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Existence and uniqueness of solution for some integral-functional equation

Let E be a Banach space with the norm $\|\cdot\|$. We denote by $C(I, R_+)$ the class of all continuous functions defined on $I \stackrel{\mathrm{df}}{=} [0, a]$ with a range in $R_+ \stackrel{\mathrm{df}}{=} [0, +\infty)$.

In the present paper we are concerned with the integral-functional equation

$$(1) \qquad x(t) = F\Big(t, \int_0^{\varphi_1((t,x(\cdot))} f\big(t, s, x(s)\big)ds, x\big(\psi_1(t, x(\cdot))\big)\Big) \stackrel{\mathrm{df}}{=} (\mathscr{F}x)(t),$$

where

$$\varphi_1\big(t, x(\cdot)\big) = \varphi\Big(t, \int_0^{\alpha_1(t)} f_1\big(t, s, x(s)\big)ds, x\big(\beta_1(t)\big)\Big),$$

$$\psi_1\big(t, x(\cdot)\big) = \psi\Big(t, \int_0^{\alpha_2(t)} f_2\big(t, s, x(s)\big)ds, x\big(\beta_2(t)\big)\Big),$$

and the functions $F\colon I\times E\times E \to E$, $f, f_j\colon I\times I\times E \to E$, $\varphi, \psi\colon I\times E\times E \to I$, $\alpha_j, \beta_j\colon I \to I$, $j = 1, 2$, are known.

In papers [2] and [3] equation (1) was discussed in the case where the functions φ and ψ do not depend on the last two variables.

The differential equation with delay dependent on the solution and its derivative

$$(2) \quad y'(t) = F\Big(t, y\Big(\varphi\big(t, y(\alpha_1(t)), y'(\beta_1(t))\big)\Big), y'\Big(\psi\big(t, y(\alpha_2(t)), y'(\beta_2(t))\big)\Big)\Big), \quad t\in I,$$

can be reduced to a particular case ($f(t, s, u) = f_j(t, s, u) = u$, $j = 1, 2$) of equation (1).

Such equation was considered in [7].

If in (2) the function F is independent of the last variable, then we have the differential-functional equation which has been considered in [4], [9] and [10]. For $\varphi(t, u, v) = \varphi(t, u)$ and $\psi(t, u, v) = \psi(t, u)$ equation (2) has been investigated in [1].

However, if F and φ are constant with respect to the second variable, then we arrive to the clasical functional equation which are discussed by many authors.

The solutions of this equation are sought in the class of functions fulfilling a Lipschitz condition (see the class $D(I, E, \bar{u}, \lambda)$ defined below).

By the use of the comparison method (see [5], [6], [8]) we shall prove the existence, uniqueness of the solution and the convergence of successive approximations for equation (1). But in the case when E has a finite dimension we shall establish the existence result under a weaker assumptions.

1. Lemma. Below we quote the lemma which we shall use in the proof of the existence of solution of equation (1).

Let us define

$$(Lu)(t) \overset{\text{df}}{=} l(t)\, u(\beta(t)), \quad (Ku)(t) \overset{\text{df}}{=} k(t) \int_0^{\alpha(t)} u(s)\, ds, \quad t \in I.$$

Put $L^n = LL^{n-1}$, $n = 1, 2, \ldots$, $L^0 = J$, where J denotes the identity operator in $C(I, R_+)$.

From the definition of the operator L it follows that

$$(L^n u)(t) = l_n(t)\, u(\beta_n(t)),$$

where

$$\beta_0(t) \overset{\text{df}}{=} t, \quad \beta_{n+1}(t) \overset{\text{df}}{=} \beta(\beta_n(t)), \quad n = 0, 1, \ldots, \ t \in I,$$

$$l_0(t) \overset{\text{df}}{=} 1, \quad l_{n+1}(t) \overset{\text{df}}{=} \prod_{k=0}^{n} l(\beta_k(t)), \quad n = 0, 1, \ldots, \ t \in I.$$

Put

$$Mu \overset{\text{df}}{=} \sum_{n=0}^{\infty} L^n u$$

with the pointwise convergence of the series in I.

LEMMA 1 [2]. *If the functions $h, l, k \in C(I, R_+)$, $\alpha, \beta \in C(I, I)$ are nondecreasing, $\alpha(t), \beta(t) \in [0, t]$, $t \in I$, and*

$$s(t) \overset{\text{df}}{=} (Mh)(t) < +\infty, \quad t \in I,$$

$$\text{(3)} \qquad \bar{s}(t) \overset{\text{df}}{=} (Mg)(t) < +\infty, \quad t \in I, \quad \sup \frac{\bar{s}(t)}{t} < +\infty,$$

where $g(t) = k(t)\, \alpha(t)$, then

(a) *there exists* $\bar{u} \in C(I, R_+)$ *which is a non-decreasing and unique solution of the equation*

$$u = MKu + Mh, \tag{4}$$

in the class $G(I, R_+)$ *of bounded and measurable functions defined on* I;

(b) *the function* $\bar{u}$ *is the non-decreasing and unique solution of the equation*

$$u = Ku + Lu + h, \tag{5}$$

in the class

$$G(I, R_+, \bar{u}) \overset{\mathrm{df}}{=} \{u\colon u \in G(I, R_+), [\![u]\!] < +\infty\},$$

where $[\![u]\!] \overset{\mathrm{df}}{=} \inf\{c\colon |u| \leqslant c\bar{u},\ c \in R_+\}$;

(c) *the function* $u = 0$ *is in the class* $G(I, R_+, \bar{u})$ *the unique solution of the inequality*

$$u \leqslant Ku + Lu.$$

Remark 1. If the assumptions of Lemma 1 are fulfilled also for $h^* \in C(I, R_+)$ and $h^*(t) \leqslant h(t)$, $t \in I$, then the suitable solution u^* of equation (4) with h^* instead of h established in Lemma 1 is the unique solution of equation (5) with h replaced by h^* in the class $G(I, R_+, \bar{u})$.

Remark 2. Some effective conditions under which conditions (3) are fulfilled can be found in [2] and [6].

2. Further lemmas and existence theorem. We introduce

ASSUMPTION H_1. Suppose that there exist non-decreasing functions $\bar{k}, \bar{l}, \tilde{k}, \tilde{k}_j, \bar{\alpha}, \bar{\beta} \in C(I, R_+)$ such that

$$\|F(t, u, v) - F(t, \bar{u}, \bar{v})\| \leqslant \bar{k}(t)\|u - \bar{u}\| + \bar{l}(t)\|v - \bar{v}\|,$$
$$\|f(t, s, v) - f(t, s, \bar{v})\| \leqslant \tilde{k}(t)\|v - \bar{v}\|,$$
$$\|f_j(t, s, v) - f_j(t, s, \bar{v})\| \leqslant \tilde{k}_j(t)\|v - \bar{v}\|, \quad j = 1, 2,$$
$$\varphi(t, u, v) \leqslant \bar{\alpha}(t), \quad \psi(t, u, v) \leqslant \bar{\beta}(t),$$

for any $t \in I$, $s \in [0, t]$, $u, v, \bar{u}, \bar{v} \in E$.

We note that from Assumption H_1 the following estimations follow:

$$\|F(t, u, v)\| \leqslant \bar{k}(t)\|u\| + \bar{l}(t)\|v\| + \gamma(t),$$
$$\|f(t, s, v)\| \leqslant \tilde{k}(t)\|v\| + \tilde{\gamma}(t),$$
$$\|f_j(t, s, v)\| \leqslant \tilde{k}_j(t)\|v\| + \gamma_j(t), \quad j = 1, 2,$$

for any $t\in I$, $s\in[0, t]$, $u, v\in E$, where

$$\gamma(t) = \max_{0\leqslant\tau\leqslant t} \|F(\tau, 0, 0)\|,$$

$$\tilde{\gamma}(t) = \max_{0\leqslant\tau\leqslant t} \max_{s\in[0,t]} \|f(\tau, s, 0)\|,$$

$$\gamma_j(t) = \max_{0\leqslant\tau\leqslant t} \max_{s\in[0,t]} \|f_j(\tau, s, 0)\|, \quad j=1, 2, \ t\in I.$$

LEMMA 2. *If Assumption* H_1 *and assumptions of Lemma* 1 *are satisfied with* $\alpha(t)=\bar{\alpha}(t)$, $\beta(t)=\bar{\beta}(t)$, $l(t)=\bar{l}(t)$, $k(t)=\bar{k}(t)\,\tilde{k}(t)$ *and* $h(t)=\bar{k}(t)\,\tilde{\gamma}(t)\,\bar{\alpha}(t)+\gamma(t)$, $t\in I$, *then the operator* $\mathscr{F}$ *defined by the right-hand side of equation* (1) *maps*

$$B(I, E, \bar{u}) \stackrel{\mathrm{df}}{=} \{x\colon\ x\in C(I, E),\ \|x(t)\|\leqslant \bar{u}(t),\ t\in I\},$$

where $\bar{u}$ *is defined in Lemma* 1, *into itself.*

Proof. If $x\in B(I, E, \bar{u})$ and $y(t)=(\mathscr{F}x)(t)$, then we have

$$\begin{aligned}\|y(t)\| &\leqslant \bar{k}(t)\int_0^{\varphi_1(t,x(\cdot))} \|f(t, s, x(s))\|\,ds+\bar{l}(t)\big\|x\big(\psi_1(t, x(\cdot))\big)\big\|+\gamma(t)\\ &\leqslant \bar{k}(t)\,\tilde{k}(t)\int_0^{\bar{\alpha}(t)} \|x(s)\|\,ds+\bar{l}(t)\,\big\|x\big(\psi_1(t, x(\cdot))\big)\big\|+\bar{k}(t)\,\tilde{\gamma}(t)\,\bar{\alpha}(t)+\gamma(t)\\ &\leqslant k(t)\int_0^{\alpha(t)} \bar{u}(s)\,ds+l(t)\,\bar{u}\big(\beta(t)\big)+h(t)=\bar{u}(t), \qquad t\in I.\end{aligned}$$

Hence it follows that $y\in B(I, E, \bar{u})$. Thus the lemma is proved.

ASSUMPTION H_2. Suppose that

1° there exist non-negative constants m, $\bar{m}$, m_j, $j=1, 2$, such that

$$\|F(t, u, v)-F(\bar{t}, u, v)\|\leqslant m|t-\bar{t}|,$$
$$\|f(t, s, v)-f(\bar{t}, s, v)\|\leqslant \bar{m}|t-\bar{t}|,$$
$$\|f_j(t, s, v)-f_j(\bar{t}, s, v)\|\leqslant m_j|t-\bar{t}|, \quad j=1, 2,$$

for

$$\|v\|\leqslant \varrho \stackrel{\mathrm{df}}{=} \max_I \bar{u}(t), \qquad \|u\|\leqslant \tilde{\varrho} \stackrel{\mathrm{df}}{=} a\varrho \max_I \tilde{k}(t)+\max_I \tilde{\gamma}(t) \quad \text{and} \quad \bar{t}, t, s\in I,$$

2° there exist non-negative constants p_j, c_j, b_j, $j = 1, 2$, such that

$$|\varphi(t, u, v) - \varphi(\bar{t}, u, v)| \leqslant p_1 |t - \bar{t}|,$$
$$|\psi(t, u, v) - \psi(\bar{t}, u, v)| \leqslant p_2 |t - \bar{t}|,$$
$$|\alpha_j(t) - \alpha_j(\bar{t})| \leqslant c_j |t - \bar{t}|,$$
$$|\beta_j(t) - \beta_j(\bar{t})| \leqslant b_j |t - \bar{t}|, \qquad j = 1, 2,$$

for $t, \bar{t} \in I$, and $\|v\| \leqslant \varrho$, $\|u\| \leqslant \tilde{\varrho}$,

3° there exist non-decreasing functions $r_j, s_j \in C(I, R_+)$ such that

$$|\varphi(t, u, v) - \varphi(t, \bar{u}, \bar{v})| \leqslant r_1(t) \|u - \bar{u}\| + s_1(t) \|v - \bar{v}\|,$$

for $t \in I$, $\|u\|, \|\bar{u}\| \leqslant \varrho_1$, $\|v\|, \|\bar{v}\| \leqslant \varrho$,

$$|\psi(t, u, v) - \psi(t, \bar{u}, \bar{v})| \leqslant r_2(t) \|u - \bar{u}\| + s_2(t) \|v - \bar{v}\|,$$

for $t \in I$, $\|u\|, \|\bar{u}\| \leqslant \varrho_2$, $\|v\|, \|\bar{v}\| \leqslant \varrho$, where

$$\varrho_i = a\varrho \max_I \tilde{k}_i(t) + \max_I \gamma_i(t), \qquad i = 1, 2.$$

Put

$$A = \bar{l}(a) \cdot s_2(a) \cdot b_2,$$
$$B = \bar{k}(a) \cdot s_1(a) \cdot b_1 [\tilde{k}(a) \bar{u}(a) + \tilde{\gamma}(a)] + $$
$$+ \bar{l}(a) \{p_2 + r_2(a) \alpha_2(a) m_2 + c_2 r_2(a) [\tilde{k}_2(a) \bar{u}(a) + \gamma_2(a)]\},$$
$$C = m + \bar{k}(a) \{m\bar{\alpha}(a) + [\tilde{k}(a) \bar{u}(a) + \tilde{\gamma}(a)] [p_1 + m_1 r_1(a) \alpha_1(a) + $$
$$+ c_1 r_1(a) (k_1(a) \bar{u}(a) + \gamma_1(a))]\}.$$

Suppose that $(B-1)^2 - 4AC > 0$. Let λ_1 and λ_2 be the non-negative roots of the equation

$$A\lambda^2 + (B-1)\lambda + C = 0.$$

We introduce the following class of functions

$$D(I, E, \bar{u}, \lambda) \stackrel{\text{df}}{=} \{x\colon x \in B(I, E, \bar{u}), \|x(t) - x(\bar{t})\| \leqslant \lambda |t - \bar{t}|\},$$

where the constant λ is fixed and it satisfies the condition

$$\lambda_1 \leqslant \lambda \leqslant \lambda_2 \quad \text{if} \quad A \neq 0, \quad \text{and} \quad \lambda \geqslant \lambda^* = C(1-B)^{-1} \quad \text{if} \quad A = 0.$$

LEMMA 3. *If Assumption* H_2 *and the assumptions of Lemma* 2 *are satisfied, and if* $B < 1$, $(B-1)^2 - 4AC > 0$, *then the operator* $\mathscr{F}$ *defined by the right-hand side of equation* (1) *maps* $D(I, E, \bar{u}, \lambda)$ *into itself.*

Proof. From Lemma 2 it follows that if $x \in D(I, E, \bar{u}, \lambda)$ and $y(t) = (\mathscr{F}x)(t)$, then $y \in B(I, E, \bar{u})$. Now we have

$$\begin{aligned}
\|y(t)-y(\bar{t})\| \leqslant\; & m|t-\bar{t}|+\bar{k}(t)\{\bar{m}\bar{\alpha}(t)|t-\bar{t}|+ \\
& +[\tilde{k}(t)\,\bar{u}(a)+\tilde{\gamma}(t)]\,|\varphi_1(t, x(\cdot))-\varphi_1(\bar{t}, x(\cdot))|\}+ \\
& +\lambda\bar{l}(t)\,|\psi_1(t, x(\cdot))-\psi_1(\bar{t}, x(\cdot))| \leqslant m|t-\bar{t}|+\bar{k}(t)\,\{\bar{m}\bar{\alpha}(t)\,|t-\bar{t}|+ \\
& +[\tilde{k}(t)\,\bar{u}(a)+\tilde{\gamma}(t)]\,[p_1|t-\bar{t}|+r_1(t)\|\int_0^{\alpha_1(t)} f_1(t, s, x(s))\,ds- \\
& -\int_0^{\alpha_1(\bar{t})} f_1(\bar{t}, s, x(s))\,ds\|+s_1(t)\,\|x(\beta_1(t))-x(\beta_2(\bar{t}))\|]\}+ \\
+\lambda\bar{l}(t)\,[p_2|t-\bar{t}|+r_2(t)\,& \|\int_0^{\alpha_2(t)} f_2(t, s, x(s))\,ds-\int_0^{\alpha_2(t)} f_2(\bar{t}, s, x(s))\,ds\|+ \\
& +s_2(t)\,\|x(\beta_2(t))-x(\beta_2(\bar{t}))\|] \leqslant m|t-\bar{t}|+\bar{k}(t)\,\{\bar{m}\bar{\alpha}(t)\,|t-\bar{t}|+ \\
& +[\tilde{k}(t)\,\bar{u}(a)+\tilde{\gamma}(t)]\,[p_1|t-\bar{t}|+ \\
& +r_1(t)\,(m_1\alpha_1(t)\,|t-\bar{t}|+C_1(\bar{k}_1(t)\,\bar{u}(a)+\gamma_1(t))\,|t-\bar{t}|)+ \\
& +\lambda s_1(t)\,b_1|t-\bar{t}|]\}+\lambda\bar{l}(t)\,[p_2|t-\bar{t}|+r_2(t)\,(m_2\alpha_2(t)\,|t-\bar{t}|+ \\
& +C_2(\bar{k}_2(t)\,\bar{u}(a)+\gamma_2(t))\,|t-\bar{t}|)+\lambda s_2(t)\,b_2|t-\bar{t}|] \\
\leqslant\; & (A\lambda^2+B\lambda+C)\,|t-\bar{t}| \leqslant \lambda|t-\bar{t}|.
\end{aligned}$$

Hence it follows that $y \in D(I, E, \bar{u}, \lambda)$. Thus the lemma is proved.

Now we can formulate

THEOREM 1. *If E is finite dimensional Banach space, i.e., $E = R^n$ and the assumptions of Lemma 3 are satisfied, then equation (1) has at least one solution $\bar{x} \in D(I, R^n, \bar{u}, \lambda)$.*

Proof. In view of Lemma 2, 3 and the Schauder fixed-point theorem the assertion of the theorem is obvious. In fact, we see that the continuous operator $\mathscr{F}$ defined by the right-hand side of equation (1), maps the bounded, closed and convex set $D(I, R^n, \bar{u}, \lambda) \subset C(I, R^n)$ into its compact subset $\mathscr{F}[D(I, R^n, \bar{u}, \lambda)]$, thus it has at least one fixed-point.

3. Theorems on the existence and uniqueness. For the general case (E is an arbitrary Banach space) we have the following result

THEOREM 2. *If the assumptions of Lemma 3 hold and $q < 1$, where*

$$q = \max_I \{\bar{k}(t)[\tilde{k}(t)\bar{\alpha}(t)+(\tilde{k}(t)\bar{u}(a)+\tilde{\gamma}(t))\cdot(r_1(t)\bar{k}_1(t)\,\alpha_1(t)+s_1(t))]+$$
$$+\lambda\bar{l}(t)[r_2(t)\bar{k}_2(t)\alpha_2(t)+s_2(t)]\},$$

then equation (1) has a unique solution in $D(I, E, \bar{u}, \lambda)$.

Proof. Under the assumptions of the theorem it is easy to find that the operator $\mathscr{F}$ defined by equation (1) is a contraction in $D(I, E, \bar{u}, \lambda)$.

But under assumptions weaker than those of Theorem 2 we shall prove below another theorem on the existence and uniqueness of solutions and the convergence of successive approximations for equation (1) in the general ($\dim E \leqslant \infty$) case.

Put

$$\begin{aligned}
k(t) &= \bar{k}(t)\tilde{k}(t)+\bar{k}(t)r_1(t)\tilde{k}_1(t)[\tilde{k}(t)\bar{u}(a)+\bar{\gamma}(t)]+\lambda\bar{l}(t)r_2(t)\tilde{k}_2(t),\\
l(t) &= \max\{\bar{k}(t)s_1(t)[\tilde{k}(t)\bar{u}(a)+\bar{\gamma}(t)]+\lambda\bar{l}(t)s_2(t),\ \bar{l}(t)\},\\
\alpha(t) &= \max_{0\leqslant s\leqslant t}\max[\bar{\alpha}(s), \alpha_1(s), \alpha_2(s)],\\
\beta(t) &= \max_{0\leqslant s\leqslant t}\max[\bar{\alpha}(s), \alpha_1(s), \alpha_2(s)],\\
h(t) &= \max_{0\leqslant s\leqslant t}\max[\|(\mathscr{F}x_0)(s)-x_0(s)\|,\ \bar{k}(s)\bar{\gamma}(s)\bar{\alpha}(s)+\gamma(s)], \qquad t\in I,
\end{aligned}\tag{6}$$

where x_0 is an arbitrary fixed element of $D(I, E, \bar{u}, \lambda)$.

In order to prove the existence of a solution of equation (1) we define the sequence $\{x_n\}$ by the relations

$$x_{n+1} = \mathscr{F}x_n, \qquad n = 0, 1, \ldots,\tag{7}$$

where the operator $\mathscr{F}$ is defined by the right-hand side of equation (1).

To prove the convergence of the sequence $\{x_n\}$ to the solution $\bar{x}$ of equation (1), we define also the sequence $\{u_n\}$ by the relations

$$u_0 = u^*, \qquad u_{n+1} = Ku_n+Lu_n, \qquad n = 0, 1, \ldots,$$

where u^* is the solution of equation (5) (which is supposed to exist) with k, l, α, β and h defined by (6).

By induction we prove the following ([2], [6]).

LEMMA 4. *If assumptions of Lemma* 1 *are satisfied with k, l, α, β and h defined by* (6), *then*

$$0 \leqslant u_{n+1} \leqslant u_n \leqslant u^*, \qquad n = 0, 1, \ldots, \qquad u_n \rightrightarrows 0 \quad \textit{for } n\to\infty,$$

where the symbol $\rightrightarrows$ denotes the uniform convergence in I.

Now from Lemmas 1, 2, 3 and 4 we infer

THEOREM 3. *If Asumptions* H_1, H_2 *and assumptions of Lemma* 1 *are satisfied for k, l, α, β and h defined by* (6) *and if $B < 1$, $(B-1)^2-4AC > 0$, then there exists a unique solution $\bar{x}\in D(I, E, \bar{u}, \lambda)$ of equation* (1). *The* *sequence $\{x_n\}$ defined by (7) converges in I uniformly to $\bar{x}$, and the following estimations*

(8) $$\|\bar{x}(t)-x_n(t)\| \leqslant u_n(t), \quad n=0,1,\ldots, \ t\in I,$$

hold.

Proof. First we note that from the assumptions of this theorem and from the definitions of the functions k, l, α, β and h (see (6)) it follows that the assumptions of Lemma 2 and 3 are satisfied. Hence $x_n \in D(I, E, \bar{u}, \lambda)$.

We prove the estimations:

(9) $$\|x_n(t)-x_0(t)\| \leqslant u^*(t), \quad n=0,1,\ldots, \ t\in I,$$

(10) $$\|x_{n+k}(t)-x_n(t)\| \leqslant u_n(t), \quad n, k=0,1,\ldots, \ t\in I.$$

It is obvious that (9) holds for $n=0$. If we suppose that (9) holds for some $n>0$, then we have

$$\begin{aligned}
\|x_{n+1}(t)-x_0(t)\| &\leqslant \bar{k}(t)\tilde{k}(t)\cdot \int_0^{\bar{\alpha}(t)} \|x_n(s)-x_0(s)\|\,ds+ \\
&\quad +\bar{k}(t)[\tilde{k}(t)\bar{u}(a)+\tilde{\gamma}(t)]\big|\varphi_1(t, x_n(\cdot))-\varphi_1(t, x_0(\cdot))\big|+ \\
&\quad +\bar{l}(t)\big\|x_n(\psi_1(t, x_n(\cdot)))-x_0(\psi_1(t, x_0(\cdot)))\big\|+h(t) \\
&\leqslant \bar{k}(t)\tilde{k}(t)\cdot \int_0^{\bar{\alpha}(t)} \|x_n(s)-x_0(s)\|\,ds+ \\
&\quad +\bar{k}(t)[\tilde{k}(t)\bar{u}(a)+\tilde{\gamma}(t)]\Big[r_1(t)\tilde{k}_1(t)\int_0^{\alpha_1(t)} \|x_n(s)-x_0(s)\|\,ds+ \\
&\quad +s_1(t)\big\|x_n(\beta_1(t))-x_0(\beta_1(t))\big\|\Big]+ \\
&\quad +\lambda\bar{l}(t)\Big[r_2(t)\tilde{k}_2(t)\int_0^{\alpha_2(t)} \|x_n(s)-x_0(s)\|\,ds+ \\
&\quad +s_2(t)\big\|x_n(\beta_2(t))-x_0(\beta_2(t))\big\|\Big]+h(t) \\
&\leqslant (Ku^*)(t)+(Lu^*)(t)+h(t)=u^*(t).
\end{aligned}$$

Now (9) follows by induction. Similarly estimation (10) is easily obtained by induction.

From Lemma 4 and (10) it follows that sequence $\{x_n\}$ is convergent to the solution $\bar{x}$ of equation (1). Obviously $\bar{x}\in D(I, E, \bar{u}, \lambda)$. If $k\to\infty$, then (10) gives estimation (8).

To prove that the solution $\bar{x}$ is the unique solution of (1) in $D(I, E, \bar{u}, \lambda)$ let us suppose that there exists another solution $\tilde{x}\in D(I, E, \bar{u}, \lambda)$. It is easy to prove that

$$\tilde{u}(t)=\max_{0\leqslant s\leqslant t}\|\tilde{x}(s)-\bar{x}(s)\| \in G(I, R_+, \bar{u})$$

and $\tilde{u}\leqslant K\tilde{u}+L\tilde{u}$, where $G(I, R_+, \bar{u})$ is defined in Lemma 1.

Hence and from Lemma 1 it follows that $\tilde{x} = \bar{x}$. Thus the proof of theorem is complete.

From Theorem 1 and by the considerations contained in [2] and [6] we have the following conclusion which gives some effective conditions under which conditions (3) are fulfilled.

CONCLUSION. If Assumptions H_1, H_2 are fulfilled and if the functions $h, l, k \in C(I, R_+)$, $\alpha, \beta \in C(I, I)$ are defined by (6) and $l(t) \leqslant \bar{l} = \text{const}$, $k(t) \leqslant \bar{k} = \text{const}$, $\alpha(t) \leqslant \bar{\alpha}t$, $\beta(t) \leqslant \bar{\beta}t$, $\bar{\alpha}, \bar{\beta} \in [0, 1]$, $h(t) \leqslant Ht^p$ for some $H, p \in R_+$, and if $\bar{l}\bar{\beta}^p < 1$, $B < 1$, $(B-1)^2 - 4AC > 0$, then the assertion of Theorem 3 holds.

One can find that this result is better than this of Theorem 2.

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