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Positive integrable solutions for nonlinear integral and differential inclusions of fractional-orders

Abstract. In this paper we study the global existence of positive integrable solution for the nonlinear integral inclusion of fractional order

$$x(t) \in p(t) + I^{\alpha} F_1(t,\ I^{\beta} f_2(t, x(\varphi(t)))),\quad t \in (0,1).$$

As an application the global existence of the solution for the initial-value problem of the arbitrary (fractional) orders differential inclusion

$$\frac{dx(t)}{dt} \in p(t) + I^{\alpha} F_1(t, D^{\gamma} x(t))),\quad a.e\ \ t > 0$$

will be studied.

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1. Introduction. The existence of integrable solution of the integral inclusion

$$x(t)\ \in\ g(t)\ +\ I^{\alpha}\ F(t, x(\varphi(t))\ a.e\ on\ (0,1),\ \alpha \in (0,1)$$

where F is lower semicontinuous from R into R and $F(.,.)$ is measurable, has been studied in [6].
Here we are concerned with the functional integral equation

$$(1)\qquad x(t) = p(t)\ +\ I^{\alpha}\ f_1(t, I^{\beta}\ f_2(t, x(\varphi(t)))\ \ a.e\ on\ (0,1),\ \alpha,\ \beta \in (0,1)$$

and we prove the existence of a positive integrable solution of (1).
As a generalization of our results we study the existence of positive integrable solution of the nonlinear functional integral inclusion

$$(2)\qquad x(t)\ \in\ p(t)\ +\ I^{\alpha}\ F_1(t, I^{\beta}\ f_2(t, x(\varphi(t)))\ a.e\ on\ (0,1),\ \alpha,\ \beta \in (0,1)$$

where the set-valued map $F(t,.)$ is lower semicontinuous from R^+ into R^+ and $F(.,.)$ is measurable.
Finally, we prove the existence of a nondecreasing solution $x \in L^1$ of the initial value problem of the differential inclusion of arbitrary (fractional) orders

$$\frac{dx(t)}{dt} \in p(t) + I^{\alpha} F_1(t, D^{\gamma} x(t))), \quad a.e \quad t > 0 \tag{3}$$

with the initial data

$$x(0) = x_{\circ}. \tag{4}$$

2. Preliminaries. Let $L^1 = L^1(I)$ be the class of Lebesgue integrable functions on the interval $I = [a,b]$, $0 \leqslant a < b < \infty$ and let $\Gamma(.)$ be the gamma function.

Definition 2.1 The fractional integral of the function $f(.) \in L^1(I)$. of order $\alpha \in R^+$ is defined by (cf. [7] , [8], [9] and [11])

$$I_a^{\alpha} \; f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; f(s) \; ds.$$

Definition 2.2 The (Caputo) fractional order derivative D^{α}, $\alpha \in (0,1]$ of the absolutely continuous function g is defined as (see [2], [8], [9] and [11])

$$D_a^{\alpha} \; g(t) = I_a^{1-\alpha} \; \frac{d}{dt} \; g(t) \quad , \qquad t \in [a,b].$$

The following two theorems will be needed in the proof of our main result.

Theorem 2.3 (Nonlinear Alternative of Leray-Shauder type [4]) *Let U be an open subset of a convex set D in a Banach space X. Assume $0 \in U$ and $T \in C(\bar{U}, D)$. Then either*

(A_1) *T has a fixed point in $\bar{U}$ or*

(A_1) *there exists $\gamma \in (0,1)$ and $x \in \partial U$ such that $x = \gamma T x$.*

Theorem 2.4 (Kolmogorov Compactness Criterion [5]) *Let $\Omega \subseteq L^p(0,1)$, $1 \leqslant p \leqslant \infty$. If*

(i) *Ω is bounded in $L^p(0,1)$ and*

(ii) *$x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^p(0,1)$, where*

$$x_h(t) = \frac{1}{h} \int_0^{t+h} \; x(s) \; ds.$$

3. Main results. In this section we present our main result by proving the existence of positive solution $x \in L^1$ for the functional integral equation (1).
To facilitate our discussion, let us first state the following assumption:

(i) $p \in L^1$.

(ii) $f_i : (0,1) \times R^+ \to R^+,\ i = 1,2$ satisfy Caratheodory condition i.e f_i are measurable in t for any $x \in R^+$ and continuous in x for almost all $t \in (0,1)$. There exists two function $a_i(.) \in L^1$ and two positive constant b_i such that

$$|f_i(t,x)| \leqslant\ a_i(t) +\ b_i|x|, \quad i = 1,2 \ \ \forall \ \ t\ \in (0,1) \ and \ x \in R.$$

(iii) $\phi : (0,1) \to (0,1)$ is absolutely continuous, and there exists a constant $M > 0$ such that $\varphi'(t) \geqslant M, \quad \forall\, t \in (0,1)$.

(iv) Assume that every solution $x(.) \in L^1$ to the equation

$$x(t) = \gamma\ (p(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\ f_2(\tau, x(\varphi(\tau)))d\tau)ds)$$

$t \in (0,1), \quad 0 < \beta < 1, \quad \gamma \in (0,1)$, satisfies $\|x\| \neq r$ ($r > 0$ is arbitrary but fixed).

Define the operator T
(5)

$$Tx(t) = p(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\ f_2(\tau, x(\varphi(\tau)))d\tau)ds, \quad t \in [0,1],$$

Now, we are in position to formulate and prove our main result.

THEOREM 3.1 *Let the assumptions (i)-(iv) are satisfied. Then equation (1) has at least one positive solution $x \in L_1$*

PROOF Let x be an arbitrary element in the open set $B_r = \{x :\ \|x\| < r, \quad r > 0\}$.

Then from assumptions (1) and (2) we have,

$$
\begin{aligned}
\|Tx\| &= \int_0^1 |(Tx)(t)|\ dt \leqslant \int_0^1 |p(t)|dt \\
&+ \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ |f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\ f_2(\tau, x(\varphi(\tau)))d\tau)|dsdt \\
&\leqslant \int_0^1 |p(t)|dt + \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ a_1(s)dsdt \\
&+ b_1 \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\ |f_2(\tau, x(\varphi(\tau)))|d\tau)dsdt \\
&\leqslant \|p\| + \int_0^1 |a_1(s)| \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dtds \\
&+ b_1 \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\ a_2(\tau)d\tau dsdt \\
&+ b_1 b_2 \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\ |x(\varphi(\tau))|d\tau dsdt \\
&\leqslant \|p\| + \int_0^1 |a_1(s)| \frac{t^\alpha}{\Gamma(\alpha+1)}ds + b_1 \int_0^1 I^{\alpha+\beta}\ a_2(t)dt \\
&+ b_1 b_2 \int_0^1 I^{\alpha+\beta}|x(\varphi(t))|dt \\
&\leqslant \|p\| + \int_0^1 |a_1(s)| \frac{t^\alpha}{\Gamma(\alpha+1)}ds + b_1 \int_0^1 \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|a_2(s)|dsdt \\
&+ b_1 b_2 \int_0^1 \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|x(\varphi(s))|dsdt \\
&\leqslant \|p\| + \frac{\|a_1\|}{\Gamma(\alpha+1)} + b_1 \int_0^1 |a_2(s)| \int_s^1 \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}dtds \\
&+ b_1 b_2 \int_0^1 |x(\varphi(s))| \int_s^1 \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}dtds \\
&\leqslant \|p\| + \frac{\|a_1\|}{\Gamma(\alpha+1)} + b_1 \int_0^1 |a_2(s)| \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}ds \\
&+ b_1 b_2 \int_0^1 |x(\varphi(s))| \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}ds \\
&\leqslant \|p\| + \frac{\|a_1\|}{\Gamma(\alpha+1)} + b_1 \frac{\|a_2\|}{\Gamma(\alpha+\beta+1)} + \frac{b_1 b_2}{M\Gamma(\alpha+\beta+1)} \int_0^1 |x(\varphi(s))|\varphi'(s)ds \\
&\leqslant \|p\| + \frac{\|a_1\|}{\Gamma(\alpha+1)} + \frac{b_1\|a_2\|}{\Gamma(\alpha+\beta+1)} + \frac{b_1 b_2}{M\Gamma(\alpha+\beta+1)} \int_{\varphi(0)}^{\varphi(1)} |x(\varphi(u))|du \\
&\leqslant \|p\| + \frac{\|a_1\|}{\Gamma(\alpha+1)} + \frac{b_1\|a_2\|}{\Gamma(\alpha+\beta+1)} + \frac{b_1 b_2}{M\Gamma(\alpha+\beta+1)} \int_0^1 |x(u)|du \\
&\leqslant \|p\| + \frac{\|a_1\|}{\Gamma(\alpha+1)} + \frac{b_1\|a_2\|}{\Gamma(\alpha+\beta+1)} + \frac{b_1 b_2\|x\|}{M\Gamma(\alpha+\beta+1)}\ .
\end{aligned}
$$

Hence the above inequality means that the operator T maps B_r into L^1.

Now, we will show that T is compact. To achieve this goal we will apply Theorem 2.4. So, let Ω be a subset of B_r. Then $T(\Omega)$ is bounded in L^1 i.e condition (i) of Theorem 2.4 is satisfied.
It remains to show that $(Tx)_h \to Tx$ in L^1 as $h \to 0$ uniformly with respect to $Tx \in \Omega$. We have the following :

$$\begin{aligned}
\|(Tx)_h - (Tx)\| &= \int_0^1 |(Tx)_h(t) - (Tx)(t)|dt \\
&= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Tx)_h(\tau)d\tau - (Tx)(t)|dt \\
&= \int_0^1 |\frac{1}{h}\int_t^{t+h} ((Tx)_h(\tau) - (Tx)(t))d\tau|dt \\
&\leqslant \int_0^1 (\frac{1}{h}\int_t^{t+h} |p(\tau) - p(t)|d\tau dt \\
&\quad + \int_0^1 \frac{1}{h}\int_t^{t+h} |I^\alpha f_1(\tau, I^\beta f_2(\tau, x(\varphi(\tau))) \\
&\qquad - I^\alpha f_1(t, I^\beta f_2(t, x(\varphi(t)))|d\tau dt.
\end{aligned}$$

Now $f_1, f_2 \in L^1$ and $I^\alpha f_1, I^\beta f_2 \in L^1$, then (cf. [12])

$$\frac{1}{h}\int_t^{t+h} |I^\alpha f_1(\tau, I^\beta f_2(\tau, x(\varphi(\tau))) - I^\alpha f_1(t, I^\beta f_2(t, x(\varphi(t)))|d\tau dt \to 0.$$

Moreover $p(.) \in L^1$. So, we have

$$\frac{1}{h}\int_t^{t+h} |p(\tau) - p(t)|d\tau \to 0$$

for a.e $t \in L^1$. Therefore, by Theorem 2.4 we have that $T(\Omega)$ is relatively compact, that is, T is compact operator.
Set $U = B_r$ and $D = X = L^1[0,1]$. Then in the view of assumption (4) condition (A_2) of Theorem 2.3 does not hold. Theorem 2.3, implies that T has a fixed point. This completes the proof. ∎

4. Functional integral inclusion. Consider now the integral inclusion (2), where $F_1 : (0,1) \times R^+ \to 2^{R^+}$ has nonempty closed convex values.
As an important consequence of the main result we can present the following:

Theorem 4.1 *Let the assumptions of Theorem 3.1 are satisfied and the multifunction F_1 satisfies the following assumptions:*

(1) *$F_1(t,x)$ are non empty, closed and convex for all $(t,x) \in [0,1] \times R^+$,*

(2) *$F_1(t,.)$ is lower semicontinuous from R^+ into R^+,*

(3) *$F_1(.,.)$ is measurable,*

(4) *There exists a function $a_1 \in L^1$ and a positive number b_1 such that*

$$|F_1(t,x)| \leqslant \; a_1(t) \; + \; b_1 \; |x| \qquad \forall \quad t \in (0,1),$$

Then there exists at least one positive solution $x \in L^1$ of the integral inclusion (2)

Proof By conditions (1)-(4) (see [1], [3], [6] [10]) we can find a selection function f_1 (Caratheodory function) $f_1 : (0,1) \times R^+ \to R^+$ such that $f_1(t,x) \in F_1(t,x)$ for all $(t,x) \in (0,1) \times R^+$, this function satisfies condition (ii) of Theorem 3.1.
Clearly all assumptions of Theorem 3.1 are hold, then there exists a positive solution $x \in L^1$ such that

$$x(t) - p(t) = I^\alpha f_1(t, \; I^\beta f_2(t, x(\varphi(t)))) \in I^\alpha F_1(t, \; I^\beta f_2(t, x(\varphi(t)))).$$

5. Differential inclusion. Consider now the initial value problem of the differential inclusion (3) with the initial data (4)

Theorem 5.1 *Let the assumption of Theorem 4.1 are satisfied, then the initial value problem* (3)-(4) *has at least one positive nondecreasing solution $x \in L^1$.*

Proof Let $y(t) = \frac{dx(t)}{dt}$, then equation (3) is transformed to the integral inclusion

$$y(t) \in p(t) + I^\alpha F_1(s, I^{1-\gamma} y(s)). \tag{6}$$

Let $\phi(t) = t$, $f_2(t,x) \; = \; x$ and $\beta \; = \; 1-\gamma$, then by Theorem 4.1 the integral inclusion (6) has at least one positive solution $y \in L^1$.
This implies that the existence of the absolutely continuous solution

$$x(t) = x_\circ + \int_0^t y(s)ds$$

of the initial-value problem (3)-(4), and this solution is positive and non decreasing. ∎

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