h_k} \left(\chi_k^{(i,j)}[z_h] \left|\varepsilon^{(i,j+e_k)}\right| + \left(1 - \chi_k^{(i,j)}[z_h]\right) \left|\varepsilon^{(i,j-e_k)}\right| \right) \right. \\
&+ \left| \delta_k^+ u^{(i,j)} \left(c^{(i,j)}[z_h] \chi_k^{(i,j)}[z_h] - c^{(i,j)}[z] \chi_k^{(i,j)}[z] \right) \right. \\
&+ \left. \left. \delta_k^- u^{(i,j)} \left(c^{(i,j)}[z_h] \left(1 - \chi_k^{(i,j)}[z_h]\right) - c^{(i,j)}[z] \left(1 - \chi_k^{(i,j)}[z]\right) \right) \right| \right\} \\
&+ h_0 \left(L_\lambda \left|\varepsilon^{(i,j)}\right| + L_z \left(x^{(j)}\right) \left|z^{(i)} - z_h^{(i)}\right| \right).
\end{aligned} \tag{22}$$

Notice that

$$\left| z^{(i)} - z_h^{(i)} \right| \le \left\| \varepsilon^{(i)} \right\|_1 + U_h^{(i)}, \tag{23}$$

where

$$U_h^{(i)} = h_1 \cdots h_n \left(\sum_{j \in \mathbb{N}^n} u^{(i,j)} - \sum_{j_1,\ldots,j_n=0}^{N_h} u^{(i,j)} \right),$$

and the remainder $U_h^{(i)}$ tends to 0 as $\|h\| \to 0$. Applying Lemma 3.3 and (23), we have the recurrence inequality

$$\left\| \varepsilon^{(i+1)} \right\|_\infty \le (1 + h_0 L_\lambda) \left\| \varepsilon^{(i)} \right\|_\infty + h_0 L_1 \left(\left\| \varepsilon^{(i)} \right\|_1 + U_h \right), \tag{24}$$

where $L_1 = 2nL_u \left\| L_c^* \right\|_\infty + \left\| L_z \right\|_\infty$, $U_h = \sup\limits_{i=0,\ldots,N_0} U_h^{(i)}$.

Multiplying by $h_1 \cdot \ldots \cdot h_n$ the both sides of (22), summing terms over $j \in \mathbb{N}^n$, applying Lemma 3.3, Assumptions $[C]$, $[L]$ and $[\Lambda]$, we obtain the following recurrence inequality

$$\left\| \varepsilon^{(i+1)} \right\|_1 \le (1 + h_0 L_2) \left\| \varepsilon^{(i)} \right\|_1 + h_0 L_3 U_h, \tag{25}$$

where $L_2 = L_\lambda + 2nL_c + 2nL_u \left\| L_c^* \right\|_1 + \left\| L_z \right\|_1$, $L_3 = 2nL_u \left\| L_c^* \right\|_1 + \left\| L_z \right\|_1$.

Writing, similarly as in the proof of Lemma 3.1, the comparision recurrence equations with respect to (24), (25) and taking into consideration the initial conditions: $\left\| \varepsilon^{(0)} \right\|_\infty = 0$, $\left\| \varepsilon^{(0)} \right\|_1 = 0$, we have the estimates

$$\left\| \varepsilon^{(i)} \right\|_1 \le e^{aL_2} \frac{L_3 U_h}{L_2} =: \hat{C}_h, \qquad \left\| \varepsilon^{(i)} \right\|_\infty \le e^{aL_\lambda} \frac{L_1(\hat{C}_h + U_h)}{L_\lambda}.$$

Since $\|h\| N_h \to \infty$, hence $U_h \to 0$ as $\|h\| \to 0$ and we have the desired assertion $\left\| \varepsilon^{(i)} \right\|_\infty \to 0$, $\left\| \varepsilon^{(i)} \right\|_1 \to 0$ as $\|h\| \to 0$. ■

Now we write the main result of our paper.

THEOREM 3.6 *If assumptions of the Lemmas 3.1 and 3.5 are satisfied, then the forward-backward scheme for (1)–(3) are stable with respect to the perturbation of the right-hand side, the initial condition and the cuts-off of the quadrature.*

PROOF The proof is a conclusion of the proofs of Lemmas 3.1 and 3.5. ■

4. Numerical experiments. In order to find approximate solutions of problem (1)–(3), we cannot apply our theoretical result directly, because it is not possible to perform practical computations in unbounded domains. Thus we cut the domain to some sufficiently large bounded subsets, and observe that global errors behave stable.

REMARK 4.1 Let $N_h, N_0 \in \mathbb{N}$, such that $N_h > N_0$ and $N_h\|h\| \to \infty$ as $\|h\| \to 0$, where $h_0 = \frac{a}{N_0}$, $h = (h_1, \ldots, h_n) \in \mathbb{R}^n_+$. At each stage $t^{(i)}$, $i = 1, \ldots, N_0$, the number of mesh points involved in computations may change, depending on the sign of the functions c_k, $k = 1, \ldots, n$. Without loss of generality we assume that the number of the mesh points is decreasing. In the set E introduce the regular mesh

$$\tilde{E}_h = \left\{ \left(t^{(i)}, x^{(j)}\right) \;:\; i = 0, \ldots, N_0,\; j = (j_1, \ldots, j_k),\; j_k = 0, \ldots, N_h - i \right\}.$$

Consider finite difference problem (4) on $\tilde{E}_h$, with $z^{(i)} = \left(Q_h^{N_h - i} u\right)_i$ and the initial condition $u^{(0,j)} = v^{(j)}$, $j = (j_1, \ldots, j_n)$, $j_k = 0, \ldots, N_h$, $k = 1, \ldots, n$. Note that the above problem is well defined. Recall that N_h was chosen in such a way that $N_h \to \infty$ as $\|h\| \to 0$. It follows from Theorem 3.6 that the discretization error for (1)–(3) tends to zero as $\|h\| \to 0$.

We present numerical tests which illustrate our theoretical results. We take $n = 1$, $a = 1$. The initial set is cut to some interval $[0, X]$, $X > 0$. Let $h_0 = 10^{-3}$. With a prescribed functions $u^{[l]} : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$, $v^{[l]}(x) = u^{[l]}(0, x)$, $x \in [0, X]$, $c^{[l]} \colon [0, 1] \times \mathbb{R}^2_+ \to \mathbb{R}$, $l = 1, 2$,

$$\begin{aligned} c^{[1]}(t,x,z) &= \frac{t \sin x \sin z}{1 + x^2}, \qquad u^{[1]}(t,x) = \frac{\cos t}{(1 + t + x)^2}, \\ z^{[1]}(t) &= \frac{\cos t}{(1 + t)}, \end{aligned}$$

and

$$\begin{aligned} c^{[2]}(t,x,z) &= t e^{-x} \sin x \sin z, \qquad u^{[2]}(t,x) = \frac{\sin^2(tx)}{1 + x^2}, \\ z^{[2]}(t) &= \frac{\pi(1 - e^{-2t})}{4}, \end{aligned}$$

we determine the respective right-hand sides of the differential equation

$$\begin{aligned} \lambda^{[l]}(t,x,p,q) &= \partial_t u^{[l]}(t,x) + \partial_x u^{[l]}(t,x) c^{[l]}(t,x,z^{[l]}(t)) \\ &\quad - \frac{t \sin\left(u^{[l]}(t,x)\right) \sin\left(z^{[l]}(t)\right)}{1 + x^2} + \frac{t \sin(p) \sin(q)}{1 + x^2}. \end{aligned}$$

Note that the functions $c^{[l]}$ and $\lambda^{[l]}$, $l = 1, 2$, satisfy Assumptions $[C]$ and $[\Lambda]$, respectively. Errors of the computations are given by the formulas

$$\Delta u^{[l]} = \max_{\substack{i=1,\ldots,N_0 \\ j=0,\ldots,N}} |\tilde{u}^{[l](i,j)} - u^{[l]}(t^{(i)}, x^{(j)})|, \;\; \Delta z^{[l]} = \max_{i=1\ldots,N_0} |\tilde{z}^{[l](i)} - z^{[l]}(t^{(i)})|,$$

where the discrete functions $\tilde{u}^{[l]}$, $\tilde{z}^{[l]}$ approximate the functions $u^{[l]}$ and $z^{[l]}$, $l = 1, 2$, on the bounded area. The results of computations for various intervals $[0, X]$ and h_0/h_1 are given in the table.

h_0/h_1	X	$\Delta u^{[1]}$	$\Delta z^{[1]}$	$\Delta u^{[2]}$	$\Delta z^{[2]}$
1	50	33.82E-4	12.97E-3	5.71E-4	10.78E-3
1	100	21.81E-4	6.65E-3	4.19E-4	5.52E-3
1	500	15.63E-4	3.17E-3	3.61E-4	3.32E-3
0.5	50	33.77E-4	13.09E-3	5.93E-4	11.16E-3
0.5	100	21.69E-4	6.59E-3	4.40E-4	5.55E-3
0.5	500	13.43E-4	2.20E-3	3.61E-4	2.41E-3
0.2	50	33.67E-4	13.53E-3	6.66E-4	11.91E-3
0.2	100	20.96E-4	6.47E-3	5.04E-4	5.75E-3
0.2	500	11.13E-4	1.01E-3	4.07E-4	1.82E-3
0.2	750	11.11E-4	0.87E-3	4.07E-4	1.82E-3
0.1	50	33.63E-4	14.49E-3	7.93E-4	13.13E-3
0.1	100	19.81E-4	6.25E-3	6.13E-4	6.04E-3
0.1	500	9.43E-4	0.55E-3	5.09E-4	1.46E-3
0.1	750	8.87E-4	25.48E-6	5.06E-4	1.43E-3
0.1	1000	8.87E-4	3.73E-6	5.06E-4	1.43E-3

Note that, for a fixed discretization parameter $h = (h_0, h_1)$, the errors of computations are decreasing as the length of the initial interval $[0, X]$ is increasing.

The computation was performed by the IBM PC computer.

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