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A note on classical quotient rings

Abstract. A characterization of reduced fully left idempotent left Goldie rings is given. Classical left quotient rings for which projectivity coincides with p-injectivity are considered.

Throughout, A represents an associative ring with identity and A-modules are unital. Z denotes the left singular ideal of A and A is called *left non-singular* if $Z = 0$. If N is a submodule of a left A-module M, and E a submodule such that $N \subseteq E$, then ${}_AE$ is an *essential extension* of ${}_AN$ in ${}_AM$ if, for any non-zero left submodule V of E, $V \cap N \neq 0$. A submodule C of ${}_AM$ is called a *complement submodule* if there exists a submodule K of ${}_AM$ such that C is maximal with respect to $C \cap K = 0$. This is equivalent to C having no proper essential extension in ${}_AM$. The concepts of complement submodule and essential extension are significant tools in ring theory (cf. for example [3], [5], [7]).

An *ideal* of A will always mean a two-sided ideal and A is called *right* (resp. *left*) *duo* if every right (resp. left) ideal of A is an ideal. A left (right) ideal is called *reduced* if it contains no non-zero nilpotent element. A is called *fully left* (resp. *right*) *idempotent* if every left (resp. right) ideal of A is idempotent. As usual, A is called a *left Goldie ring* if it satisfies the maximum condition on left annihilators and complement left ideals. A well-known theorem [5, Theorem 3.35] states that A is semiprime left Goldie iff A has a semisimple Artinian classical left quotient ring. (Recall that Q is a *classical left quotient ring* of A if (a) $A \subseteq Q$; (b) every non-zero-divisor of A is invertible in Q; (c) for any $q \in Q$, $q = b^{-1}a$, $a, b \in A$, b being a non-zero-divisor.) A result of R. E. Johnson states that A is left non-singular iff A has a von Neumann regular maximal left quotient ring Q. In that case, Q is a left self-injective ring and ${}_AQ$ is the injective hull of ${}_AA$. If A has a semisimple Artinian classical left quotient ring Q, then Q coincides with the maximal left quotient ring of A. For results concerning classical quotient and maximal quotient rings, consult, for example, [5]. J. H. Cozzens' domains [3, p. 105] are simple principal left ideal V-domains but not division rings. Also, simple left Ore domains need not be right Ore.

Consequently, there is no inclusion relation between the classes of simple left Ore domains and reduced Artinian rings. But simple left Ore domains and reduced Artinian rings are reduced fully left idempotent left Goldie rings. In ring theory, various generalizations of commutative rings (for example, duo rings) have been studied.

A generalization of right duo rings, namely rings whose non-zero right ideals contain non-zero ideals, called *strongly right bounded rings*, play an important role in the study of FPF (GFC) rings (cf. [1], [4]). (A is called *right FPF* (resp. *GFC*) if every finitely generated (resp. cyclic) faithful right A-module generates the category Mod-A.) Strongly right bounded rings are right GFC [1]. In [4, Theorem 3.38], reduced fully right idempotent right FPF right Goldie rings are considered. In the first section of this note, I study reduced fully left idempotent left Goldie rings. In particular, such rings are characterized in terms of complement left ideals and ideals.

§ 1. Fully left idempotent left Goldie rings. It is well known that fully left idempotent left Goldie rings need not be Artinian (even in the case of integral domains). In this section, I shall characterize reduced fully left idempotent left Goldie rings in terms of the following class of rings (cf. Theorem 1.6).

DEFINITION. A is called a *left WCT ring* if every complement left ideal of A is an ideal and every ideal of A is a complement left ideal.

Right WCT rings are similarly defined.

Our first result will lead to a first information on the maximal left quotient ring of a left WCT ring.

PROPOSITION 1.1. *The following conditions are equivalent:*

(1) *A is a left WCT ring;*

(2) *Every non-zero complement left ideal of A contains a non-zero ideal of A and every ideal of A is a complement left ideal;*

(3) *If I is either a complement left ideal of A or an ideal of A, then I is generated by a central idempotent;*

(4) *A is a fully left idempotent ring whose complement left ideals and ideals are finitely generated right ideals.*

Proof. Obviously, (1) implies (2).

Assume (2). Suppose T is an ideal of A such that $T^2 = 0$.

If V is a complement left ideal of A such that $r(T) \oplus V$ is an essential left ideal of A, then $TV \subseteq V \cap r(T) = 0$ implies $V \subseteq r(T)$, whence $r(T)$ is an essential left ideal of A. By hypothesis, $r(T)$ is a complement left ideal, which implies that $r(T) = A$, yielding $T = 0$.

This shows that A is semiprime. Then, for any ideal I of A, $I \cap l(I) = 0$. Let K be a complement left ideal of A such that $(I \oplus l(I)) \oplus K$ is an essential left ideal of A. Then $IK \subseteq I \cap K = 0$ implies $K \subseteq r(I) = l(I)$, whence $I \oplus l(I)$ is an essential left ideal. Now since $I \oplus l(I)$ is an ideal of A, it follows that $I \oplus l(I)$ is a complement left ideal, which yields $I \oplus l(I) = A$. This proves that every ideal .

of A is generated by a central idempotent (inasmuch as A is semiprime). If $0 \neq b \in A$ such that $b^2 = 0$, since $Z = 0$, there exists a non-zero complement left ideal U of A with $l(b) \cap U = 0$. Now U contains a non-zero central idempotent u, which implies that $ub \in l(b) \cap U = 0$, whence $u \in l(b) \cap U = 0$, a contradiction! This proves that A must be reduced. Let C be a non-zero complement left ideal such that $C \neq l(r(C))$. Then there exists a non-zero complement left subideal U of $l(r(C))$ such that $C \cap U = 0$. Since $l(r(C))$ is a complement left ideal of A, it follows that U is a complement left ideal of A and therefore contains a non-zero central idempotent v. Therefore $Cv \subseteq C \cap V = 0$ implies $v \in r(C)$, whence $v^2 = 0$ (because $v \in l(r(C))$), contradicting A being reduced. This proves that C is a left annihilator ideal, and therefore an ideal of A (inasmuch as A is reduced). But then, C is generated by a central idempotent. Thus (2) implies (3).

Assume (3). Then A is biregular, and therefore fully left (and right) idempotent. Also, any complement left ideal or ideal of A is a principal right ideal. Therefore (3) implies (4).

Assume (4). Let T be either a complement left ideal or an ideal of A. Then T is a finitely generated right ideal and for any $t \in T$, $t \in AtAt \subseteq Tt$, which implies that A/T_A is flat. Therefore A/T is a finitely related flat right A-module, which implies that A/T_A is projective. It follows that T_A is a direct summand of A_A. Then we immediately have $Z = 0$ and since every complement left ideal of A is an ideal, we know that A must be reduced. Thus T is generated by a central idempotent and (4) implies (1).

COROLLARY 1.1.1. *A left WCT ring is a reduced Baer ring.*

COROLLARY 1.1.2. *If A is left WCT, then A has a maximal left quotient ring Q which is left and right self-injective strongly regular.*

Proof. We have seen A is reduced. Since A is left non-singular, it has a regular maximal left quotient ring Q, where ${}_AQ$ is the injective hull of ${}_AA$. For any principal left ideal P of Q, if $C = P \cap A$, then ${}_AP$ is the injective hull of ${}_AC$ and since ${}_AP$ is an injective submodule of ${}_AQ$, it follows that C is a complement left ideal of A.

From (3) of Proposition 1.1, $C = Ae$, where e is a central idempotent in A, which yields $P = QC = Qe$. For any $q \in Q$, since ${}_AA$ is essential in ${}_AQ$, there exists an essential left ideal L of A such that $Lq \subseteq A$. Now for every $b \in L$, $b(qe-eq) = (bq)e-(be)q = ebq-ebq = 0$ (e being central in A), which yields $qe = eq$ (inasmuch as ${}_AQ$ is non-singular). This proves that e is central in Q. Therefore Q is left and right self-injective strongly regular.

COROLLARY 1.1.3. *If A is left and right WCT, then the maximal left quotient ring Q coincides with the maximal right quotient ring of A.* (Apply [5, Theorem 2.38].)

COROLLARY 1.1.4. *The following conditions are equivalent*: (1) *A is a simple* *left Ore domain*; (2) *A is a prime left WCT ring*; (3) *A is an indecomposable left WCT ring.*

Applying [3, Lemma 18.34B] to Corollary 1.1.3, we get

COROLLARY 1.1.5. *The following conditions are equivalent*: (1) *A is either a simple left Noetherian domain or a reduced Artinian ring*; (2) *A is a left WCT left Noetherian ring whose proper prime factor rings are Artinian.*

Recall that a left A-module M is *divisible* if $M = cM$ for each non-zero-divisor c of A. A left A-module M is called *p-injective* if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M. A is *von Neumann regular* iff every left (right) A-module is flat iff every left (right) A-module is p-injective. Flatness and p-injectivity are distinct concepts. However, (a) if I is a p-injective left ideal of A, then A/I is a flat left A-module; (b) if M is a maximal left ideal of A which is an ideal, then ${}_AA/M$ is flat iff A/M_A is injective iff A/M_A is p-injective (cf. also [7, p. 271, ex. 14]). The class of p-injective modules is strictly between the classes of injective and divisible modules (cf. Remark 6 at the end). A is called *left p-injective* if ${}_AA$ is p-injective. Continuous rings considered here are in the sense of Y. Utumi [11]. Recall that A is *ELT* (resp. *ERT*) iff every essential left (resp. right) ideal of A is an ideal [15]. ERT rings are right bounded [3, p. 49]. The next theorem will follow from [7, Corollary 11.3.2], [11, Lemma 4.1], Proposition 1.1 and the following results: (a) Reduced left p-injective rings are strongly regular; (b) Left continuous reduced rings are strongly regular; (c) If A is left WCT, then any prime factor ring of A is simple.

THEOREM 1.2. *The following conditions are equivalent for a left WCT ring A*:

(1) *A is reduced Artinian*;
(2) *A is von Neumann regular*;
(3) *A is left continuous*;
(4) *A is left p-injective*;
(5) *Every simple left A-module is flat*;
(6) *Every primitive factor ring of A is either ELT or ERT*;
(7) *A is semiperfect.*

If A is a left Ore domain, then A has a classical left quotient ring Q which is a division ring. In that case, Q is the maximal left quotient ring of A. It is known that simple left Ore domains need not be right Ore [3, p. 105]. Consequently, Proposition 1.1 yields our first remark.

Remark 1. Left WCT rings need not be right WCT.

THEOREM 1.3. *If A is left WCT, then the maximal left quotient ring of A is a finite direct sum of division rings.*

Proof. If Q is the maximal left quotient ring of A, then Q is strongly regular by Corollary 1.1.2. If T is a left ideal of Q, then T is an ideal, which implies that $U = T \cap A$ is an ideal of A. Therefore $U = Ae$, where e is a central idempotent in A (Proposition 1.1), which implies that e is central in Q. Since ${}_AA$ is essential in ${}_AQ$, it follows that ${}_AU$ is essential in ${}_AT$, which implies that QU is an essential submodule of ${}_QT$. But $QU = Qe$, which yields $T = Qe$. Therefore every left ideal of Q is a direct summand of ${}_QQ$ and by Corollary 1.1.2, Q is reduced Artinian, which proves the theorem.

COROLLARY 1.3.1. *If A is left WCT, then the maximal left quotient ring is left and right WCT.*

COROLLARY 1.3.2. *A left WCT ring is left Goldie.* (Apply [10, Theorem 1.6].)

COROLLARY 1.3.3. *If A is left WCT, then A is right WCT iff A satisfies the maximum condition on complement right ideals.*

Combining [2, Theorem 2.3], [5, Theorem 2.38], Proposition 1.1 and Corollary 1.3.2, we get

PROPOSITION 1.4. *Let A be a left WCT ring with maximal left quotient ring Q. Then A is right WCT iff ${}_AQ$ is flat.*

(Note that if ${}_AQ$ is projective, then A is a finite direct sum of division rings).

As before, Z denotes the left singular ideal of A.

PROPOSITION 1.5. *If A is a fully left idempotent ring satisfying the maximum condition on left annihilators and every complement left ideal of A is an ideal, then A is left WCT.*

Proof. Since A is semiprime with maximum condition on left annihilators, we have $Z = 0$. If $b \in A$ such that $b^2 = 0$, there exists a complement left ideal C of A such that $L = l(b) \oplus C$ is an essential left ideal. Since $Cb \subseteq C \cap l(b) = 0$, we have $C \subseteq l(b)$, which implies $L = l(b)$, whence $b \in Z = 0$. This proves that A is reduced. Let T be an ideal of A. Since A is reduced, $T \cap l(T) = 0$ and it is easily seen that $E = T \oplus l(T)$ is an essential left ideal of A. By [6, Theorem], E contains a non-zero-divisor c. Since A is fully left idempotent, $c = dc$ for some $d \in AcA$, which implies that $1 = d \in E$, and hence $E = A$. This proves that any ideal of A is generated by a central idempotent. Thus A is left WCT.

Recall that a left A-module M is *torsionfree* if, for any non-zero-divisor c of A and any non-zero element v of M, $cv \neq 0$.

We are now in a position to characterize reduced fully left idempotent left Goldie rings.

THEOREM 1.6. *The following conditions are equivalent*:

(1) *A is a reduced fully left idempotent left Goldie ring*;

(2) *A is a fully left idempotent ring having a reduced Artinian classical left quotient ring*;

(3) *A is left WCT.*

Proof. If A is a reduced ring having a classical left quotient ring Q, then Q must be reduced [16, Proposition 1.5]. Therefore (1) implies (2) by [5, Theorem 3.35].

Assume (2). If A has a reduced Artinian classical left quotient ring Q, then A satisfies, in particular, the maximum condition on left annihilators. Also, Q is the maximal left quotient ring of A. Let C be a non-trivial complement left ideal of A. If H is the injective hull of $_AC$ in $_AQ$, then $_AC$ is an essential submodule of $_AH \cap A$, which implies that $C = H \cap A$. Now $_AH$ is non-singular and since A has a classical left quotient ring, $_AH$ is torsionfree, and since $_AH$ is divisible (because it is injective), therefore H must be a left Q-module (cf. [8, Theorem 3.3]). Consequently, H is an injective left ideal of Q and H is therefore an ideal of Q. Finally, for any $c \in C$ and $a \in A$, we have $ca \in H \cap A = C$, which proves that C is an ideal of A. Then (2) implies (3) by Proposition 1.5.

Assume (3). By Proposition 1.1, A is biregular, and therefore fully left (and right) idempotent. A is left Goldie by Corollary 1.3.2, and by Corollary 1.1.1, (3) implies (1).

COROLLARY 1.6.1. *If every simple left A-module is either p-injective or projective, then A has a reduced Artinian classical left quotient ring iff A is left WCT.* (Apply [13, Proposition 6].)

COROLLARY 1.6.2. *If A admits a reduced Artinian classical left quotient ring, then the following are equivalent*: (a) *Any complement left ideal or ideal of A is generated by an idempotent*; (b) *A is fully left idempotent.* (Apply Proposition 1.1.)

It may be noted that in (1) and (2) of Theorem 1.6, the "fully left idempotent" property cannot be dropped (otherwise, any reduced ELT left Goldie ring would be necessarily Artinian!).

V-rings have been extensively studied by many authors (cf. [3]).

Remark 2. The following conditions are equivalent for a left V-ring A: (a) A is reduced left Goldie; (b) A is left WCT; (c) All complement left ideals and ideals of A are finitely generated right ideals.

We add some more properties of WCT rings.

Remark 3. Let A be left WCT.

(1) The maximal left quotient of A is a flat right A-module and every p-injective non-singular left A-module is injective;

(2) If A is left hereditary, then A is left Noetherian;

(3) If A is right WCT, then A is left hereditary iff every divisible left A-module is injective;

(4) If every finitely generated non-singular left A-module is projective, then A is right WCT;

(5) If every p-injective right A-module is injective, then A is right hereditary, right Noetherian, right WCT.

Following [14], A is called a *left WP-ring* if every left ideal of A not isomorphic to ${}_AA$ is p-injective. Applying [14, Proposition 1.9], Proposition 1.1 and Theorem 1.2, we get

Remark 4. The following conditions are equivalent: (a) A is either reduced Artinian or a simple principal left ideal domain; (b) A is a left WP and left WCT ring.

The next question is motivated by Proposition 1.1.

QUESTION: If every non-zero complement left ideal of A contains a non-zero ideal and every non-zero ideal of A contains a non-zero complement left ideal, is A left WCT?

§ 2. Projective modules and p-injective modules. I now turn to rings whose projective modules coincide with p-injective modules by looking at a particular situation when a classical left quotient ring is quasi-Frobeniusean.

PROPOSITION 2.1. *Let A have a classical left quotient ring Q. If every p-injective torsionfree left A-module is a flat injective left Q-module, then Q is quasi-Frobeniusean.*

Proof. Since any p-injective left A-module is divisible [12], any p-injective torsionfree left A-module must be a left Q-module [8, p. 140]. Let M be a p-injective left Q-module. If $P = Ay$, $y \in A$, $f\colon P \to M$ a left A-homomorphism, setting $u = f(y)$, we may define $h\colon Qy \to M$ by $h(qy) = qu$ for all $q \in Q$. Suppose that $q_1 y = q_2 y$, $q_1, q_2 \in Q$. If $q_1 = b_1^{-1} a_1$, $q_2 = b_2^{-1} a_2$, $a_i, b_i \in A$ $(i = 1, 2)$, then there exists a non-zero-divisor b_3 in A such that with $c = b_3 b_2 b_1$, we get $cq_1 \in A$, $cq_2 \in A$, which yields $ch(q_1 y) = cq_1 u = cq_1 f(y) = f(cq_1 y) = f(cq_2 y) = cq_2 f(y) = cq_2 u = ch(q_2 y)$, whence $h(q_1 y) = h(q_2 y)$ (inasmuch as c is a non-zero-divisor). This proves that h is a well defined left Q-homomorphism. Since ${}_QM$ is p-injective, there exists $w \in M$ such that $h(qy) = qyw$ for all $q \in Q$, which implies $f(ay) = ayw$ for all $a \in A$. This proves that ${}_AM$ is p-injective. Since ${}_AM$ is torsionfree, by hypothesis, M is an injective flat left Q-module. We know that any direct sum of p-injective left Q-modules is p-injective. Then any direct sum of injective left Q-modules is a p-injective left Q-module and therefore is injective. It is well known that Q is left Noetherian in this case. It follows that Q is a left Noetherian ring whose injective left modules are flat, which yields Q being quasi-Frobeniusean (for IF-rings, consult [9]).

Classical left quotient rings in which projectivity coincides with p-injectivity may now be characterized.

THEOREM 2.2. *If A admits a classical left quotient ring Q, then the following conditions are equivalent*:

(1) *Projective left Q-modules coincide with p-injective left Q-modules*;

(2) *Every p-injective torsionfree left A-module is a flat injective left Q-module.*

Proof. Assume (1). Let N be a p-injective torsionfree left A-module. Then ${}_AN$ is divisible torsionfree, which implies that N is a left Q-module [8]. If $P = Qt$, $t \in Q$, $t = b^{-1}d$, b, $d \in A$, then $P = Qd$. Let $g\colon P \to N$ be a left Q-homomorphism. If $f\colon Ad \to N$ is defined by $f(ad) = g(ad)$ for all $a \in A$, then f is a left A-homomorphism, which implies that $f(d) = du$ for some $u \in N$. Now for any $q \in Q$, $g(qd) = qg(d) = qf(d) = qdu$, which proves that ${}_QN$ is p-injective. By hypothesis, ${}_QN$ is projective, which implies that ${}_QN$ is flat. Since any injective left Q-module is projective, Q is quasi-Frobeniusean and N is therefore an injective left Q-module [3, Theorem 24.20]. Thus (1) implies (2).

Assume (2). Then Q is quasi-Frobeniusean by Proposition 2.1. Now every projective left Q-module is injective [3, Theorem 24.20] and hence p-injective. Since any left Q-module is a torsionfree left A-module, it follows that every p-injective left Q-module, which is a p-injective torsionfree left A-module, is an injective left Q-module. Therefore every p-injective left Q-module is projective and hence (2) implies (1).

COROLLARY 2.2.1. *The following conditions are equivalent*:

(1) *A is a left continuous ring such that every p-injective torsionfree left A-module is an injective flat left A-module*;

(2) *Projective left A-modules coincide with p-injective left A-modules.*

It may be noted that A is quasi-Frobeniusean iff every flat left A-module is injective (cf. [3, Theorem 24.20]). But if every injective left (right) A-module is flat, then A not be quasi-Frobeniusean (cf. [9, p. 397]).

Remark 5. Let A be a left V-ring having a classical left quotient ring Q. Then A is left WCT iff A is a reduced ring such that every p-injective torsionfree left A-module is a flat injective left Q-module.

The next result generalizes the well-known fact that if A is an integral domain having a classical left quotient ring Q, then Q is a division ring.

PROPOSITION 2.3. *If A is a reduced ring having a classical left quotient ring Q, then the following conditions are equivalent*:

(1) *Q is left and right continuous strongly regular*;

(2) *Q is a Baer ring.*

Proof. Obviously, (1) implies (2).

Assume (2). For any $q \in Q$, $l_Q(q) = Qe$, $e^2 = e \in Q$. Since Q is reduced [16, Proposition 1.5], $c = q+e$ is a non-zero-divisor in Q, and therefore is invertible in Q. Thus $q = qcc^{-1} = q^2c^{-1}$, which proves that Q is strongly regular. Then every complement left ideal of Q, being an ideal of Q, is a left annihilator, and therefore a direct summand of ${}_QQ$. Since Q is regular, it is left continuous, and since Q is reduced, therefore (2) implies (1).

If A is semiprime left Goldie, then it is well known that every essential left ideal contains a non-zero-divisor. This motivates our last result.

PROPOSITION 2.4. *Let A be a ring with a classical left quotient ring Q and satisfying the following condition* (∗): *if K is an essential left ideal of A and $f\colon K \to A$ a left A-homomorphism, there exist a non-zero-divisor c of A and a left A-homomorphism $h\colon K+Ac \to A$ which extends f. Then ${}_AQ$ is injective and hence Q is the maximal left quotient ring of A.*

Proof. Let E be an essential left ideal of A, and $g\colon E \to Q$ a left A-homomorphism. If $K = \{y \in E \mid g(y) \in A\}$, then K is a left ideal of A. For any $u \in E$, $g(u) = b^{-1}d$, $b, d \in A$, and $g(bu) = bg(u) = d$ implies that $bu \in K$. If f is the restriction of g to K, then $f\colon K \to A$ is a left A-homomorphism and for any $v \in E$, $v \notin K$, we have $g(v) \neq 0$ and there exists a non-zero-divisor s of A such that $sg(v) \in A$. Thus $0 \neq sv \in K$, which proves that ${}_AK$ is essential in ${}_AE$. Inasmuch as K is an essential left ideal of A, by hypothesis, there exists a non-zero-divisor c of A such that with $L = K+Ac$, a left A-homomorphism $h\colon L \to A$ extending f can be found. Now $QL = Q$ and if $\hat{h}\colon QL \to Q$ is defined by $\hat{h}(qw) = qh(w)$ for all $q \in Q$, $w \in L$, then the left Q-endomorphism $\hat{h}$ extends h. If $\hat{h}(1) = t \in Q$, for any $v \in E$, there exists a non-zero-divisor z of A such that $zv \in K$ and $zg(v) = g(zv) = f(zv) = h(zv) = \hat{h}(zv) = zv\hat{h}(1) = zvt$, whence $g(v) = vt$. This proves that ${}_AQ$ is injective and Q is therefore the maximal left quotient ring of A.

COROLLARY 2.4.1. *The following conditions are equivalent*:

(1) *A is semisimple Artinian*;

(2) *A satisfies condition* (∗) *with a classical left quotient ring Q such that every essential left ideal of Q is a projective left A-module.*

Remark 6. Let A be a reduced fully left idempotent left Goldie ring. Then any divisible left (or right) A-module is p-injective. Consequently, if A is a reduced fully left idempotent principal left ideal ring, then the three concepts of divisibility, p-injectivity and injectivity coincide for left A-modules. (But A need not be von Neumann regular.)

Let me conclude by noting that in homological terms, the following connections between von Neumann regularity, flatness and p-injectivity hold.

Remark 7. The following conditions are equivalent: (1) A is von Neumann regular; (2) For any cyclic singular left A-module C, $\mathrm{Tor}_1^A(A/I, C) = 0$ for every right ideal I of A; (3) For any cyclic singular left A-module C and any principal left ideal P of A, $\mathrm{Ext}_A^1(A/P, C) = 0$ and $\mathrm{Ext}_A^1(A/P, A) = 0$.

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