ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXX (1991)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXX (1991)

DARIUSZ BUGAJEWSKI (Poznań)

On the structure of the L^{p_1,p_2}-solution sets of Volterra integral equations in Banach spaces

Abstract. It was proved by Szufla in [6] that the set of L^p-solutions of a Volterra integral equation in Banach spaces is a compact R_δ, i.e. it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. We prove a similar theorem for the set of L^{p_1,p_2}-solutions.

Let $D_1 = [0, d_1]$, $D_2 = [0, d_2]$ be compact intervals in $\mathbf{R}$, $D = D_1 \times D_2$, and let E, F be Banach spaces. For a pair $p = (p_1, p_2)$ of real numbers $p_1, p_2 > 1$ we denote by $L^p(D, E)$ the space of all strongly measurable functions $u\colon D \to E$ with

$$\|u\|_p = \Big(\int_{D_2} \Big(\int_{D_1} \|u(t_1, t_2)\|^{p_1} dt_1\Big)^{p_2/p_1} dt_2\Big)^{1/p_2} < \infty,$$

provided with the norm $\|u\|_p$.

Consider the integral equation

$$x(t) = f(t) + \int_{D(t)} K(t, s) g(s, x(s)) ds, \tag{1}$$

where $D(t) = \{\tau \in \mathbf{R}^2\colon 0 \leqslant \tau_1 \leqslant t_1, 0 \leqslant \tau_2 \leqslant t_2\}$ and $t = (t_1, t_2) \in D$.

We assume that

1° $p = (p_1, p_2)$, $q = (q_1, q_2)$, $p_1 \geqslant q_1 > 1$ and $p_2 \geqslant q_2 > 1$; let $r = (r_1, r_2)$ be such that $1/q_1 + 1/r_1 = 1$ and $1/q_2 + 1/r_2 = 1$, and let $m = (m_1, m_2)$ be such that $1/p_1 + 1/r_1 + 1/m_1 = 1$ and $1/p_2 + 1/r_2 + 1/m_2 = 1$ (if $p_i = q_i$ we put $m_i = \infty$).

2° $f \in L^p(D, E)$.

3° $(s, x) \to g(s, x)$ is a function from $D \times E$ into F such that

(i) g is strongly measurable in s and continuous in x;

(ii) $\|g(s, x)\| \leqslant a(s) + b\|x\|$ for $s \in D$ and $x \in E$, where $a \in L^q(D, \mathbf{R})$ and $b \geqslant 0$.

4° K is a strongly measurable function from $D \times D$ into the space $\mathscr{L}(F, E)$ of continuous linear mappings $F \to E$ such that $\|K(t, \cdot)\| \in L^r(D, \mathbf{R})$ for a.e. $t \in D$ and the function $t \to k(t) = \|K(t, \cdot)\|_r$ belongs to $L^p(D, \mathbf{R})$.

5° H is a nonnegative function belonging to $L^m(D, \mathbf{R})$ and $\alpha(g(s, X)) \leqslant \leqslant H(s)\alpha(X)$ for $s \in D$ and for each bounded subset X of E, where α denotes the Kuratowski measure of noncompactness.

The purpose of this paper is to prove the following Aronszajn-type

THEOREM. *If 1°–5° hold, then the set S of all solutions $x \in L^p(D, E)$ of (1) is a compact R_δ, i.e. S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.*

This result extends a similar theorem from [6] concerning the case when p is a real number > 1.

Proof. For simplicity put $L^p = L^p(D, E)$. Note that from 1° it follows that each $x \in L^p$ belongs to $L^q(D, E)$ and

$$\|x\|_q \leqslant \omega \|x\|_p, \tag{2}$$

where $\omega = d_1^{1/q_1 - 1/p_1} d_2^{1/q_2 - 1/p_2}$.

Consider now a mapping F defined by

$$F(x)(t) = \int_{D(t)} K(t, s)g(s, x(s))ds \qquad (x \in L^p,\ t \in D).$$

By 1°, 3°, 4°, (2) and the Hölder inequality, we get

$$\begin{aligned}\|F(x)(t)\| &\leqslant \int_{D(t)} \|K(t, s)\|\big(a(s) + b\|x(s)\|\big)ds \\ &\leqslant k(t)(\|a\|_q + b\|x\chi_{D(t)}\|_q) \leqslant k(t)(\|a\|_q + b\omega\|x\chi_{D(t)}\|_p)\end{aligned}$$

for $t \in D$ and $x \in L^p$, so that $F(x) \in L^p$ for $x \in L^p$.

We shall show that F is a continuous mapping $L^p \to L^p$. Let $x_n, x_0 \in L^p$ and $\lim_{n\to\infty} \|x_n - x_0\|_p = 0$. Suppose that $\|F(x_n) - F(x_0)\|_p$ does not converge to 0 as $n \to \infty$. Then there are $\varepsilon > 0$ and a subsequence (x_{n_j}) such that

$$\|F(x_{n_j}) - F(x_0)\|_p > \varepsilon \qquad \text{for } j = 1, 2, \ldots \tag{3}$$

and

$$\lim_{j\to\infty} x_{n_j}(t) = x_0(t) \qquad \text{for a.e. } t \in D.$$

By 3°(i) we have

$$\lim_{j\to\infty} \|g(t, x_{n_j}(t)) - g(t, x_0(t))\| = 0 \qquad \text{for a.e. } t \in D.$$

Moreover, as $\lim_{n\to\infty} \|x_n - x_0\|_p = 0$, the sequence (x_n) has equi-absolutely continuous norms in L^p. By 3°(ii) and (2) this implies that the sequence $(g(\cdot, x_n))$ has equi-absolutely continuous norms in $L^q(D, F)$. Thus

$$\lim_{j\to\infty} \|g(\cdot, x_{n_j}) - g(\cdot, x_0)\|_q = 0.$$

Since

$$\|F(x_{n_j})-F(x_0)\|_p \leqslant \|k\|_p \|g(\cdot, x_{n_j})-g(\cdot, x_0)\|_q,$$

this proves that $\lim_{j\to\infty} \|F(x_{n_j})-F(x_0)\|_p = 0$, which contradicts (3).

For any positive integer n and $x \in L^p$ put

$$F_n(x)(t) = \begin{cases} 0 & \text{if } t \in D(d^n), \\ \int\limits_{D(t-d^n)} K(t, s) g(s, x(s)) ds & \text{if } t \in D \backslash D(d^n), \end{cases}$$

where $d^n = (d_1/n, d_2/n)$. Arguing as for F, it can be shown that F_n is a continuous mapping $L^p \to L^p$ and

(4) $$\|F_n(x)(t)\| \leqslant k(t)(\|a\|_q + b\omega \|x\chi_{D(t)}\|_p) \quad \text{for } x \in L^p \text{ and } t \in D.$$

Moreover,

(5) $$\|F(x)(t) - F_n(x)(t)\| \leqslant k_n(t)(\|a\|_q + b\omega\|x\chi_{D(t)}\|_p)$$

for $x \in L^p$ and $t \in D$, where

$$k_n(t) = \begin{cases} k(t) & \text{if } t \in D(d^n), \\ \|K(t, \cdot)\chi_{D(t)\backslash D(t-d^n)}\|_r & \text{if } t \in D\backslash D(d^n). \end{cases}$$

Choose $\delta > 0$ such that $\|k\chi_A\|_p < 1/(2b\omega)$ for each measurable subset A of D with $\mu(A) < \delta$ (μ the Lebesgue measure), and choose points $a^0, a^1, \ldots, a^j \in D$ in such a way that $0 = a^0 \leqslant a^1 \leqslant \ldots \leqslant a^j = d$ and $\mu(D(a^{i+1})\backslash D(a^i)) < \delta$ for $i = 0, \ldots, j-1$. Let c be a positive number, $\varrho = (2^{j+1}-2)(c + \|f\|_p + \|a\|_q/(2b\omega))$, $U = \{x \in L^p\colon \|x\|_p \leqslant c\}$, and let $B = \{x \in L^p\colon \|x\|_p \leqslant \varrho\}$.

Put $G(x) = f + F(x)$ and $G_n(x) = f + F_n(x)$ for $x \in B$. Then G and G_n are continuous mappings of B into L^p and, by (5),

(6) $$\lim_{n\to\infty} \|G(x) - G_n(x)\|_p = 0 \quad \text{uniformly in } x \in B.$$

Now we shall prove that $I - G_n\colon B \to L^p$ is an into homeomorphism (I is the identity mapping).

It is easy to see that for any $x, y \in B$

$$x - G_n(x) = y - G_n(y) \Rightarrow x = y.$$

It is enough to prove the continuity of $(I-G_n)^{-1}$. Suppose that $x_i, x_0 \in B$ and

$$\lim_{i\to\infty} \|x_i - G_n(x_i) - x_0 + G_n(x_0)\|_p = 0.$$

Since

$G_n(x_i)(t) = G_n(x_0)(t) = f(t)$ for $t \in D(d^n)$, we have $\lim\limits_{i\to\infty} \|(x_i - x_0)\chi_{D(d^n)}\|_p = 0$.

Further

$$x_i(t)-x_0(t) = \big(x_i(t)-G_n(x_i)(t)-x_0(t)+G_n(x_0)(t)\big) + \big(F_n(x_i\chi_{D(d^n)})(t)-F_n(x_0\chi_{D(d^n)})(t)\big)$$

for $t\in D(2d^n)\backslash D(d^n)$ and $i = 1, 2, \ldots$ By the continuity of F_n this proves that

$$\lim_{i\to\infty} \|(x_i-x_0)\chi_{D(2d^n)\backslash D(d^n)}\|_p = 0$$

and, consequently, $\lim_{i\to\infty} \|(x_i-x_0)\chi_{D(2d^n)}\|_p = 0$. Arguing similarly we get $\lim_{i\to\infty} \|(x_i-x_0)\chi_{D(ld^n)}\|_p = 0$ for $l = 3, \ldots, n$, so that $\lim_{i\to\infty} \|x_i-x_0\|_p = 0$. This proves the continuity of $(I-G_n)^{-1}$.

Fix n. For a given $y\in U$ we define a sequence of functions x_i, $i = 1, \ldots, n$, by

$$x_1(t) = y(t)+f(t) \quad \text{for } t\in D(d^n),$$

$$\bar{x}_i(t) = \begin{cases} x_i(t) & \text{for } t\in D(id^n), \\ 0 & \text{for } t\in D\backslash D(id^n), \end{cases}$$

$$x_{i+1}(t) = \begin{cases} x_i(t) & \text{for } t\in D(id^n), \\ y(t)+f(t)+F_n(\bar{x}_i)(t) & \text{for } t\in D((i+1)d^n)\backslash D(id^n). \end{cases}$$

Then $x_n\in L^p$ and

$$x_n(t) = y(t)+f(t)+F_n(x_n)(t) \quad \text{for } t\in D.$$

In view of (4) we have

$$\|x_n(t)\| \leqslant \|y(t)\| + \|f(t)\| + k(t)(\|a\|_q+b\omega\|x_n\chi_{D(t)}\|_p) \quad \text{for } t\in D. \tag{7}$$

We shall show that

$$\|x_n\chi_{D(a^i)}\|_p \leqslant (2^{i+1}-2)\left(c+\|f\|_p+\frac{\|a\|_q}{2b\omega}\right) \quad (i = 1, 2, \ldots, j). \tag{8}$$

It follows from (7) that

$$\|x_n\chi_{D(a^1)}\|_p \leqslant \|y\|_p+\|f\|_p+\|k\chi_{D(a^1)}\|_p(\|a\|_q+b\omega\|x_n\chi_{D(a^1)}\|_p)$$
$$\leqslant c+\|f\|_p+\frac{\|a\|_q}{2b\omega}+\tfrac{1}{2}\|x_n\chi_{D(a^1)}\|_p,$$

i.e.

$$\|x_n\chi_{D(a^1)}\|_p \leqslant 2\left(c+\|f\|_p+\frac{\|a\|_q}{2b\omega}\right).$$

Suppose that (8) holds for some i, $1\leqslant i<j$. Then by (7)

$$\|x_n\chi_{D(a^{i+1})\backslash D(a^i)}\|_p \leqslant \|y\|_p+\|f\|_p+\|k\chi_{D(a^{i+1})\backslash D(a^i)}\|_p(\|a\|_q+b\omega\|x_n\chi_{D(a^{i+1})}\|_p)$$

$$\leqslant c+\|f\|_p+\frac{1}{2b\omega}(\|a\|_q+b\omega\|x_n\chi_{D(a^{i+1})\setminus D(a^i)}\|_p+b\omega\|x_n\chi_{D(a^i)}\|_p)$$

$$\leqslant c+\|f\|_p+\frac{\|a\|_q}{2b\omega}+\tfrac{1}{2}\|x_n\chi_{D(a^{i+1})\setminus D(a^i)}\|_p+(2^i-1)\left(c+\|f\|_p+\frac{\|a\|_q}{2b\omega}\right)$$

$$=2^i\left(c+\|f\|_p+\frac{\|a\|_q}{2b\omega}\right)+\tfrac{1}{2}\|x_n\chi_{D(a^{i+1})\setminus D(a^i)}\|_p,$$

so that

$$\|x_n\chi_{D(a^{i+1})\setminus D(a^i)}\|_p\leqslant 2^{i+1}\left(c+\|f\|_p+\frac{\|a\|_q}{2b\omega}\right).$$

Consequently,

$$\|x_n\chi_{D(a^{i+1})}\|_p\leqslant\|x_n\chi_{D(a^{i+1})\setminus D(a^i)}\|_p+\|x_n\chi_{D(a^i)}\|_p$$
$$\leqslant(2^{i+1}-2)\left(c+\|f\|_p+\frac{\|a\|_q}{2b\omega}\right).$$

This proves (8). From (8) it is clear that $x_n\in B$. This shows that

(10) $$U\subset(I-G_n)(B)\quad\text{for all } n.$$

Before passing to further considerations we shall quote two lemmas. For a given set V of functions from D into E we define a function v by $v(t)=\alpha(V(t))$ for $t\in D$, where $V(t)=\{x(t)\colon x\in V\}$.

LEMMA 1 (Heinz [2]). *Let V be a countable set of strongly measurable functions $D\to E$ such that there exists $\mu\in L^1(D,\mathbf{R})$ such that $\|x(t)\|\leqslant\mu(t)$ for $x\in V$ and $t\in D$. Then the corresponding function v is integrable and*

$$\alpha(\{\int_D x(t)dt\colon x\in V\})\leqslant 2\int_D v(t)dt.$$

Let α_1 denote the Kuratowski measure of noncompactness in $L^1(D,E)$.

LEMMA 2 (Szufla [5]). *Let V be a countable set of strongly measurable functions $D\to E$ such that*

(i) *there exists $\mu\in L^1(D,E)$ such that $\|x(t)\|\leqslant\mu(t)$ for $x\in V$ and $t\in D$;*

(ii) $$\lim_{h\to 0}\sup_{x\in V}\int_D\|x(t+h)-x(t)\|dt=0.$$

Then $\alpha_1(V)\leqslant 2\int_D v(t)dt$.

Now we shall prove that

(11) $(I-G)^{-1}(C)$ is compact for each compact subset C of L^p.

Let C be a given compact subset of L^p and let (u_n) be a sequence in $(I-G)^{-1}(C)$.

Since $u_n - G(u_n) \in C$, we can find a subsequence (u_{n_k}) and $z \in C$ such that

$$\lim_{k\to\infty} \|u_{n_k} - G(u_{n_k}) - z\|_p = 0.$$

By passing to a subsequence if necessary, we may assume that

$$\lim_{k\to\infty} \big(u_{n_k}(t) - G(u_{n_k})(t)\big) = z(t) \quad \text{for a.e. } t \in D.$$

Moreover, by the inequality

$$\|F(x)(t)\| \leqslant k(t)(\|a\|_q + b\omega\varrho) \quad \text{for } x \in B \text{ and } t \in D \tag{12}$$

and the Egorov and Lusin theorems, for each $\varepsilon > 0$ there exists a closed subset D_ε of D and $M_\varepsilon > 0$ such that $\mu(D \setminus D_\varepsilon) < \varepsilon$ and $\|u_{n_k}(t)\| \leqslant M_\varepsilon$ for all k and $t \in D_\varepsilon$. Let $V = \{u_{n_k}\colon k = 1, 2, \ldots\}$ and let $W = F(V)$. It is clear that $\alpha_1(V) = \alpha_1(W)$ and $\alpha\big(V(t)\big) = \alpha\big(W(t)\big)$ for a.e. $t \in D$.

Observe that for $x \in B$ and $t \in D$ we have

$$\|F(x)(t+h) - F(x)(t)\| \leqslant d(t, h),$$

where

$$d(t, h) = \begin{cases} k(t)(\|a\|_q + b\omega\varrho) & \text{if } t+h \notin D, \\ \|K(t+h, \cdot) - K(t, \cdot)\|_r(\|a\|_q + b\omega\varrho) & \text{if } t+h \in D. \end{cases}$$

From 4° it is clear that $\|K\|$ belongs to the mixed norm space $L^{r,1} = L^1[L^r]$ (see [3], pp. 401–402). Since $C(D \times D, \mathscr{L}(F, E))$ is dense in the space $L^{r,1}(\mathscr{L}(F, E))$ of all strongly measurable functions $u\colon D \times D \to \mathscr{L}(F, E)$ such that $\|u\| \in L^{r,1}$, we have

$$\lim_{h\to 0} \int_D \|K(t+h, \cdot) - K(t, \cdot)\|_r dt = 0.$$

Consequently,

$$\lim_{h\to 0} \sup_{x \in W} \int_D \|x(t+h) - x(t)\| dt = 0.$$

From the above and (12) it follows that the function $t \to v(t) = \alpha\big(V(t)\big)$ satisfies all assumptions of Lemmas 1 and 2. Hence v is integrable on D and

$$\alpha_1(V) = \alpha_1(W) \leqslant 2 \int_D v(t) dt. \tag{13}$$

Fix now $t \in D$ such that $k(t) < \infty$, and put $T = D(t) \cap D_\varepsilon$ and $P = D(t) \setminus D_\varepsilon$. Let $Z = \{K(t, \cdot) g(\cdot, u_{n_i})\colon i = 1, 2, \ldots\}$. As

$$\|K(t, s) g(s, u_{n_i}(s))\| \leqslant \|K(t, s)\| \big(a(s) + b\omega M_\varepsilon\big)$$

for $s \in T$ and $i = 1, 2, \ldots$, by 5° and Lemma 1 we get

$$\begin{aligned} \alpha\Big(\int_T Z(s) ds\Big) &\leqslant 2 \int_T \alpha\big(Z(s)\big) ds \leqslant 2 \int_T \|K(t, s)\| \alpha\big(g(s, V(s))\big) ds \\ &\leqslant 2 \int_T \|K(t, s)\| H(s) v(s) ds \leqslant 2 \int_{D(t)} \|K(t, s)\| H(s) v(s) ds. \end{aligned}$$

Moreover, by 3° and the Hölder inequality

$$\int_P \|K(t, s)\| \, \|g(s, u_{n_i}(s))\| \, ds \leqslant \|K(t, \cdot)\chi_P\|_r(\|a\|_q + b\omega\varrho)$$

for $i = 1, 2, \ldots$ Since

$$v(t) \leqslant \alpha\big(\int_T Z(s)ds\big) + \alpha\big(\int_P Z(s)ds\big),$$

we obtain

$$v(t) \leqslant 2 \int_{D(t)} \|K(t, s)\| H(s)v(s)ds + 2\|K(t, \cdot)\chi_P\|_r(\|a\|_q + b\omega\varrho).$$

As ε is arbitrary, this shows that

$$v(t) \leqslant 2 \int_{D(t)} \|K(t, s)\| H(s)v(s)ds \quad \text{for a.e. } t \in D. \tag{14}$$

Now we shall prove that $v(t) = 0$ for a.e. $t \in D$. Choose $\eta > 0$ such that $2\|H\|_m \|k\chi_A\|_p < 1$ for each measurable subset A of D with $\mu(A) < \eta$, and choose $b^0, b^1, \ldots, b^k \in D$ in such a way that $0 = b^0 \leqslant b^1 \leqslant \ldots \leqslant b^k = d$ and $\mu(D(b^{l+1}) \setminus D(b^l)) < \eta$ for $l = 0, \ldots, k-1$. From (14) it follows that

$$v(t) \leqslant 2k(t)\|H\|_m \|v\chi_{D(b^1)}\|_p \quad \text{for } t \in D(b^1),$$

so that

$$\|v\chi_{D(b^1)}\|_p \leqslant 2\|k\chi_{D(b^1)}\|_p \|H\|_m \|v\chi_{D(b^1)}\|_p.$$

This implies that $\|v\chi_{D(b^1)}\|_p = 0$, i.e. $v(t) = 0$ for a.e. $t \in D(b^1)$. Again from (14) we obtain

$$v(t) \leqslant 2k(t) \int_{D(t)\setminus D(b^1)} \|K(t, s)\| H(s)v(s)ds \quad \text{for } t \in D,$$

so that

$$v(t) \leqslant 2k(t)\|H\|_m \|v\chi_{D(b^2)\setminus D(b^1)}\|_p \quad \text{for } t \in D(b^2).$$

Hence

$$\|v\chi_{D(b^2)\setminus D(b^1)}\|_p \leqslant 2\|k\chi_{D(b^2)\setminus D(b^1)}\|_p \|H\|_m \|v\chi_{D(b^2)\setminus D(b^1)}\|_p,$$

which proves that $\|v\chi_{D(b^2)/D(b^1)}\|_p = 0$ and, consequently, $v(t) = 0$ for a.e. $t \in D(b^2)$. Arguing similarly we deduce that $v(t) = 0$ for a.e. $t \in D(b^l)$, $l = 3, \ldots, k$, i.e. $v(t) = 0$ for a.e. $t \in D$.

Therefore, by (13), $\alpha_1(W) = 0$, i.e. W is relatively compact in L^1. On the other hand, from (12) it follows that W has equi-absolutely continuous norms in L^p. Thus W is relatively compact in L^p. From the above it follows that V is relatively compact in L^p, which proves (11).

From (6), (10) and (11) we deduce that the mapping $I - G$ satisfies all assumptions of Theorem 7 of [1]. Therefore, the set $(I-G)^{-1}(0)$ is a compact R_δ. On the other hand, if $x \in S$, then analogously as for x_n in the proof of (10), it can be shown that $x \in B$. Consequently, $S = (I-G)^{-1}(0)$. This completes the proof.

The author wishes to express his thanks to Dr. S. Szufla for valuable suggestions.

References

[1] F. E. Browder and C. P. Gupta, *Topological degree and nonlinear mappings of analytical type in Banach spaces*, J. Math. Anal. Appl. 26 (1969), 390–402.
[2] H. P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. 7 (1983), 1351–1371.
[3] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Moskva 1977 (in Russian).
[4] B. N. Sadovskii, *Limit-compact and condensing operators*, Russian Math. Surveys 27 (1972), 85–155.
[5] S. Szufla, *Appendix to the paper „An existence theorem for the Urysohn integral equation in Banach spaces"*, Comment. Math. Univ. Carolin. 25 (1984), 763–764.
[6] —, *Appendix to the paper "Existence theorems for L^p-solutions of integral equation in Banach spaces"*, Publ. Inst. Math. 43 (57), 1988, 113–116.

INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY
J. MATEJKI 48/49, 60-769 POZNAŃ, POLAND