ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XVI (1972)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XVI (1972)

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# Some remarks on Toeplitz methods and continuity *

**1.** Definition. Let $f$ be a real function, $f\colon R \to R$, and let $A$ be a Toeplitz method of summability. We shall say that $f$ is *$A$-continuous* if, for every $A$-summable sequence $(t_n)$, the sequence $\big(f(t_n)\big)$ is also $A$-summable.

The following facts are simple consequences of the above definition:

**1.1.** Given a Toeplitz method $A$, the set of all $A$-continuous functions contains all linear functions and is closed with respect to superposition of functions.

**1.2.** A function $f$ is continuous iff it is continuous in the sense of identity method ($I$-continuous).

Moreover, for a wide class of Toeplitz methods $A$-continuity implies continuity:

Theorem 1. *Let $A$ be a permanent Toeplitz method. If $f$ is $A$-continuous, then $f$ is continuous in ordinary sense.*

Proof. Suppose the contrary. Let $x_0$ be a point of discontinuity of $f$. Suppose, first, that there exist two sequences $(x_n)$ and $(y_n)$, each converging to $x_0$, and such that $f(x_n) \to a$ and $f(y_n) \to b$, $a \neq b$. In view of 1.1 we can assume without loss of generality that $a = 0$ and $b = 1$. As it is well known, [1], for every permanent Toeplitz method there exists a sequence $(\varepsilon_n)$ which is not $A$-summable and such that

$$\varepsilon_n = \begin{cases} 0 & \text{for } n = n_k, \\ 1 & \text{for } n = n_k', \end{cases} \qquad k = 0, 1, \ldots,$$

the sequences $(n_k)$, $(n_k')$ exhausting all non-negative integers. Let $(t_n)$ be defined as follows:

$$t_n = \begin{cases} x_k & \text{for } n = n_k, \\ y_k & \text{for } n = n_k', \end{cases} \qquad k = 0, 1, \ldots$$

* This paper was performed under scientific guidance of prof. dr S. Mazur as a master thesis.

Since the sequence $(t_n)$ is convergent (with $x_0$ as its limit), it is also $A$-summable. The sequence $\big(f(t_n)\big)$ can be written in the form

$$f(t_n) = \varepsilon_n + c_n,$$

where $c_n \to 0$. Therefore $\big(f(t_n)\big)$ is not $A$-summable, what contradicts the hypothesis. To complete the proof, we suppose now that there exists a sequence $(x_n)$ such that $x_n \to x_0$ and $f(x_n) \to \infty$. The method $A$ being permanent, it is easy to find a sequence $(t_n)$ of the form

$$x_0, \ldots, x_0, x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k, \ldots$$

(obviously convergent), such that $\big(f(t_n)\big)$ is not $A$-summable (it is $A$-summable to infinity). Therefore $f$ must be continuous at $x_0$.

THEOREM 2. *If a function $f$ is $A$-continuous for every permanent Toeplitz method $A$, then $f$ is a linear function.*

Proof. The theorem is implied by the following

EXAMPLE 1. There exists a permanent Toeplitz method $A$ such that the only $A$-continuous functions are linear functions. Let $A$ be defined as follows:

$$\begin{aligned} A_{3k}\big((t_n)\big) &= -rt_{3k} + rt_{3k+1} + t_{3k+2}, \\ A_{3k+1}\big((t_n)\big) &= A_{3k+2}\big((t_n)\big) = t_{3k+1}, \end{aligned} \qquad k = 0, 1, \ldots;\ 0 \neq r \neq 1.$$

It is easy to verify that a sequence $(t_n)$ is $A$-summable iff it can be written in the form

$$\begin{aligned} t_{3k} &= s_k, \qquad t_{3k+1} = b_k, \\ t_{3k+2} &= (1-r)b_k + rs_k + d_k, \end{aligned} \qquad k = 0, 1, \ldots,$$

where $(s_k)$ is arbitrary sequence, $(b_k)$ — convergent sequence, and $(d_k)$ is a sequence convergent to 0. Since $f$ is $A$-continuous, we can find (for every $(s_k)$, $(b_k)$, $(d_k)$) a sequence $(d_k')$, $d_k' \to 0$, such that

$$f\big((1-r)b_k + rs_k + d_k\big) = (1-r)f(b_k) + rf(s_k) + d_k'.$$

In particular, let $b_k = b$, $d_k = 0$ $(k = 0, 1, \ldots)$ and let $(s_k)$ be dense in $R$. For any real $s$ we can find a sequence of indices $(n_k)$ such that $s_{n_k} \to s$. Taking into account that $f$ is continuous (Theorem 1) and that $d'_{n_k} \to 0$, we obtain

$$f\big((1-r)b + rs\big) = (1-r)f(b) + rf(s).$$

Without loss of generality we can assume that $f(0) = 0$. Simple reasoning leads now to the conclusion, that the identity

$$f(x+y) = f(x) + f(y)$$

holds for arbitrary reals $x$, $y$. Since the function $f$ is continuous, it must be linear.

The question arises if there exist a permanent Toeplitz method $A \neq I$ and a non-linear function $f$, such that $f$ is $A$-continuous. An answer to this is given by the following

EXAMPLE 2. Let $A = A_{(2^n)}$ be so called *single-sequence method*, i.e. a permanent Toeplitz method such that the only $A$-summable sequences $(d_n)$ are

$$d_n = a2^n + c_n,$$

where $a$ denotes arbitrary constant, $(c_n)$ — arbitrary convergent sequence (cf. [2], p. 48). Let $g_0$ be an arbitrary continuous function defined on the interval $\langle -1, 1\rangle$, vanishing at each end of the interval. Let

$$g(t) = \begin{cases} g_0(t-2^k) & \text{for } t \in \langle 2^k-1, 2^k+1\rangle,\ k = 1, 2, \ldots, \\ 0 & \text{elsewhere.} \end{cases}$$

It is easy to verify that the sequence $(y_n)$, $y_n = g(a2^n + c_n)$, is convergent for any $a$ and $(c_n)$. Therefore $g$ is $A$-continuous.

The function constructed in the above example is $A$-continuous because of its very special behaviour at infinity. Now we shall state some necessary conditions a function must satisfy in order to be continuous in the sense of some non-trivial Toeplitz method.

**2.** A Toeplitz method $A$ is said *to have a strict rate of growth* if there exists a sequence $(\vartheta_n)$ such that the following conditions are fulfilled:

— $(\vartheta_n t_n)$ is bounded for every $A$-summable sequence $(t_n)$;

— if for some sequence $(\sigma_n)$ the sequence $(\sigma_n t_n)$ is bounded for every $A$-summable $(t_n)$, then $\sigma_n = O(\vartheta_n)$.

LEMMA 1. *The following conditions can not be satisfied simultaneously*:

(a) *A Toeplitz method $A$ has a strict rate of growth*;

(b) *a function $f$ is $A$-continuous*;

(c) *there exists an $A$-summable sequence $(t_n)$ such that for some sequence of indices $(n_k)$, $t_{n_k} \to +\infty$, and the quotient $f(t_{n_k})/t_{n_k}$ tends to infinity.*

Proof. Let $(\vartheta_n)$ be a strict rate of growth for $A$. Since the function $f$ is $A$-continuous, there exist such constants $M_1$ and $M_2$ that for every $n$ we have $|\vartheta_n t_n| \leqslant M_1$, $|f(t_n)\vartheta_n| \leqslant M_2$. Let $\theta_k = f(t_{n_k})/t_{n_k}$, and

$$\sigma_n = \begin{cases} \vartheta_n & \text{for } n \neq n_k \\ \vartheta_{n_k}|\theta_k| & \text{for } n = n_k \end{cases} \qquad (k = 0, 1, 2, \ldots).$$

Since $|t_n \sigma_n| \leqslant \max(M_1, M_2)$ and $(\vartheta_n)$ is a strict rate of growth, there must exist a constant $K$ such that $\sigma_n \leqslant K\vartheta_n$. This, however, contradicts condition (c).

We shall say that two Toeplitz methods are equivalent if their fields are equal. A method $A$ is said to be equivalent to a method extracted from $I$, if there exists an increasing sequence of indices $(n_k)$ such that a sequence $(t_n)$ is $A$-summable iff its subsequence $(t_{n_k})$ is convergent.

The following are simple consequences of Lemma 1:

Corollary 2.1. *Let $A$ be a Toeplitz method such that the method $\bar{A}$ obtained from $A$ by omitting all zero-columns is convergence preserving Toeplitz method with a strict rate of growth. Let $f$ be $A$-continuous real function such that $|f(t)/t| \to \infty$ when $t \to +\infty$ $(t \to -\infty)$. Then $A$ is equivalent to a method extracted from $I$.*

Proof. It is sufficient to observe that $\bar{A}$ is equivalent to $I$. Suppose the contrary. Then there exists an unbounded from above $\bar{A}$-summable sequence $(t_n)$ and a sequence of indices $(n_k)$ such that $t_{n_k} \to +\infty$. Now we can apply Lemma 1.

Corollary 2.2. *Let $A$ have a strict rate of growth. If there exists an $A$-summable sequence $(t_n)$ dense in a half-line, then every $A$-continuous function $f$ must satisfy the condition $\overline{\lim}|f(t)/t| < +\infty$ when $t \to +\infty$.*

Proof. The hypotheses of Lemma 1 are satisfied.

The following weaker results are valid for permanent methods which not necesarilly have a strict rate of growth.

Lemma 2. *If $A = (a_{kn})$ is permanent for null-sequences (i.e. sequences convergent to zero) and if $f$ is $A$-continuous function, then for every $A$-summable sequence $(t_n)$ and every convergent to zero sequence $\tau_n$ the following equality holds:*

$$\lim_k \sum_n a_{kn}\big(f(t_n+\tau_n)-f(t_n)\big) = 0.$$

Proof. Let $(\varepsilon_n)$ be an arbitrary sequence with terms equal to 0 or 1, let $(t_n)$ and $(\tau_n)$ satisfy the hypotheses of the lemma. Since $f(t_n+\varepsilon_n\tau_n)-f(t_n) = \varepsilon_n\big(f(t_n+\tau_n)-f(t_n)\big)$ and the sequence $\big(f(t_n+\varepsilon_n\tau_n-f(t_n)\big)$ is $A$-summable, the limit

$$\lim_k \sum_n a_{kn}\big(f(t_n+\tau_n)-f(t_n)\big)\varepsilon_n$$

does exist. Applying Schur lemma (cf. [1], p. 133) to the matrix $(b_{kn})$, $b_{kn} = a_{kn}\big(f(t_n+\tau_n)-f(t_n)\big)$ we conclude that $\sum_n |b_{kn}| \underset{k}{\to} 0$. The proof is completed.

Now we can prove what follows:

Corollary 2.3. *Let $A = (a_{kn})$ be a permanent Toeplitz method such that there exists an $A$-summable sequence $(t_n)$, $t_n \to +\infty$. If a function $f$ satisfies the condition: $f'(t) \to +\infty$ when $t \to +\infty$, and if $f'$ is increasing for $t > T$, then $f$ is not $A$-continuous function.*

Proof. Let $(t_n)$, $t_n \to +\infty$, be an $A$-summable sequence. Assuming $\tau_n = \big(f'(t_n)\big)^{-1}$ and taking into account that $f(t_n+\tau_n)-f(t_n) = f'(t_n+$

$+\theta_n\tau_n)\tau_n \geqslant f'(t_n)\tau_n = 1$, we conclude in view of Lemma 2 that $\sum_n |a_{kn}| \to 0$. This contradicts the permanency of the method $A$.

In all we have proved above some additionary assumptions about the structure of the field of the method under consideration were made. Now we shall prove

THEOREM 3. *Let a differentiable function $f$ satisfy the following condition: there exist $\varepsilon > 0$ and $T > 0$ such that $|f'(t)/t| \geqslant \varepsilon$ for $t \geqslant T$. If $f$ is $A$-continuous for a convergence preserving Toeplitz method $A = (a_{kn})$, then $A$ is equivalent to a method extracted from $I$.*

Proof. We can assume that the matrix $(a_{kn})$ contains no zero-columns, and under this assumption we shall prove that $A$ is equivalent to $I$.

(1) We shall prove first, that for every $A$-summable sequence $(t_n)$ there exists a constant $M = M((t_n))$ such that $\sum_n |a_{kn}t_n| \leqslant M$ for $k = 0, 1, 2, \ldots$ Let $f$ satisfy the assumptions of the theorem, let $\varrho$ be arbitrary real number such that $|\varrho| \leqslant 1/N$, where $N = (T+1)/\varepsilon T$, and let $|t| \geqslant T+1$. Since the function $f$ is increasing (or decreasing) on the interval $\langle t-1, t+1\rangle$ ($f'(t)$ exists and does not vanish), we conclude that one of the numbers

$$\frac{f(t+N\varrho)-f(t)}{t}, \quad \frac{f(t-N\varrho)+f(t)}{t}$$

is positive, and the other is negative. It is easy to verify that absolute value of each of them is not smaller than $|\varrho|$. It means that the equation

$$\frac{f(t+\tau)-f(t)}{t} = \varrho$$

(with unknown $\tau$, and fixed $\varrho$ and $t$) has at least one solution in the interval $|\tau| \leqslant N|\varrho|$.

Let $(t_n)$ be an arbitrary $A$-summable sequence, let $(t_{n_i})$ be a subsequence consisting of all $t_n$ such that $t_n \geqslant T+1$. If $(\varrho_i)$ is an arbitrary sequence convergent to zero and such that $|\varrho_i| \leqslant N^{-1}$ for every $i$, then for every equation

$$\frac{f(t_{n_i}+\tau)-f(t_{n_i})}{t_{n_i}} = \varrho_i$$

we can find a solution $\tau_{n_i}$, $|\tau_{n_i}| \leqslant N\varrho_i$. Obviously $\tau_{n_i} \underset{i}{\to} 0$. Take now $\tau_n = 0$ for $n \neq n_i$ $(i = 0, 1, 2, \ldots)$. Since $f$ is $A$-continuous, the sequence $(s_n)$,

$$s_n = f(t_n+\tau_n)-f(t_n),$$

is $A$-summable and the limit

$$\lim_k \sum_n a_{kn}\big(f(t_n+\tau_n)-f(t_n)\big) = \lim_k \sum_i a_{kn_i}t_{n_i}\varrho_i$$

does exist. But the sequence $(\varrho_i)$ was arbitrarily chosen, and the existence of the right-hand limit means that every null-sequence $(\varrho_i)$ (with $|\varrho_i| \leqslant N^{-1}$) is summable by the method $(b_{ki})$, $b_{ki} = a_{kn_i} t_{n_i}$. Therefore every null-sequence is $(b_{ki})$-summable and there exists a positive number $K = K\big((t_{n_i})\big)$ such that $\sum_i |a_{kn_i} t_{n_i}| \leqslant K$ for every $k$. Since there exists a constant $L$ such that $\sum_n |a_{kn}| \leqslant L$ for every $k$, we obtain

$$\sum_n |a_{kn} t_n| \leqslant K + L(T+1) = M,$$

what was to be proved.

(2) Now we shall prove that $A$ is equivalent to $I$ (this part of the proof is a modification of the proof of Theorem 3 of [1]). Let $\alpha_n = \sup_k |a_{kn}|$, $\gamma = \inf_n \alpha_n$. It is sufficient to prove that $\gamma$ is bigger than 0. Indeed, if $\gamma > 0$, then by the first part of the proof we have $|t_n| \leqslant M/\alpha_n \leqslant \gamma^{-1} M$. That means that every $A$-summable sequence is bounded, and $A$ must be equivalent to $I$ (cf. Theorem 7 of [1]). It remains to prove that $\gamma > 0$. Suppose, then, that $\gamma = 0$. There exists an increasing sequence of indices $(n_m)$ such that $\sum_m \alpha_{n_m}^{1/3} < +\infty$. From what we know of $f$ it follows that there exists such a number $T' > 0$ that for every $t \geqslant T'$ we have: $f(t) \geqslant t^2 \varepsilon/2 + C$ for some constant $C$. We can assume that $\alpha_{n_m}^{-2/3} \geqslant T'$, $m = 0, 1, 2, \ldots$ Let

$$t_n = \begin{cases} \alpha_{n_m}^{-2/3} & \text{for } n = n_m,\ m = 0, 1, \ldots, \\ 0 & \text{elsewhere.} \end{cases}$$

The sequence $(t_n)$ is $A$-summable, for the series $\sum_m |a_{kn_m} \alpha_{n_m}^{-2/3}|$ converges uniformly with respect to $k$. The sequence $(v_n)$, $v_n = f(t_n) - C$ is also $A$-summable then. In view of (1) we have: $\sum_n |a_{kn}\big(f(t_n) - C\big) \leqslant M\big((v_n)\big) = M_1$. The last inequality leads to $\alpha_{n_m} \geqslant (\varepsilon/2M_1)^3$, what contradicts the hypothesis that $\alpha_{n_m} \underset{m}{\to} 0$, and that completes the proof.

Some minor generalisations of the theorem are possible, e.g.:

Let for a differentiable function $f$ there exist $a > 0$, $\varepsilon > 0$ and $T > 0$ such that $\big|f'(t)/|t|^a\big| \geqslant \varepsilon$ for $|t| \geqslant T$. If $f$ is $A$-continuous for a convergence preserving Toeplitz method $A$, then $A$ is equivalent to a method extracted from $I$.

Proof. The function $f^m = f \circ f \circ \ldots \circ f$, for sufficiently large positive integer $m$, satisfies the assumptions of the preceding theorem.

**4.** A fairly good characterisations of some classes of $A$-continuous functions related to Toeplitz methods of some types can be given (cf. Example 1). However, in all such examples known to the author, the existence of non-linear $A$-continuous functions is connected with the fact that the method $A$ does not sum any bounded divergent sequence.

It is open question whether there exists a Toeplitz method $A$ which sums some divergent bounded sequences and which yields a non-linear $A$-continuous function. Author believes that the answer is negative. In any case, the notion of $A$-continuity does not seem to be interesting.

**References**

[1] S. Mazur and W. Orlicz, *On linear methods of summability*, Studia Math. 14 (1954).

[2] K. Zeller, *Theorie der Limitierungsverfahren*, Berlin 1958.