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Line method approximations to the initial-boundary value problem of Neumann type for parabolic differential-functional equations

Abstract. The non-linear differential-functional equation

(i) $$D_x z(x, y) = f(x, y, z(x, y), z, D_y z(x, y), D_{yy} z(x, y))$$

with initial boundary conditions of Neumann type is treated with the longitudinal method of lines. The corresponding line method has the form

(ii) $$D_x w^{(m)}(x) = \Phi_h(x, y^{(m)}, Aw^{(m)}(x), w, \Delta w^{(m)}(x), \Delta^{(2)} w^{(m)}(x))$$

where Δ and $\Delta^{(2)}$ are difference operators with respect to the spatial variable. We prove that if the method (ii) satisfies a consistency condition with respect to (i) and is stable then it is convergent. The proof of the convergence of the line method is based on differential inequalities.

I. Introduction. Let $\mathscr{F}(X, Y)$ denote the set of all functions defined on X with values in Y, where X, Y are arbitrary sets. Assume that X_0 and Y_0 are metric spaces. We denote by $C(X_0, Y_0)$ the class of continuous mappings from X_0 into Y_0.

Let $b = (b_1, \ldots, b_n) \in \mathbf{R}^n$, $b_i > 0$ for $i = 1, \ldots, n$ and $a, \tau_0 \in \mathbf{R}$, $a > 0$, $\tau_0 \geqslant 0$. We define $E = (0, a] \times (-b, b)$, $E^{(0)} = [-\tau_0, 0] \times [-b, b]$ and $\bar{E}$ is the closure of E. If $z\colon E^{(0)} \cup \bar{E} \to \mathbf{R}$ is a function of the variables $(x, y) = (x, y_1, \ldots, y_n)$ and the derivatives $D_{y_i} z$, $D_{y_i y_j} z$ exist for $i, j = 1, \ldots, n$ then we write $D_y z = (D_{y_1} z, \ldots, D_{y_n} z)$ and $D_{yy} z = [D_{y_i y_j} z]_{i,j=1}^n$. Let $\Omega = \bar{E} \times \mathbf{R} \times C(E^{(0)} \cup \bar{E}, \mathbf{R}) \times \mathbf{R}^n \times \mathbf{R}^{n^2}$ and $f\colon \Omega \to \mathbf{R}$, $\omega\colon E^{(0)} \to R$. Let us denote by $\partial E_j^{(-)}$ and $\partial E_j^{(+)}$, $j = 1, \ldots, n$, the sets

$$\partial E_j^{(-)} = \{(x, y) \in (0, a] \times [-b, b]\colon\ y_j = -b_j\},$$

$$\partial E_j^{(+)} = \{(x, y) \in (0, a] \times [-b, b]\colon\ y_j = b_j\}.$$

Suppose that $\varphi_j\colon \partial E_j^{(-)} \to \mathbf{R}$, $\psi_j\colon \partial E_j^{(+)} \to \mathbf{R}$ for $j = 1, \ldots, n$.

We consider the differential-functional problem

$$
(1)\quad
\begin{aligned}
&D_x z(x, y) = f(x, y, z(x, y), z, D_y z(x, y), D_{yy} z(x, y)), && (x, y) \in E,\\
&D_{y_j} z(x, y) = \varphi_j(x, y), && (x, y) \in \partial E_j^{(-)},\ j = 1, \ldots, n,\\
&D_{y_j} z(x, y) = \psi_j(x, y), && (x, y) \in \partial E_j^{(+)},\ j = 1, \ldots, n,\\
&z(x, y) = \omega(x, y), && (x, y) \in E^{(0)}.
\end{aligned}
$$

If $x \in [-\tau_0, a]$ then we set

$$H_x = \{(\xi, \eta) = (\xi, \eta_1, \ldots, \eta_n) \in E^{(0)} \cup \bar{E}:\ \xi \leqslant x\}.$$

For $z \in C(E^{(0)} \cup \bar{E}, \mathbf{R})$ we define $\|z\|_x = \max\{|z(\xi, \eta)|: (\xi, \eta) \in H_x\}$.

Assume that the differential-functional problem (1) is of Volterra type, i.e. if $x \in (0, a]$, $z, \bar{z} \in C(E^{(0)} \cup \bar{E}, \mathbf{R})$ and $z(\xi, \eta) = \bar{z}(\xi, \eta)$ for $(\xi, \eta) \in H_x$ then $f(x, y, p, z, q, r) = f(x, y, p, \bar{z}, q, r)$ for $y \in [-b, b]$, $p \in \mathbf{R}$, $q = (q_1, \ldots, q_n) \in \mathbf{R}^n$, $r = [r_{ij}]_{i,j=1}^n$, $r \in \mathbf{R}^{n^2}$.

The so-called longitudinal method of lines for parabolic equations consists in replacing spatial derivatives by difference operators. Then the initial boundary value problem for a parabolic equation is replaced by a sequence of initial-value problems for ordinary differential equations. Line methods for nonlinear parabolic differential or differential-functional equations with initial boundary conditions of Dirichlet type were considered in [3], [11], [15]. An error estimate implying the convergence of line methods is obtained in these papers by using differential inequalities. In [1], [10], [13], [15] the authors study the error due to the discretization in spatial variables of the Cauchy problem for parabolic equations. In [13], [15] the approximated solutions satisfy the growth-restricting condition $|u(x, y)| \leqslant \text{const}\, e^{B|y|}$, $y \in \mathbf{R}$. Similar results for the Cauchy problem under the assumption $|u(x, y)| \leqslant \text{const} \times e^{B|y|^{2-\delta}}$, $y \in \mathbf{R}$, $\delta > 0$, were proved in [1]. In [10] the solutions of the Cauchy problem are allowed to belong to a natural class of fast increasing functions. In [14], [15] the author has used the line method as a tool for proving existence theorems for the first boundary value problem and the Cauchy problem for a non-linear parabolic equation in two independent variables. [6] deals with the Cauchy problem for non-linear hyperbolic systems in two independent variables; the author studies convergence conditions and an existence theorem based on the line method.

The main problem in these investigations is to find a difference approximation which satisfies some consistency conditions with respect to the differential problem and which is stable. The stability problems for line methods were investigated by means of differential inequalities.

Finite difference approximation of the initial boundary problem of Neumann type is found in [2], [7]–[9].

For further bibliography, see [5], [12], [15], [16].

The paper is organized as follows. We introduce a general class of line methods for the problem (1). Using theorems on differential-functional inequalities, we establish sufficient conditions for the stability of the line method. In the next part of the paper we prove that if the line method is stable and satisfies a consistency condition with respect to (1) then it is convergent. We give examples of line methods which are convergent.

The following comparison result is needed in our discussion.

LEMMA 1. *Suppose that*

1° *the function* $F = (F_1, \ldots, F_{n_0})$: $[0, a] \times \mathbf{R}^{n_0} \times C([-\tau_0, a], \mathbf{R}^{n_0}) \to \mathbf{R}^{n_0}$ *of the variables* (x, ξ, η), $\xi = (\xi_1, \ldots, \xi_{n_0})$, *is non-decreasing with respect to the functional argument and satisfies the Volterra condition,*

2° *for each* i, $1 \leqslant i \leqslant n_0$, F_i *is non-decreasing in* ξ_j *for* $j = 1, \ldots, n_0, j \neq i$,

3° *there exists a function* $\sigma = (\sigma_1, \ldots, \sigma_{n_0}) \in C([0, a] \times \mathbf{R}_+^{n_0}, \mathbf{R}_+^{n_0})$, $\mathbf{R}_+ = [0, +\infty)$, *of the variables* (x, ξ) *such that*

(i) *for each* i, $1 \leqslant i \leqslant n_0$, σ_i *is non-decreasing in* ξ_j, *for* $j = 1, \ldots, n_0, j \neq i$,

(ii) *the maximum solution of the problem* $\eta'(x) = \sigma(x, \eta(x))$, $\eta(0) = 0$, *is* $\eta(x) = 0$ *for* $x \in [0, a]$,

4° *if* $x \in [0, a]$, $\eta, \bar{\eta} \in C([-\tau_0, a], \mathbf{R}^{n_0})$ *and* $\eta(t) \leqslant \bar{\eta}(t)$ *for* $t \in [-\tau_0, x]$ *then*

$$F(x, \eta(x), \eta) - F(x, \bar{\eta}(x), \bar{\eta}) \geqslant -\sigma(x, [\![\bar{\eta} - \eta]\!]_{[-\tau_0, x]}),$$

where

$$[\![\bar{\eta} - \eta]\!]_{[-\tau_0, x]} = (\max\{[\bar{\eta}_1(t) - \eta_1(t)]:\ t \in [-\tau_0, x]\}, \ldots,$$
$$\max\{[\bar{\eta}_{n_0}(t) - \eta_{n_0}(t)]:\ t \in [-\tau_0, x]\}),$$

5° *the functions* $\alpha, \beta \in C([-\tau_0, a], \mathbf{R}^{n_0})$ *satisfy the initial inequality* $\alpha(x) \leqslant \beta(x)$ *for* $x \in [-\tau_0, 0]$ *and the differential-functional inequality*

$$D_-\alpha(x) - F(x, \alpha(x), \alpha) \leqslant D_-\beta(x) - F(x, \beta(x), \beta), \qquad x \in (0, a]$$

(here $D_-\eta(x) = (D_-\eta_1(x), \ldots, D_-\eta_{n_0}(x))$ *is the left-hand lower Dini derivative of* η *at* x*).*

Under these assumptions we have $\alpha(x) \leqslant \beta(x)$ *for* $x \in [-\tau_0, a]$.

Lemma 1 can be proved by the method used in [4], [9], [15].

II. Discretization. For $y = (y_1, \ldots, y_n)$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$, $y, \bar{y} \in \mathbf{R}^n$, we define $y * \bar{y} = (y_1\bar{y}_1, \ldots, y_n\bar{y}_n)$. We shall be using vector inequalities, with the understanding that the same inequalities are satisfied between their corresponding components. Let $d = (d_1, \ldots, d_n) \in \mathbf{R}^n$ and $d_i > 0$ for $i = 1, \ldots, n$. Suppose that for an $h = (h_1, \ldots, h_n) \in (0, d]$ there exists $N = (N_1, \ldots, N_n)$ such that N_i, $i = 1, \ldots, n$, are natural numbers and $N * h = b$. Denote by $I_0 \subset (0, d]$ the set of all h having the above property. In the next part of the paper we adopt additional assumptions on I_0. For $h = (h_1, \ldots, h_n) \in I_0$ we define $\|h\| = \max_{1 \leqslant i \leqslant n} h_i$.

Let $m = (m_1, \ldots, m_n)$ where m_i, $i = 1, \ldots, n$, are integers and $\bar{J} = \{m: -N \leqslant m \leqslant N\}$, $J = \{m: -N+1 \leqslant m \leqslant N-1\}$ where $-N+1 = (-N_1+1, \ldots, -N_n+1)$, $N-1 = (N_1-1, \ldots, N_n-1)$. For $h \in I_0$ we write $y^{(m)} = (y_1^{(m_1)}, \ldots, y_n^{(m_n)}) = m*h$ where $m \in \bar{J}$. For $1 \leqslant j \leqslant n$ we define

$$j(m) = (m_1, \ldots, m_{j-1}, m_j+1, m_{j+1}, \ldots, m_n),$$

$$-j(m) = (m_1, \ldots, m_{j-1}, m_j-1, m_{j+1}, \ldots, m_n).$$

Let $j^0(m) = -j^0(m) = m$ and $j^{i+1}(m) = j(j^i(m))$, $-j^{i+1}(m) = -j(-j^i(m))$ for $i = 0, 1, 2, \ldots$ Set

$$E_h^{(0)} = \{(x, y^{(m)}): x \in [-\tau_0, 0], m \in \bar{J}\}, \quad E_h = \{(x, y^{(m)}): x \in [0, a], m \in \bar{J}\}.$$

For a function $w: E_h^{(0)} \cup E_h \to \mathbf{R}$ we write $w^{(m)}(x) = w(x, y^{(m)})$. Let $S = \{s = (s_1, \ldots, s_n): s_j \in \{-1, 0, 1\}$ for $j = 1, \ldots, n\}$ and $S' = S \setminus \{\theta\}$ where $\theta = (0, \ldots, 0) \in \mathbf{R}^n$. We define the following operators A, $\Delta = (\Delta_1, \ldots, \Delta_n)$, $\Delta^{(2)} = [\Delta_{ij}^{(2)}]_{i,j=1}^n$.

If $w: E_h^{(0)} \cup E_h \to \mathbf{R}$, $m \in J$ and $x \in [0, a]$ then

$$Aw^{(m)}(x) = \sum_{s \in S} a_s w^{(m+s)}(x), \tag{2}$$

$$\Delta_i w^{(m)}(x) = \frac{1}{h_i} \sum_{s \in S} b_s^{(i)} w^{(m+s)}(x), \quad i = 1, \ldots, n,$$

(3)

$$\Delta_{ij}^{(2)} w^{(m)}(x) = \frac{1}{h_i h_j} \sum_{s \in S} c_s^{(i,j)} w^{(m+s)}(x), \quad i, j = 1, \ldots, n,$$

where a_s, $b_s^{(i)}$, $c_s^{(i,j)} \in \mathbf{R}$.

We denote by $\mathscr{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$ the class of all functions w defined on $E_h^{(0)} \cup E_h$ taking values in $\mathbf{R}$ such that $w(\cdot, y^{(m)}) \in C([-\tau_0, a], \mathbf{R})$ for each $m \in \bar{J}$. We set $\tilde{E}_h = \{(x, y^{(m)}): x \in [0, a], m \in J\}$ and $\hat{w} = w|_{E_h^{(0)} \cup E_h}$ where $w \in \mathscr{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$. The class of all elements $\hat{w}$ with $w \in \mathscr{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$ is denoted by $\mathscr{F}_c(E_h^{(0)} \cup \tilde{E}_h, \mathbf{R})$ and $\Omega_h = \tilde{E}_h \times \mathbf{R} \times \mathscr{F}_c(E_h^{(0)} \cup \tilde{E}_h, \mathbf{R}) \times \mathbf{R}^n \times \mathbf{R}^{n^2}$.

Let $\Phi_h: \Omega_h \to \mathbf{R}$, $h \in I_0$. Put

$$J_j^{(-)} = \{m \in \bar{J}: m_j = -N_j, \text{ if } 1 \leqslant i < j \text{ then } m_i \neq -N_i \text{ and } m_i \neq N_i\},$$

$$J_j^{(+)} = \{m \in \bar{J}: m_j = N_j, \quad \text{if } 1 \leqslant i < j \text{ then } m_i \neq -N_i \text{ and } m_i \neq N_i\},$$

$$\partial E_{h,j}^{(-)} = \{(x, y^{(m)}): x \in [0, a], m \in J_j^{(-)}\},$$

$$\partial E_{h,j}^{(+)} = \{(x, y^{(m)}): x \in [0, a], m \in J_j^{(+)}\} \quad \text{where } j = 1, \ldots, n.$$

Let $\|y\| = \max_{1 \leqslant i \leqslant n} |y_i|$ denote the norm of $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$. If $\Lambda = [\lambda_{ij}]_{i,j=1}^n$ is an $n \times n$ matrix then we define

$$[\![\Lambda]\!] = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |\lambda_{ij}|.$$

Assume that for each $h \in I_0$ we have $\varphi_h = (\varphi_{h,1}, \ldots, \varphi_{h,n}): E_h \to \mathbf{R}^n$, $\psi_h = (\psi_{h,1}, \ldots, \psi_{h,n}): E_h \to \mathbf{R}^n$ and $\omega_h: E_h^{(0)} \to \mathbf{R}$. Suppose that $1 \leqslant k \leqslant \leqslant \min_{1 \leqslant i \leqslant n} N_i$ and $\alpha = [\alpha_{ij}]$ $\tilde{\alpha} = [\tilde{\alpha}_{ij}]$, $\beta = [\beta_{ij}]$, $\tilde{\beta} = [\tilde{\beta}_{ij}]$ where α_{ij}, β_{ij}: $[0, a] \to \mathbf{R}_+$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$ and $\tilde{\alpha}_{ij}, \tilde{\beta}_{ij}$: $[0, a] \to \mathbf{R}$ for $i = 1, \ldots, n$, $j = 0, 1, \ldots, k$.

For $w \in \mathscr{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$, $x \in [0, a]$ we define

$$A_i w^{(m)}(x) = \sum_{j=1}^{k} \alpha_{ij}(x) w^{(ij(m))}(x) + h_i \sum_{j=0}^{k} \tilde{\alpha}_{ij}(x) \varphi_{h,i}(x, y^{(ij(m))}), \tag{4}$$

$$m \in J_i^{(-)}, \; i = 1, \ldots, n, \quad \text{and}$$

$$B_i w^{(m)}(x) = \sum_{j=1}^{k} \beta_{ij}(x) w^{(-ij(m))}(x) + h_i \sum_{j=0}^{k} \tilde{\beta}_{ij}(x) \psi_{h,i}(x, y^{(-ij(m))}), \tag{5}$$

$$m \in J_i^{(+)}, \; i = 1, \ldots, n.$$

We consider the following line method for the problem (1):

$$\begin{aligned} & D_x w^{(m)}(x) = \Phi_h(x, y^{(m)}, Aw^{(m)}(x), \hat{w}, \Delta w^{(m)}(x), \Delta^{(2)} w^{(m)}(x)), \; x \in [0, a], \, m \in J, \\ & w^{(m)}(x) = \omega_h(x, y^{(m)}) \quad \text{for } x \in [-\tau_0, 0], \, m \in \bar{J}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} & w^{(m)}(x) = A_i w^{(m)}(x) \quad \text{for } x \in [0, a], \, m \in J_i^{(-)}, \, i = 1, \ldots, n, \\ & w^{(m)}(x) = B_i w^{(m)}(x) \quad \text{for } x \in [0, a], \, m \in J_i^{(+)}, \, i = 1, \ldots, n. \end{aligned} \tag{7}$$

If $x \in [-\tau_0, a]$ then we set $H_{h,x} = \{(\xi, y^{(m)}) \in E_h^{(0)} \cup \tilde{E}_h: \; \xi \leqslant x\}$. The function Φ_h is said to satisfy the *Volterra condition* if

$$\Phi_h(x, y, p, w, q, r) = \Phi_h(x, y, p, \bar{w}, q, r)$$

where $w, \bar{w} \in \mathscr{F}_c(E_h^{(0)} \cup \tilde{E}_h, \mathbf{R})$, $w|_{H_{h,x}} = \bar{w}|_{H_{h,x}}$, $(x, y, p, q, r) \in \tilde{E}_h \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n^2}$. For $w \in \mathscr{F}_c(E_h^{(0)} \cup \tilde{E}_h, \mathbf{R})$ we define $\|w\|_{h,x} = \max\{|w(\xi, y^{(m)})|: \; (\xi, y^{(m)}) \in H_{h,x}\}$.

Suppose that there exists a solution v of (1). We give sufficient conditions for the convergence of a sequence $\{w_h\}$, where w_h are solutions of (6), (7), to the solution v when the step size tends to zero.

III. Stability of the line method. The following assumptions will be needed throughout the paper.

ASSUMPTION H_1. *Suppose that*

1° *the function* Φ_h: $\Omega_h \to \mathbf{R}$ *of the variables* (x, y, p, w, q, r) *is non-de-*

creasing with respect to the functional argument and satisfies the Volterra condition,

2° *for each* $h \in I_0$, $m \in J$, *the function*

$$\Phi_h(\cdot, y^{(m)}, \cdot)\colon [0, a] \times \mathbf{R} \times \mathcal{F}_c(E_h^{(0)} \cup \tilde{E}_h, \mathbf{R}) \times \mathbf{R}^n \times \mathbf{R}^{n^2} \to \mathbf{R}$$

is continuous,

3° *the derivatives* $D_p\Phi_h$, $D_q\Phi_h = (D_{q_1}\Phi_h, \ldots, D_{q_n}\Phi_h)$, $D_r\Phi_h = [D_{r_{ij}}\Phi_h]_{i,j=1}^n$, $h \in I_0$, *exist on* Ω_h *and for each* $(x, y, w) \in \tilde{E}_h \times \mathcal{F}_c(E_h^{(0)} \cup \tilde{E}_h, \mathbf{R})$ *the functions* $D_p\Phi_h(x, y, \cdot, w, \cdot, \cdot)$, $D_q\Phi_h(x, y, \cdot, w, \cdot, \cdot)$, $D_r\Phi_h(x, y, \cdot, w, \cdot, \cdot)$ *are continuous on* $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n^2}$,

4° *for each* $h \in I_0$ *the matrix* $D_r\Phi_h$ *is symmetric on* Ω_h,

5° *for* $P = (x, y, p, w, q, r) \in \Omega_h$, $h \in I_0$, *and for* $s \in S'$ *we have*

$$a_s D_p\Phi_h(P) + \sum_{i=1}^{n} \frac{1}{h_i} b_s^{(i)} D_{q_i}\Phi_h(P) + \sum_{i,j=1}^{n} \frac{1}{h_i h_j} c_s^{(i,j)} D_{r_{ij}}\Phi_h(P) \geqslant 0, \tag{8}$$

6° *there exist constants* $L, L_0 \geqslant 0$ *such that* $|D_p\Phi_h(P)| \leqslant L$, $\|D_q\Phi_h(P)\| \leqslant L_0$, $[\![D_r\Phi_h(P)]\!] \leqslant L_0$, $P \in \Omega_h$, $h \in I_0$ *and*

$$|\Phi_h(x, y, p, w, q, r) - \Phi_h(x, y, p, w^*, q, r)| \leqslant L\|w^* - w\|_{h,x} \quad \text{on } \Omega_h,$$

7° *there exists* $c_0 > 0$ *such that* $h_i h_j^{-1} \leqslant c_0$, $i, j = 1, \ldots, n$, *for* $h \in I_0$.

Assumption H_2. *Suppose that the operators* A, Δ *and* $\Delta^{(2)}$ *satisfy the following conditions*:

1° *for each* $i, j = 1, \ldots, n$ *we have* $\sum_{s\in S} b_s^{(i)} = 0$, $\sum_{s\in S} c_s^{(i,j)} = 0$,

2° $a_s \geqslant 0$ *for* $s \in S$ *and* $\sum_{s\in S} a_s = 1$.

We define

$$\bar{b} = \max\Big[\max_{1\leqslant j\leqslant n} \sum_{s\in S} |b_s^{(j)}|,\ \max_{1\leqslant i,j\leqslant n} \sum_{s\in S} |c_s^{(i,j)}|\Big].$$

Assumption H_3. *Suppose that the operators* A_i, B_i, $i = 1, \ldots, n$, *satisfy the following conditions*:

1° $\varphi_h, \psi_h\colon E_h \to \mathbf{R}^n$,

2° $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{ik})\colon [0, a] \to \mathbf{R}_+^k$, $\beta_i = (\beta_{i1}, \ldots, \beta_{ik})\colon [0, a] \to \mathbf{R}_+^k$, $i = 1, \ldots, n$, *and* $\sum_{j=1}^k \alpha_{ij}(x) = 1$, $\sum_{j=1}^k \beta_{ij}(x) = 1$ *for* $x \in [0, a]$,

3° $\tilde{\alpha}_i = (\tilde{\alpha}_{i0}, \ldots, \tilde{\alpha}_{ik})\colon [0, a] \to \mathbf{R}^{k+1}$, $\tilde{\beta}_i = (\tilde{\beta}_{i0}, \ldots, \tilde{\beta}_{ik})\colon [0, a] \to \mathbf{R}^{k+1}$, *where* $i = 1, \ldots, n$.

Now we state a result on the stability of the method (6), (7).

Theorem 1. *Suppose that*

1° *Assumptions* H_1–H_3 *are satisfied,*

2° $w_h \in \mathcal{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$ *is a solution of* (6), (7), 3° $v_h \in \mathscr{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$ *and there exist functions* $\gamma, \tilde{\gamma}, \gamma_0 \colon I_0 \to \mathbf{R}_+$ *such that*

$$(9)\quad \begin{aligned} &\left|D_x v_h^{(m)}(x) - \Phi_h\big(x, y^{(m)}, Av_h^{(m)}(x), \hat{v}_h, \Delta v_h^{(m)}(x), \Delta^{(2)} v_h^{(m)}(x)\big)\right| \leqslant \gamma(h), \\ &\qquad\qquad\qquad\qquad\qquad\qquad m \in J,\ x \in [0, a], \\ &|v_h^{(m)}(x) - A_j v_h^{(m)}(x)| \leqslant h_j^2 \tilde{\gamma}(h), \quad m \in J_j^{(-)},\ j = 1, \ldots, n,\ x \in [0, a], \\ &|v_h^{(m)}(x) - B_j v_h^{(m)}(x)| \leqslant h_j^2 \tilde{\gamma}(h), \quad m \in J_j^{(+)},\ j = 1, \ldots, n,\ x \in [0, a], \end{aligned}$$

and

$$(10)\qquad |v_h(x, y^{(m)}) - \omega_h(x, y^{(m)})| \leqslant \gamma_0(h) \quad \textit{for } (x, y^{(m)}) \in E_h^{(0)}.$$

Then

$$(11)\qquad \|v_h - w_h\|_{h,x} \leqslant \eta_h(x), \qquad x \in [0, a],\ \textit{where}$$

$$(12)\quad \begin{aligned} \eta_h(x) &= \begin{cases} \gamma_0(h) & \textit{for } x \in [-\tau_0, 0],\ L > 0, \\ \left(\gamma_0(h) + \dfrac{\bar{\gamma}(h)}{2L}\right) e^{2Lx} - \dfrac{\bar{\gamma}(h)}{2L} & \textit{for } x \in [0, a],\ L > 0, \end{cases} \\ \eta_h(x) &= \begin{cases} \gamma_0(h) & \textit{for } x \in [-\tau_0, 0],\ L = 0, \\ \bar{\gamma}(h)x + \gamma_0(h) & \textit{for } x \in [0, a],\ L = 0, \end{cases} \end{aligned}$$

$$(13)\qquad \bar{\gamma}(h) = \gamma(h) + 2L\|h\|^2 \tilde{\gamma}(h) + L_0 c_0 \bar{b} n \tilde{\gamma}(h)(1 + \|h\|), \qquad h \in I_0.$$

Proof. It follows from Assumptions H_1 and H_3 that the solution w_h of (6), (7) exists on $[-\tau_0, a]$. We prove that

$$(14)\qquad w_h^{(m)}(x) \leqslant v_h^{(m)}(x) + \eta_h(x), \qquad x \in [0, a],\ m \in J.$$

To do this, we apply Lemma 1. Let

$$(15)\quad \begin{aligned} \tilde{v}_h(x, y^{(m)}) &= v_h(x, y^{(m)}) + \eta_h(x) \quad \text{for } x \in [0, a],\ m \in J, \\ \tilde{v}_h(x, y^{(m)}) &= v_h(x, y^{(m)}) + \eta_h(0) \quad \text{for } x \in [-\tau, 0],\ m \in \bar{J}, \end{aligned}$$

and

$$(16)\quad \begin{aligned} \tilde{v}_h(x, y^{(m)}) &= A_j \tilde{v}_h^{(m)}(x) \quad \text{for } x \in [0, a],\ m \in J_j^{(-)},\ j = 1, \ldots, n, \\ \tilde{v}_h(x, y^{(m)}) &= B_j \tilde{v}_h^{(m)}(x) \quad \text{for } x \in [0, a],\ m \in J_j^{(+)},\ j = 1, \ldots, n. \end{aligned}$$

Then we have $\tilde{v}_h \colon E_h^{(0)} \cup E_h \to \mathbf{R}$. We first show that

$$(17)\quad D_x \tilde{v}_h^{(m)}(x) \geqslant \Phi_h\big(x, y^{(m)}, A\tilde{v}_h^{(m)}(x), \bar{v}_h, \Delta \tilde{v}_h^{(m)}(x), \Delta^{(2)} \tilde{v}_h^{(m)}(x)\big),$$
$$x \in [0, a],\ m \in \bar{J},$$

where $\bar{v}_h = \tilde{v}_h|_{E_h^0 \cup \bar{E}_h}$. Write

$$(18)\quad \begin{aligned} R_{h,0}^{(m)}(x) &= D_x\tilde{v}_h^{(m)}(x) - \Phi_h\big(x, y^{(m)}, A\tilde{v}_h^{(m)}(x), \bar{v}_h, \Delta\tilde{v}_h^{(m)}(x), \Delta^{(2)}\tilde{v}_h^{(m)}(x)\big), \\ R_{h,1}^{(m)}(x) &= D_x v_h^{(m)}(x) - \Phi_h\big(x, y^{(m)}, Av_h^{(m)}(x), \hat{v}_h, \Delta v_h^{(m)}(x), \Delta^{(2)}v_h^{(m)}(x)\big), \\ R_{h,2}^{(m)}(x) &= \Phi_h\big(x, y^{(m)}, Av_h^{(m)}(x), \hat{v}_h, \Delta v_h^{(m)}(x), \Delta^{(2)}v_h^{(m)}(x)\big) \\ &\quad - \Phi_h\big(x, y^{(m)}, A\tilde{v}_h^{(m)}(x), \bar{v}_h, \Delta\tilde{v}_h^{(m)}(x), \Delta^{(2)}\tilde{v}_h^{(m)}(x)\big), \end{aligned}$$

where $x \in [0, a]$, $m \in J$. Then we have

$$(19)\qquad R_{h,0}^{(m)}(x) = R_{h,1}^{(m)}(x) + R_{h,2}^{(m)}(x) + \eta_h'(x), \qquad x \in [0, a],\ m \in J.$$

It follows from (9) that

$$(20)\qquad R_{h,1}^{(m)}(x) \geqslant -\gamma(h), \qquad m \in J,\ x \in [0, a].$$

Our next concern will be the estimate of $R_{h,2}^{(m)}(x)$ for $x \in [0, a]$, $m \in J$. Write

$$J^* = \{m = (m_1, \ldots, m_n):\ -N_j + 2 \leqslant m_j \leqslant N_j - 2 \text{ for } j = 1, \ldots, n\}.$$

We need only consider two cases.

(i) Suppose that $m \in J^*$. It follows from (15), (16) and from Assumption H_2 that $\Delta v_h^{(m)}(x) = \Delta\tilde{v}_h^{(m)}(x)$, $\Delta^{(2)}v_h^{(m)}(x) = \Delta^{(2)}\tilde{v}_h^{(m)}(x)$, $x \in [0, a]$, and

$$(21)\qquad \|v_h - \tilde{v}_h\|_{h,x} \leqslant \eta_h(x) + \|h\|^2\tilde{\gamma}(h), \qquad x \in [0, a],$$

and consequently

$$(22)\qquad \begin{aligned} R_{h,2}^{(m)}(x) &\geqslant -L[\eta_h(x) + \|\tilde{v}_h - v_h\|_{h,x}] \\ &\geqslant -2L\eta_h(x) - L\|h\|^2\bar{\gamma}(h), \qquad x \in [0, a],\ m \in J^*. \end{aligned}$$

(ii) Assume that $m \in J \backslash J^*$. We introduce the following notations:

$$S[m] = \{s \in S:\ m + s \in J\}, \qquad S_i^{(+)}[m] = \{s \in S:\ m + s \in J_i^{(+)}\},$$

$$S_i^{(-)}[m] = \{s \in S:\ m + s \in J_i^{(-)}\} \qquad \text{where } i = 1, \ldots, n.$$

Then we have

$$(23)\qquad \begin{aligned} |Av_h^{(m)}(x) - A\tilde{v}_h^{(m)}(x)| &\leqslant \eta_h(x) \sum_{s \in S[m]} a_s \\ &\quad + \sum_{i=1}^{n} \sum_{s \in S_i^{(-)}[m]} a_s[|A_i v_h^{(m+s)}(x) - A_i\tilde{v}_h^{(m+s)}(x)| + \|h\|^2\tilde{\gamma}(h)] \\ &\quad + \sum_{i=1}^{n} \sum_{s \in S_i^{(+)}[m]} a_s[|B_i v_h^{(m+s)}(x) - B_i\tilde{v}_h^{(m+s)}(x)| + \|h\|^2\tilde{\gamma}(h)] \\ &\leqslant \eta_h(x) + \|h\|^2\tilde{\gamma}(h), \qquad x \in [0, a]. \end{aligned}$$

In a similar way we obtain

$$(24)\quad \Delta_j v_h^{(m)}(x)-\Delta_j \tilde{v}_h^{(m)}(x)=\frac{1}{h_j}\sum_{i=1}^{n}\Big[\sum_{s\in S_i^{(-)}[m]} b_s^{(j)}\big(v_h^{(m+s)}(x)-A_i v_h^{(m+s)}(x)\big)+\sum_{s\in S_i^{(+)}[m]} b_s^{(j)}\big(v_h^{(m+s)}(x)-B_i v_h^{(m+s)}(x)\big)\Big].$$

Hence, by (9), (24) and by Assumption H_2 we get

$$(25)\quad |\Delta_j v_h^{(m)}(x)-\Delta_j \tilde{v}_h^{(m)}(x)|\leqslant c_0\bar{b}\|h\|\tilde{\gamma}(h),\quad j=1,\ldots,n,\ x\in[0,a].$$

Our next aim is to show that

$$(26)\quad |\Delta_{ij}^{(2)} v_h^{(m)}(x)-\Delta_{ij}^{(2)}\tilde{v}_h^{(m)}(x)|\leqslant c_0^2\bar{b}\tilde{\gamma}(h),\quad i,j=1,\ldots,n,\ x\in[0,a].$$

To show (26), we derive from Assumption H_2 the relation

$$\Delta_{ij}^{(2)} v_h^{(m)}(x)-\Delta_{ij}^{(2)}\tilde{v}_h^{(m)}(x)$$
$$=\frac{1}{h_i h_j}\sum_{j'=1}^{n}\Big[\sum_{s\in S_{j'}^{(-)}[m]} c_s^{(i,j)}\big(v_h^{(m+s)}(x)-A_{j'}v_h^{(m+s)}(x)\big)+\sum_{s\in S_{j'}^{(+)}[m]} c_s^{(i,j)}\big(v_h^{(m+s)}(x)-A_{j'}v_h^{(m+s)}(x)\Big],\quad i,j,\ldots,n,\ x\in[0,a].$$

Now, (26) follows from (9).

It follows from (21), (23)–(26) and from Assumption H_1 that

$$R_{h,2}^{(m)}(x)\geqslant -2L\eta_h(x)-2L\|h\|^2\tilde{\gamma}(h)-L_0c_0\bar{b}n\tilde{\gamma}(h)(1+\|h\|),\quad x\in[0,a],\ m\in J\backslash J^*.$$

Thus we see that

$$R_{h,1}^{(m)}(x)+R_{h,2}^{(m)}(x)+\eta_h'(x)\geqslant -\bar{\gamma}(h)-2L\eta_h(x)+\eta_h'(x),\qquad x\in[0,a],\ m\in J.$$

The above inequality and (12), (13), (18), (19) imply (17).

It follows from condition 1° of Assumption H_1 that the right-hand sides of (6), (7) are non-decreasing with respect to the functional argument and satisfy the Volterra condition. Now we prove that they possess the quasi-monotone property.

Suppose that $w^*\in\mathscr{F}_c(E_h^{(0)}\cup\tilde{E}_h,\mathbf{R})$ and $w,\bar{w}\in\mathscr{F}(E_h^{(0)}\cup E_h,\mathbf{R})$. Assume that $x\in[0,a]$, $m\in J$ are fixed and $w^{(m)}(x)=\bar{w}^{(m)}(x)$, $w^{(\bar{m})}(x)\leqslant\bar{w}^{(\bar{m})}(x)$ for $\bar{m}\in J$. Suppose that $m\in J^*$. By standard manipulations we find that

$$\Phi_h\big(x,y^{(m)},Aw^{(m)}(x),w^*,\Delta w^{(m)}(x),\Delta^{(2)}w^{(m)}(x)\big)$$
$$-\Phi_h\big(x,y^{(m)},A\bar{w}^{(m)}(x),w^*,\Delta\bar{w}^{(m)}(x),\Delta^{(2)}\bar{w}^{(m)}(x)\big)$$
$$=\sum_{s\in S'}\big(w^{(m+s)}(x)-\bar{w}^{(m+s)}(x)\big)\Bigg[a_sD_p\Phi_h(Q)+\sum_{i=1}^{n}\frac{1}{h_i}b_s^{(i)}D_{q_i}\Phi_h(Q)+\sum_{i,j=1}^{n}\frac{1}{h_ih_j}c_s^{(i,j)}D_{r_{ij}}\Phi_h(Q)\Bigg]\leqslant 0,$$

where Q is an intermediate point. Suppose that $m\in J\backslash J^*$. Write $S_0[m] = \{s\in S'\colon m+s\in J\}$ and $\tilde{w} = w-\bar{w}$. Then

$$\begin{aligned}
&\Phi_h\big(x, y^{(m)}, Aw^{(m)}(x), w^*, \Delta w^{(m)}(x), \Delta^{(2)}w^{(m)}(x)\big)\\
&\quad -\Phi_h\big(x, y^{(m)}, A\bar{w}^{(m)}(x), w^*, \Delta\bar{w}^{(m)}(x), \Delta^{(2)}\bar{w}^{(m)}(x)\big)\\
&\quad = \Big\{\sum_{s\in S_0[m]} \tilde{w}^{(m+s)}(x) + \sum_{j=1}^{n}\Big[\sum_{s\in S_j^{(-)}[m]}\sum_{i=1}^{k}\alpha_{ji}(x)\tilde{w}^{(j^i(m+s))}(x)\\
&\quad + \sum_{s\in S_j^{(+)}[m]}\sum_{i=1}^{k}\beta_{ji}\tilde{w}^{(-j^i(m))}(x)\Big]\Big\}\Big[a_s D_p\Phi_h(\tilde{Q}) + \sum_{i=1}^{n}\frac{1}{h_i}D_{q_i}\Phi_h(\tilde{Q})b_s^{(i)}\\
&\quad + \sum_{i,j=1}^{n}\frac{1}{h_ih_j}c_s^{(i,j)}D_{r_{ij}}\Phi_h(\tilde{Q})\Big] \leqslant 0,
\end{aligned}$$

where $\tilde{Q}$ is an intermediate point. This completes the proof of the quasi-monotone property of the right-hand sides of (6), (7).

Since w_h is a solution of (6), (7), using (10), (17) and Lemma 1 we have the estimate (14). In a similar way we obtain the inequality

$$v_h^{(m)}(x)-\eta_h(x) \leqslant w_h^{(m)}(x), \qquad x\in[0, a],\ m\in J. \tag{27}$$

Now, (11) follows from (14), (27), which completes the proof.

Remark 1. Theorem 1 enables us to get estimates of the difference between solutions of two problems of the form (6), (7). Suppose that

1° $\tilde{\Phi}_h\colon \Omega_h\to\mathbf{R}$, $\tilde{\omega}_h\colon E_h^{(0)}\to\mathbf{R}$ and $\tilde{A}$, $\tilde{\Delta}$, $\tilde{\Delta}^{(2)}$ are operators given by (2), (3) with $\tilde{a}_s$, $\tilde{b}_s^{(i)}$, $\tilde{c}_s^{(i,j)}$ instead of a_s, $b_s^{(i)}$, $c_s^{(i,j)}$,

2° $\tilde{\varphi}_h$, $\tilde{\psi}_h\colon E_h\to\mathbf{R}^n$, $\tilde{\varphi}_h = (\tilde{\varphi}_{h,1}, \ldots, \tilde{\varphi}_{h,n})$, $\tilde{\psi}_h = (\tilde{\psi}_{h,1}, \ldots, \tilde{\psi}_{h,n})$ and

$$\begin{aligned}
\alpha_i^* &= (\alpha_{i1}^*, \ldots, \alpha_{ik}^*)\colon [0, a]\to\mathbf{R}_+^k,\\
\beta_i^* &= (\beta_{i1}^*, \ldots, \beta_{ik}^*)\colon [0, a]\to\mathbf{R}_+^k,\\
\tilde{\alpha}_i^* &= (\tilde{\alpha}_{i0}^*, \ldots, \tilde{\alpha}_{ik}^*)\colon [0, a]\to\mathbf{R}^{k+1},\\
\tilde{\beta}_i^* &= (\tilde{\beta}_{i0}^*, \ldots, \tilde{\beta}_{ik}^*)\colon [0, a]\to\mathbf{R}^{k+1}, \qquad \text{where } i = 1, \ldots, n,
\end{aligned}$$

3° Assumptions H_1–H_3 are satisfied.

Consider the problem (6), (7) together with the following one:

$$(\tilde{6})\qquad \begin{aligned}
&D_x w^{(m)}(x) = \tilde{\Phi}_h\big(x, y^{(m)}, \tilde{A}w^{(m)}(x), \hat{w}, \tilde{\Delta}w^{(m)}(x), \tilde{\Delta}^{(2)}w^{(m)}(x)\big),\\
&\hspace{20em} x\in[0, a],\ m\in J,\\
&w^{(m)}(x) = \tilde{\omega}_h(x, y^{(m)}) \qquad \text{for } x\in[-\tau_0, a],\ m\in\bar{J},
\end{aligned}$$

where

$$(\tilde{7})\qquad \begin{aligned}
w^{(m)}(x) &= \tilde{A}_i w^{(m)}(x) \qquad \text{for } x\in[0, a],\ m\in J_i^{(-)},\ i = 1, \ldots, n,\\
w^{(m)}(x) &= \tilde{B}_i w^{(m)}(x) \qquad \text{for } x\in[0, a],\ m\in J_i^{(+)},\ i = 1, \ldots, n,
\end{aligned}$$

and $\tilde{A}_i$, $\tilde{B}_i$, $i = 1, \ldots, n$, are defined by (4), (5) with α_i^*, β_i^*, $\tilde{\alpha}_i^*$, $\tilde{\beta}_i^*$, $\tilde{\varphi}_h$, $\tilde{\psi}_h$ instead of α_i, β_i, $\tilde{\alpha}_i$, $\tilde{\beta}_i$, φ_h, ψ_h.

Denote by $w_h \in \mathcal{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$ the solution of (6), (7) and assume that there exists a solution $\tilde{w}_h \in \mathcal{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$ of $(\tilde{6})$, $(\tilde{7})$. Suppose that there exist $\gamma, \tilde{\gamma}, \gamma_0 \colon I_0 \to \mathbf{R}_+$ such that

$$\big|\Phi_h(x, y^{(m)}, A\tilde{w}_h^{(m)}(x), \tilde{w}_h, \Delta\tilde{w}_h^{(m)}(x), \Delta^{(2)}\tilde{w}_h^{(m)}(x)) \\ -\tilde{\Phi}_h(x, y^{(m)}, \tilde{A}\tilde{w}_h^{(m)}(x), \tilde{w}_h, \tilde{\Delta}\tilde{w}_h^{(m)}(x), \tilde{\Delta}^{(2)}\tilde{w}_h^{(m)}(x))\big| \leqslant \gamma(h), \quad x \in [0, a],\ m \in \bar{J},$$

and

$$|w_h^{(m)}(x) - \tilde{w}_h^{(m)}(x)| \leqslant \gamma_0(h), \qquad x \in [-\tau_0, 0],\ m \in J,$$
$$|A_j \tilde{w}_h^{(m)}(x) - \tilde{A}_j \tilde{w}_h^{(m)}(x)| \leqslant h_j^2 \tilde{\gamma}(h), \qquad m \in J_j^{(-)},\ j = 1, \ldots, n,\ x \in [0, a],$$
$$|B_j \tilde{w}_h^{(m)}(x) - \tilde{B}_j \tilde{w}_h^{(m)}(x)| \leqslant h_j^2 \tilde{\gamma}(h), \qquad m \in J_j^{(+)},\ j = 1, \ldots, n,\ x \in [0, a].$$

Then $\|w_h - \tilde{w}_h\|_{h,x} \leqslant \eta_h(x)$, $x \in [0, a]$, where η_h is given by (12), (13). This estimate follows from Theorem 1.

IV. The convergence of the line method. We prove that if the method (6), (7) is stable and satisfies the consistency condition with respect to (1) then it is convergent.

ASSUMPTION H_4. *Suppose that*

1° $f \in C(\Omega, \mathbf{R})$, $\omega \in C(E^{(0)}, \mathbf{R})$ *and* $\varphi_j \in C(\partial E_j^{(-)}, \mathbf{R})$, $\psi_j \in C(\partial E_j^{(+)}, \mathbf{R})$, $j = 1, \ldots, n$,

2° *there exists a solution* v *of* (1) *such that* $v|_{\bar{E}}$ *is of class* C^3,

3° *there exists* $\bar{\gamma} \colon I_0 \to \mathbf{R}_+$ *such that*

$$(28)\qquad \big|\Phi_h(x, y^{(m)}, Av_h^{(m)}(x), \hat{v}_h, \Delta v_h^{(m)}(x), \Delta^{(2)} v_h^{(m)}(x)) \\ - f(x, y^{(m)}, Av^{(m)}(x), v, \Delta v^{(m)}(x), \Delta^{(2)} v^{(m)}(x))\big| \leqslant \bar{\gamma}(h), \qquad x \in [0, a],\ m \in J,$$

where $v_h = v|_{E_h^{(0)} \cup E_h}$ *and*

$$(29)\qquad \lim_{\|h\| \to 0} \bar{\gamma}(h) = 0,$$

3° *there exist* $\tilde{\gamma}, \gamma_0 \colon I_0 \to \mathbf{R}_+$ *such that*

$$|\omega_h(x, y^{(m)}) - \omega(x, y^{(m)})| \leqslant \gamma_0(h), \qquad (x, y^{(m)}) \in E_h^{(0)},$$
$$|v_h^{(m)}(x) - A_j v_h^{(m)}(x)| \leqslant h_j^2 \tilde{\gamma}(h), \qquad m \in J_j^{(-)},\ j = 1, \ldots, n,\ x \in [0, a],$$
$$|v_h^{(m)}(x) - B_j v_h^{(m)}(x)| \leqslant h_j^2 \tilde{\gamma}(h), \qquad m \in J_j^{(+)},\ j = 1, \ldots, n,\ x \in [0, a],$$
$$\lim_{\|h\| \to 0} \gamma_0(h) = 0, \qquad \lim_{\|h\| \to 0} \tilde{\gamma}(h) = 0.$$

ASSUMPTION H_5. *Suppose that*

1° *for* $i, j = 1, \ldots, n$ *we have* $\sum_{s \in S} s_j b_s^{(i)} = \delta_{ij}$ *where* δ_{ij} *is the Kronecker symbol,* 2° *for* $i, j, i', j' = 1, \ldots, n$ *we have*

$$\sum_{s\in S} s_{j'} c_s^{(i,j)} = 0, \quad \sum_{s\in S} s_{i'} s_{j'} c_s^{(i,j)} = \delta_{i'i}\delta_{j'j}, \quad \text{for } i \neq j,$$

$$\sum_{s\in S} s_{i'} s_{j'} c_s^{(i,j)} = 2\delta_{ii'}\delta_{jj'}, \quad \text{for } i = j.$$

THEOREM 2. *Suppose that*

1° *Assumptions* H_1–H_5 *are satisfied,*
2° $w_h \in \mathscr{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$ *is a solution of* (6), (7).
Then $\lim_{\|h\|\to 0} \|\hat{w}_h - \hat{v}_h\|_{h,x} = 0$ *uniformly in* $x \in [0, a]$.

Proof. It follows from condition 2° of Assumption H_4 and from Assumptions H_2, H_5 that there exist constants c_0, c_1, c_2 such that

$$(30)\qquad \begin{aligned} &|Av^{(m)}(x) - v^{(m)}(x)| \leqslant c_0\|h\|, \\ &|\Delta_i v^{(m)}(x) - D_{y_i} v^{(m)}(x)| \leqslant c_1\|h\|, \quad i = 1, \ldots, n, \\ &|\Delta_{ij} v^{(m)}(x) - D_{y_i y_j} v^{(m)}(x)| \leqslant c_2\|h\|, \quad i, j = 1, \ldots, n, \end{aligned}$$

where $x \in [0, a]$, $m \in J$. We define

$$\begin{aligned} \tilde{R}_{h,0}^{(m)}(x) = {} & f\big(x, y^{(m)}, Av^{(m)}(x), v, \Delta v^{(m)}(x), \Delta^{(2)} v^{(m)}(x)\big) \\ & - \Phi_h\big(x, y^{(m)}, Av_h^{(m)}(x), \hat{v}_h, \Delta v_h^{(m)}(x), \Delta^{(2)} v_h^{(m)}(x)\big), \\ \tilde{R}_{h,1}^{(m)}(x) = {} & f\big(x, y^{(m)}, v^{(m)}(x), v, D_y v^{(m)}(x), D_{yy} v^{(m)}(x)\big) \\ & - f\big(x, y^{(m)}, Av^{(m)}(x), v, \Delta v^{(m)}(x), \Delta^{(2)} v^{(m)}(x)\big), \quad x \in [0, a], m \in J. \end{aligned}$$

It follows from (30) that there exists $\gamma^*\colon I_0 \to \mathbf{R}_+$ such that $|\tilde{R}_{h,1}^{(m)}(x)| \leqslant \gamma^*(h)$, $x \in [0, a]$, $m \in J$, and $\lim_{\|h\|\to 0} \gamma^*(h) = 0$. Define $\gamma(h) = \bar{\gamma}(h) + \gamma^*(h)$, $h \in I_0$. Then

$$\begin{aligned} &\big|D_x v_h^{(m)}(x) - \Phi_h\big(x, y^{(m)}, Av_h^{(m)}(x), \hat{v}_h, \Delta v_h^{(m)}(x), \Delta^{(2)} v_h^{(m)}(x)\big)\big| \\ &\qquad \leqslant |\tilde{R}_{h,0}^{(m)}(x)| + |\tilde{R}_{h,1}^{(m)}(x)| \leqslant \gamma(h), \quad x \in [0, a], m \in J, \end{aligned}$$

and $\lim_{\|h\|\to 0} \gamma(h) = 0$. It follows from Theorem 1 that $\|w_h - v_h\| \leqslant \eta_h(x)$, $x \in [0, a]$, $h \in I_0$, where η_h is given by (12), (13). Since $\lim_{\|h\|\to 0} \eta_h(x) = 0$ uniformly in $x \in [0, a]$, we have our assertion.

V. Examples of line methods.

EXAMPLE 1. Let $M = \{(i, j)\colon i, j = 1, \ldots, n, i \neq j\}$ and suppose that $M^{(-)}, M^{(+)} \subset M$ satisfy the following conditions:

(i) $M^{(-)} \cap M^{(+)} = \emptyset$, $M^{(-)} \cup M^{(+)} = M$,
(ii) if $(i, j) \in M^{(-)}$ then $(j, i) \in M^{(-)}$.

Assume that the operators A, $\Delta = (\Delta_1, \ldots, \Delta_n)$, $\Delta^{(2)} = [\Delta_{ij}^{(2)}]_{i,j=1}^n$ are defined by

$$(31)\qquad Aw^{(m)}(x) = w^{(m)}(x),$$

(32) $$\Delta_i w^{(m)}(x) = (2h_i)^{-1}[w^{(i(m))}(x) - w^{(-i(m))}(x)], \quad i = 1, \ldots, n,$$

and

$$\Delta_{ij}^{(2)} w^{(m)}(x) = (2h_i h_j)^{-1}[-w^{(i(m))}(x) - w^{(j(m))}(x) - w^{(-i(m))}(x) - w^{(-j(m))}(x) + 2w^{(m)}(x) + w^{(i(j(m)))}(x) + w^{(-i(-j(m)))}(x)], \quad (i, j) \in M^{(+)},$$

(33) $$\Delta_{ij}^{(2)} w^{(m)}(x) = (2h_i h_j)^{-1}[w^{(i(m))}(x) + w^{(j(m))}(x) + w^{(-i(m))}(x) + w^{(-j(m))}(x) - 2w^{(m)}(x) - w^{(i(-j(m)))}(x) - w^{(-i(j(m)))}(x)], \quad (i, j) \in M^{(-)},$$

$$\Delta_{ii}^{(2)} w^{(m)}(x) = (h_i)^{-2}[w^{(i(m))}(x) - 2w^{(m)}(x) + w^{(-i(m))}(x)], \quad i = 1, \ldots, n,$$

where $m \in J$, $x \in [0, a]$. Then Assumptions H_2 and H_5 hold.

The condition (8) for the operators A, Δ, $\Delta^{(2)}$ defined by (31)–(33) is equivalent to

(34) $$-\tfrac{1}{2}|D_{q_j}\Phi_h(P)| + \frac{1}{h_j} D_{r_{jj}}\Phi_h(P) - \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{1}{h_i} |D_{r_{ij}}\Phi_h(P)| \geqslant 0, \quad j = 1, \ldots, n, \ P \in \Omega_h,$$

and

(35) $$D_{r_{ij}}\Phi_h(P) \geqslant 0 \quad \text{for } (i, j) \in M^{(+)}, \ P \in \Omega_h,$$
$$D_{r_{ij}}\Phi_h(P) \leqslant 0 \quad \text{for } (i, j) \in M^{(-)}, \ P \in \Omega_h.$$

Example 2. Suppose that the operators Δ and $\Delta^{(2)}$ are given by (32) and (33) respectively. Assume that

$$A w^{(m)}(x) = \frac{1}{2n} \sum_{j=1}^{n} \left(w^{(j(m))}(x) + w^{(-j(m))}(x)\right), \quad m \in J, \ x \in [0, a].$$

If

$$\frac{1}{n} D_p \Phi_h(P) - \frac{1}{h_j} |D_{q_j}\Phi_h(P)| + \frac{2}{h_j^2} D_{r_{jj}}\Phi_h(P) - 2\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{1}{h_i h_j} |D_{r_{ij}}\Phi_h(P)| \geqslant 0, \quad j = 1, \ldots, n, \ P \in \Omega_h,$$

and (35) hold then (8) is satisfied.

Example 3. Suppose the operator T_h: $\mathscr{F}(E_h^{(0)} \cup E_h, \mathbf{R}) \to \mathscr{F} \times (E^{(0)} \cup \bar{E}, \mathbf{R})$ satisfies the following conditions:

(i) if $w \in \mathscr{F}_c(E_h^{(0)} \cup E_h, \mathbf{R})$ then $T_h w \in C(E^{(0)} \cup \bar{E}, \mathbf{R})$,

(ii) if $z \in C(E^{(0)} \cup E, \mathbf{R})$ and $z_h = z|_{E_h^{(0)} \cup E_h}$ then $\lim_{\|h\| \to 0}(T_h z_h - z) = 0$ uniformly on $E^{(0)} \cup \bar{E}$.

Suppose that $f \in C(\Omega, \mathbf{R})$ and

$$\Phi_h(x, y, p, \hat{w}, q, r) = f(x, y, p, T_h w, q, r) \quad \text{on } \Omega_h.$$

If the operators A, Δ, $\Delta^{(2)}$ satisfy Assumptions H_2 and H_5, then the consistency condition (28), (29) holds.

Remark 2. Theorems 1 and 2 can easily be extended to weakly coupled systems of parabolic differential-functional systems.

References

[1] L. I. Kamynin, *On the applicability of a finite-difference method for solution of the heat equation, II*, Izv. Akad. Nauk SSSR Ser. Mat. 17 (1953), 249–268 (in Russian).

[2] Z. Kamont and M. Kwapisz, *Difference methods for nonlinear parabolic differential-functional systems with initial boundary conditions of the Neumann type*, Comment. Math. Prace Mat. 28 (1989), 223–248.

[3] Z. Kamont and S. Zacharek, *The line method for parabolic differential-functional equations with initial boundary conditions of the Dirichlet type*, Atti Sem. Mat. Fis. Univ. Modena 35 (1987), 249–262.

[4] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Academic Press, New York and London 1969.

[5] O. A. Liskovets, *The method of lines*, Differentsial'nye Uravneniya 1 (1965), 1662–1678 (in Russian).

[6] S. Łojasiewicz, *Sur le problème de Cauchy pour les systèmes d'équations aux dérivées partielles du premier ordre dans le cas hyperbolique de deux variables indépendantes*, Ann. Polon. Math. 3 (1957), 87–117.

[7] M. Malec, *Sur une famille biparamétrique de schémas des différences finies pour un système d'équations paraboliques aux dérivées mixtes et avec des conditions aux limites du type de Neumann*, ibid. 32 (1976), 33–42.

[8] —, *Schéma explicite des différences finies pour un système d'équations non linéaires du type parabolique avec des conditions aux limites non linéaires*, ibid. 41 (1983), 185–192.

[9] J. Szarski, *Differential Inequalities*, PWN, Warszawa 1969.

[10] A. Voigt, *Line method approximations to the Cauchy problem for nonlinear parabolic differential equations*, Numer. Math. 23 (1974), 23–36.

[11] —, *The method of lines for nonlinear parabolic differential equations with mixed derivatives*, ibid. 32 (1979), 197–207.

[12] W. Walter, *Die Linienmethode bei nichtlinearen parabolischen Differentialgleichungen*, ibid. 12 (1968), 307–321.

[13] —, *Approximation für das Cauchy problem bei parabolischen Differentialgleichungen mit der Linienmethode*, Internat. Ser. Numer. Math. 10, Abstract Spaces and Approximation, 135–145, Basel, Birkhäuser 1969.

[14] —, *Existenzsätze im Großen für das Cauchyproblem bei nichtlinearen parabolischen Differentialgleichungen mit der Linienmethode*, Math. Ann. 183 (1969), 254–274.

[15] —, *Differential and Integral Inequalities*, Springer, Berlin–Heidelberg–New York 1970.

[16] —, *Existence of solutions of a nonlinear boundary value problem via the method of lines*, Nonlinear Anal. 2 (1978), 519–535.

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