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On the comparison of the density type topologies generated by sequences and by functions

Abstract. In the paper we investigate density type topologies generated by functions f satisfying condition $\liminf_{x\to 0^+} \frac{f(x)}{x} > 0$, which are not generated by any sequence.

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Through the paper we shall use standard notation: $\mathbb{R}$ will be the set of real numbers, $\mathbb{N}$ the set of positive integers, $\mathcal{L}$ the family of Lebesgue measurable subsets of $\mathbb{R}$ and $|E|$ the Lebesgue measure of a measurable set E. By $\Phi_d(E)$ we shall denote the set of all Lebesgue density points of measurable set E (i.e. $\Phi_d(E) = \left\{x \in \mathbb{R}; \lim_{h\to 0^+} \frac{|(x-h;x+h)\cap E|}{2h} = 1\right\}$) and by $\mathcal{T}_d$ the density topology consists of measurable sets satisfying $E \subset \Phi_d(E)$. For any operators $\Phi, \Psi : \mathcal{L} \to \mathcal{L}$ we write $\Phi \subset \Psi$ if $\Phi(E) \subset \Psi(E)$ for every $E \in \mathcal{L}$. If $\Phi \subset \Psi$ and $\Phi \neq \Psi$ then we write $\Phi \subsetneq \Psi$.

We will consider two generalizations of Lebesgue density. First of them, called a density generated by a sequence, was introduced by J. Hejduk and M. Filipczak in [6]. For a convenience we will formulate definitions using decreasing sequences tending to zero, instead of nondecreasing sequences going to infinity.

Let $\widetilde{\mathcal{S}}$ be the family of all decreasing sequences tending to zero. We will denote sequences from $\widetilde{\mathcal{S}}$ by (a_n) or by $\langle a\rangle$. Let $\langle a\rangle \in \widetilde{\mathcal{S}}$, $E \in \mathcal{L}$ and $x \in \mathbb{R}$. We shall say that x is an $\langle a\rangle$*-density point* of E (a *right-hand* $\langle a\rangle$*-density point* of E) if

$$\lim_{n\to\infty} \frac{|E \cap [x-a_n; x+a_n]|}{2a_n} = 1 \ \Big(\lim_{n\to\infty} \frac{|E\cap [x; x+a_n]|}{a_n} = 1\Big).$$

By $\Phi_{\langle a\rangle}(E)$ $(\Phi^+_{\langle a\rangle}(E))$ we will denote the set of all $\langle a\rangle$-density (right-hand $\langle a\rangle$-density) points of E. In the same way one may define *left-hand* $\langle a\rangle$*-density point* of E and the set $\Phi^-_{\langle a\rangle}(E)$. Evidently, $\Phi_{\langle a\rangle}(E) = \Phi^+_{\langle a\rangle}(E) \cap \Phi^-_{\langle a\rangle}(E)$.

In [6] it was proved that $\Phi_{\langle a\rangle}$ is a lower density operator and the family

$$\mathcal{T}_{\langle a\rangle} = \left\{E \in \mathcal{L}; E \subset \Phi_{\langle a\rangle}(E)\right\}$$

is a topology containing the density topology $\mathcal{T}_d$. Moreover for $a_n = \frac{1}{n}$, $\Phi_{\langle a\rangle} = \Phi_d$ and $\mathcal{T}_{\langle a\rangle} = \mathcal{T}_d$.

For any pair of sequences $\langle a\rangle$ and $\langle b\rangle$ from $\widetilde{\mathcal{S}}$ we denote by $\langle a\rangle \cup \langle b\rangle$ the decreasing sequence consisting of all elements from $\langle a\rangle$ and $\langle b\rangle$. It is clear that $\langle a\rangle \cup \langle b\rangle \in \widetilde{\mathcal{S}}$ and that

Proposition 1 *If* $\Phi_{\langle a\rangle} \subset \Phi_{\langle b\rangle}$ *then* $\Phi_{\langle a\rangle\cup\langle b\rangle} = \Phi_{\langle a\rangle}$.

The second type of density we will observe are densities generated by functions. We denote by $\mathcal{A}$ the family of all functions $f : (0;\infty) \to (0;\infty)$ such that

(A1) $\lim\limits_{x\to 0^+} f(x) = 0$,

(A2) $\liminf\limits_{x\to 0^+} \frac{f(x)}{x} < \infty$,

(A3) f is nondecreasing.

Let $f \in \mathcal{A}$. We say that x is a *right-hand f-density point* of a measurable set E if

$$\lim_{h\to 0^+} \frac{|(x;x+h) \setminus E|}{f(h)} = 0.$$

By $\Phi_f^+(E)$ we denote the set of all right-hand f-density points of E. In the same way one may define *left-hand f-density points* of E and the set $\Phi_f^-(E)$. We say that x is an *f-density point* of E if it is a right and a left-hand f-density point of E. By $\Phi_f(E)$ we denote the set of all f-density points of E, i.e. $\Phi_f(E) = \Phi_f^+(E) \cap \Phi_f^-(E)$. For any $f \in \mathcal{A}$, the family

$$\mathcal{T}_f = \{E \in \mathcal{L}; E \subset \Phi_f(E)\}$$

forms a topology stronger than the natural topology on the real line (see [1, Th. 7] and [4, Th. 1]).

Proposition 2 ([4, Prop. 4]) *For each $f, g \in \mathcal{A}$, an inclusion $\mathcal{T}_f \subset \mathcal{T}_g$ holds if and only if $\Phi_f \subset \Phi_g$.*

In [2] and [3] it has been shown that properties of f-density operator Φ_f and f-density topology $\mathcal{T}_f$ are strictly depended on $\liminf\limits_{x\to 0^+} \frac{f(x)}{x}$. In our paper we are interested in topologies generated by functions $f \in \mathcal{A}$ for which $\liminf\limits_{x\to 0^+} \frac{f(x)}{x} > 0$. The family of all such functions we will denote by $\mathcal{A}^1$.

Topologies generated by functions from $\mathcal{A}^1$ and from $\mathcal{A} \setminus \mathcal{A}^1$ have quite different properties (for example they satisfy different separation axioms - see [3, Th. 5 and

Th. 7)]). On the other hand for any function f from $\mathcal{A}^1$ properties of $\mathcal{T}_f$ are similar to properties of topologies generated by sequences, so similar to properties of the density topology $\mathcal{T}_d$ (compare [4, Th. 3], [9], [7] and [8]). Moreover

PROPOSITION 3 ([4, TH. 5]) *For any sequence $\langle s\rangle \in \widetilde{\mathcal{S}}$, the function f defined by a formula*

$$f(x) = s_n \quad \text{for} \quad x \in (s_{n+1}; s_n]$$

belongs to $\mathcal{A}^1$ and $\Phi_{\langle s\rangle} = \Phi_f$.

COROLLARY 4 *For each $\langle a\rangle, \langle b\rangle \in \widetilde{\mathcal{S}}$, an inclusion $\mathcal{T}_{\langle a\rangle} \subset \mathcal{T}_{\langle b\rangle}$ holds if and only if $\Phi_{\langle a\rangle} \subset \Phi_{\langle b\rangle}$.*

In [4] we have constructed a function $f \in \mathcal{A}^1$ such that $\Phi_f \neq \Phi_{\langle s\rangle}$ for each $\langle s\rangle \in \widetilde{\mathcal{S}}$. We will remain the definition of that function, omitting the proof, because we will construct a similar one in Theorem 11.

EXAMPLE 5 ([4, TH. 6]) Let us define sequences

$$\begin{aligned}
\langle w\rangle &= (2,2,3,3,3,4,4,4,4,\ldots), \\
\langle r\rangle &= (1,2,1,2,3,1,2,3,4,\ldots), \\
a_0 &= 1,\ a_n = \frac{a_{n-1}}{w_n^2} \text{ for } n \geqslant 1, \\
b_n &= a_n w_n = \frac{a_{n-1}}{w_n} \text{ for } n \geqslant 1.
\end{aligned}$$

Evidently

$$\lim_{n\to\infty} \frac{a_{n-1}}{b_n} = \lim_{n\to\infty} \frac{b_n}{a_n} = \infty.$$

The function f defined by a formula

$$f(x) = \begin{cases} a_{n-1} & \text{for} \quad x \in (b_n; a_{n-1}], \\ b_n r_n & \text{for} \quad x \in (a_n; b_n]. \end{cases}$$

belongs to $\mathcal{A}^1$ and $\Phi_f \neq \Phi_{\langle s\rangle}$ for each $\langle s\rangle \in \widetilde{\mathcal{S}}$.

The function constructed above proves that the family $\left\{\mathcal{T}_f; f \in \mathcal{A}^1\right\}$ is bigger than $\left\{\mathcal{T}_{\langle s\rangle}; \langle s\rangle \in \widetilde{\mathcal{S}}\right\}$. We will show that there exist continuum topologies from $\left\{\mathcal{T}_f; f \in \mathcal{A}^1\right\} \setminus \left\{\mathcal{T}_{\langle s\rangle}; \langle s\rangle \in \widetilde{\mathcal{S}}\right\}$ and that for any pair of sequences $\langle a\rangle, \langle b\rangle \in \widetilde{\mathcal{S}}$ satisfying $\mathcal{T}_{\langle a\rangle} \subsetneq \mathcal{T}_{\langle b\rangle}$ there is a function $f \in \mathcal{A}^1$ such that $\mathcal{T}_{\langle a\rangle} \subsetneq \mathcal{T}_f \subsetneq \mathcal{T}_{\langle b\rangle}$ and $\mathcal{T}_f \neq \mathcal{T}_{\langle s\rangle}$ for $\langle s\rangle \in \widetilde{\mathcal{S}}$.

Set

$$f_\alpha(x) = f\left(\frac{x}{\alpha}\right) \quad \text{and} \quad s_\alpha(n) = \alpha s(n)$$

for $f \in \mathcal{A}^1$, $\langle s\rangle \in \widetilde{\mathcal{S}}$ and $\alpha > 0$. Obviously $f_\alpha \in \mathcal{A}^1$ and $\langle s_\alpha\rangle \in \widetilde{\mathcal{S}}$.

PROPOSITION 6 *If* $\Phi_f = \Phi_{\langle s\rangle}$ *then* $\Phi_{f_\alpha} = \Phi_{\langle s_\alpha\rangle}$.

PROOF It is sufficient to prove that for any measurable set E

$$0 \in \Phi^+_{f_\alpha}(E) \iff 0 \in \Phi^+_{\langle s_\alpha\rangle}(E).$$

If $0 \in \Phi^+_{f_\alpha}(E)$ then

$$\frac{\left|(0;x)\cap \frac{1}{\alpha}E'\right|}{f(x)} = \frac{\frac{1}{\alpha}\left|(0;\alpha x)\cap E'\right|}{f_\alpha(\alpha x)} \overset{x\to 0+}{\longrightarrow} 0,$$

and so $0 \in \Phi^+_f\left(\frac{1}{\alpha}E\right) = \Phi^+_{\langle s\rangle}\left(\frac{1}{\alpha}E\right)$. Hence

$$\frac{\left|(0;s_\alpha(n))\cap E'\right|}{s_\alpha(n)} = \frac{\left|(0;s(n))\cap \frac{1}{\alpha}E'\right|}{s(n)} \overset{n\to\infty}{\longrightarrow} 0,$$

which gives $0 \in \Phi^+_{\langle s_\alpha\rangle}(E)$.

Conversely, if $0 \in \Phi^+_{\langle s_\alpha\rangle}(E)$ then

$$\frac{\left|(0;s(n))\cap \frac{1}{\alpha}E'\right|}{s(n)} = \frac{\left|(0;s_\alpha(n))\cap E'\right|}{s_\alpha(n)} \overset{n\to\infty}{\longrightarrow} 0,$$

and consequently $0 \in \Phi^+_{\langle s\rangle}\left(\frac{1}{\alpha}E\right) = \Phi^+_f\left(\frac{1}{\alpha}E\right)$. Thus

$$\frac{\left|(0;x)\cap E'\right|}{f_\alpha(x)} = \frac{\alpha\left|\left(0;\frac{x}{\alpha}\right)\cap \frac{1}{\alpha}E'\right|}{f\left(\frac{x}{\alpha}\right)} \overset{x\to 0+}{\longrightarrow} 0,$$

which implies $0 \in \Phi^+_{f_\alpha}(E)$. ∎

COROLLARY 7 *If* $\Phi_f \notin \left\{\Phi_{\langle s\rangle}; \langle s\rangle \in \widetilde{\mathcal{S}}\right\}$ *then* $\Phi_{f_\alpha} \notin \left\{\Phi_{\langle s\rangle}; \langle s\rangle \in \widetilde{\mathcal{S}}\right\}$ *for any* $\alpha > 0$.

THEOREM 8 *If* f *is the function defined in Example 5 then* $\Phi_{f_\alpha} \subsetneq \Phi_{f_\beta}$ *for any* $\alpha > \beta > 1$.

PROOF Since f is nondecreasing, we have $f_\alpha \leqslant f_\beta$ and $\Phi_{f_\alpha} \subset \Phi_{f_\beta}$. Let (n_i) be an increasing sequence of positive integers such that $r_{n_i} = 1$ for every i. Define

$$E = \mathbb{R} \setminus \bigcup_{i=1}^{\infty} \left(\beta b_{n_i}; \beta b_{n_i}\sqrt{w_{n_i}}\right).$$

It is sufficient to show that

$$0 \in \Phi_{f_\beta}(E) \setminus \Phi_{f_\alpha}(E).$$

Without loss of generality, we can assume that for any n

$$\beta a_n < b_n \quad \text{and} \quad \beta b_n < \alpha b_n < \beta b_n \sqrt{w_n} < a_{n-1}.$$

If $x \in (\beta b_{n_i}; \beta a_{n_i-1}]$ for some i then

$$\frac{|(0;x) \cap E'|}{f_\beta(x)} \leqslant \frac{\beta b_{n_i} \sqrt{w_{n_i}}}{f\left(\frac{x}{\beta}\right)} = \frac{\beta b_{n_i} \sqrt{w_{n_i}}}{a_{n_i-1}} = \frac{\beta}{\sqrt{w_{n_i}}}. \tag{1}$$

If $x \in (\beta a_{n_i}; \beta b_{n_i}]$ for some i then

$$\frac{|(0;x) \cap E'|}{f_\beta(x)} < \frac{a_{n_i}}{f\left(\frac{x}{\beta}\right)} = \frac{a_{n_i}}{r_{n_i} b_{n_i}} = \frac{1}{w_{n_i}}. \tag{2}$$

If $x \in (\beta a_p; \beta b_{p-1}]$ and $n_i < p < n_{i+1}$ for some i then

$$\frac{|(0;x) \cap E'|}{f_\beta(x)} < \frac{a_p}{r_p b_p} \leqslant \frac{1}{w_p}. \tag{3}$$

From (1)-(3) we conclude that $0 \in \Phi_{f_\beta}(E)$.

On the other hand for $x_i = \alpha b_{n_i}$

$$\frac{|(0;x_i) \cap E'|}{f_\alpha(x_i)} \geqslant \frac{|(\beta b_{n_i}; \alpha b_{n_i})|}{f(b_{n_i})} = \frac{(\alpha - \beta) b_{n_i}}{r_{n_i} b_{n_i}} = \alpha - \beta > 0,$$

and consequently $0 \notin \Phi_{f_\alpha}(E)$. ∎

In [4, Th. 1] it has been proved that for each function $f \in \mathcal{A}$ there is a continuous function $g \in \mathcal{A}$ such that $\Phi_f = \Phi_g$. Thus there is at most continuum different f-density topologies. Combining this result with Corollary 4, Corollary 7 and Theorem 8 we obtain

Corollary 9 *The family* $\{\mathcal{T}_f; f \in \mathcal{A}^1\} \setminus \left\{\mathcal{T}_{\langle s \rangle}; \langle s \rangle \in \widetilde{\mathcal{S}}\right\}$ *has cardinality continuum.*

In the next theorem we will compare densities generated by sequences. Let us formulate a useful lemma from [5]. Fix $\langle a \rangle$ and $\langle b \rangle$ from $\widetilde{\mathcal{S}}$. There is an unique sequence (k_n) of positive integers such that

$$b_n \in (a_{k_n+1}; a_{k_n}]$$

for each n with $b_n < a_1$. From now on (k_n) will denote this unique sequence.

Lemma 10 *([5, Th. 7]). The following conditions are equivalent*

(a) $\Phi_{\langle a \rangle} \subset \Phi_{\langle b \rangle}$.

(b) For arbitrary increasing sequence (n_i) *of positive integers*

$$\liminf_{i\to\infty} \frac{a_{k_{n_i}}}{b_{n_i}} < \infty \textit{ or } \liminf_{i\to\infty} \frac{b_{n_i}}{a_{k_{n_i}+1}} = 1.$$

THEOREM 11 *Let* $\langle a\rangle, \langle b\rangle \in \widetilde{\mathcal{S}}$*. If* $\Phi_{\langle b\rangle} \subsetneq \Phi_{\langle a\rangle}$ *then there is* $f \in \mathcal{A}^1$ *such that* $\Phi_{\langle b\rangle} \subset \Phi_f \subset \Phi_{\langle a\rangle}$ *and* $\Phi_f \neq \Phi_{\langle s\rangle}$ *for* $\langle s\rangle \in \widetilde{\mathcal{S}}$*.*

PROOF By Proposition 1 we can asume that $\langle a\rangle$ is a subsequence of $\langle b\rangle$. Since $\Phi_{\langle a\rangle} \not\subseteq \Phi_{\langle b\rangle}$, there exists an increasing sequence (n_i) of positive integers such that

$$\lim_{i\to\infty} \frac{a_{k_{n_i}}}{b_{n_i}} = \infty \quad \text{and} \quad M = \liminf_{i\to\infty} \frac{b_{n_i}}{a_{k_{n_i}+1}} > 1.$$

Let us denote

$$\beta = \begin{cases} \sqrt[4]{M} & ; \quad M < \infty \\ 2 & ; \quad M = \infty \end{cases}.$$

Replacing (n_i) by a subsequence we can assume that (k_{n_i}) is increasing and for every i

$$\frac{a_{k_{n_i}}}{b_{n_i}} > i^2 \quad , \quad \frac{b_{n_i}}{a_{k_{n_i}+1}} > \beta^3 \quad \text{and} \quad \frac{b_{n_i}}{a_{k_{n_i}+1}} < \beta^5 \text{ if } M < \infty. \tag{4}$$

Let us define

$$\begin{aligned}
\langle c\rangle &= \langle a\rangle \cup (b_{n_i}), \\
\langle r\rangle &= (1,2,1,2,3,1,2,3,4,\ldots), \\
g_1(x) &= a_n \quad \text{for} \quad x \in (a_{n+1}; a_n], \\
g_2(x) &= c_n \quad \text{for} \quad x \in (c_{n+1}; c_n], \\
g_3(x) &= b_n \quad \text{for} \quad x \in (b_{n+1}; b_n], \\
f(x) &= \begin{cases} r_i b_{n_i} & \text{for} \quad x \in \left(a_{k_{n_i}+1}; b_{n_i}\right], \\ g_1(x) & \text{for} \quad x \notin \bigcup_{i=1}^{\infty} \left(a_{k_{n_i}+1}; b_{n_i}\right]. \end{cases}
\end{aligned}$$

Since $1 \leqslant r_i \leqslant i^2$, $f \in \mathcal{A}^1$ and $g_3 \leqslant g_2 \leqslant f \leqslant g_1$. Thus

$$\Phi_{\langle b\rangle} = \Phi_{g_3} \subset \Phi_{\langle c\rangle} = \Phi_{g_2} \subset \Phi_f \subset \Phi_{g_1} = \Phi_{\langle a\rangle}.$$

We will show that $\Phi_f \neq \Phi_{\langle s\rangle}$ for $\langle s\rangle \in \widetilde{\mathcal{S}}$. Suppose, contrary to our claim, that there is $\langle s\rangle \in \widetilde{\mathcal{S}}$ such that

$$\Phi_f = \Phi_{\langle s\rangle}. \tag{5}$$

Let us consider the sets

$$\begin{aligned}
T &= \left\{i \in \mathbb{N}; \exists_{m\in\mathbb{N}}\ s_m \in \left[\beta^3 a_{k_{n_i}+1}; i b_{n_i}\right]\right\}, \\
R &= \{r_i; i \in T\}.
\end{aligned}$$

We will prove that the set R is bounded. Suppose it is not true and so T is infinite. Let (t_i) be an increasing sequence consisting of all elements from T, i.e.

$$T = \{t_i; i \in \mathbb{N}\}.$$

Let m_i be a fixed positive integer such that

$$s_{m_i} \in \left[\beta^3 a_{k_{n_{t_i}}+1}; t_i b_{n_{t_i}}\right].$$

We will show that

$$s_{m_i} < 2b_{n_{t_i}} \tag{6}$$

for almost every i. Suppose on the contrary that there is an increasing sequence (p_i) of positive integers such $s_{m_{p_i}} \geqslant 2b_{n_{t_{p_i}}}$ for every i. Hence

$$\frac{a_{k_{n_{t_{p_i}}}}}{s_{m_{p_i}}} \geqslant \frac{t_{p_i}^2 b_{n_{t_{p_i}}}}{t_{p_i} b_{n_{t_{p_i}}}} = t_{p_i} \geqslant p_i \geqslant i$$

and so

$$\lim_{i\to\infty} \frac{a_{k_{n_{t_{p_i}}}}}{s_{m_{p_i}}} = \infty$$

and

$$\liminf_{i\to\infty} \frac{s_{m_{p_i}}}{b_{n_{t_{p_i}}}} \geqslant 2.$$

This, by Lemma 10, contradicts the assumption that $\Phi_{\langle c\rangle} \subset \Phi_{\langle s\rangle}$ and finishes the proof of (6).

Now we show that there are real numbers $\varepsilon > \delta > 0$ such that

$$a_{k_{n_{t_i}}+1} < \delta b_{n_{t_i}} < \varepsilon b_{n_{t_i}} < s_{m_i} \tag{7}$$

for almost every i. If $M = \liminf_{i\to\infty} \frac{b_{n_i}}{a_{k_{n_i}+1}} < \infty$ then by (4) it is sufficient to set $\varepsilon = \frac{1}{\beta^2}$ and $\delta = \frac{1}{\beta^3}$. If $M = \infty$, then obviously it is sufficient to show that for some positive ε inequality

$$\varepsilon b_{n_{t_i}} < s_{m_i}$$

holds for almost every i. Suppose on the contrary that there is an increasing sequence (p_i) of positive integers such $i s_{m_{p_i}} < b_{n_{t_{p_i}}}$ for every i. Hence

$$\lim_{i\to\infty} \frac{b_{n_{t_{p_i}}}}{s_{m_{p_i}}} = \infty.$$

and

$$\liminf_{i\to\infty} \frac{s_{m_{p_i}}}{a_{k_{n_{t_{p_i}}}+1}} \geqslant \beta^3 > 1.$$

which also contradicts the assumption that $\Phi_{\langle c\rangle} \subset \Phi_{\langle s\rangle}$ and ends the proof of (7).

By our assumptions, the set R is not bounded. Thus there exists an increasing sequence (p_i) of positive integers such that

$$\lim_{i\to\infty} r_{t_{p_i}} = \infty.$$

Put

$$A=\bigcup_{i=1}^{\infty}\left(\delta b_{n_{t_{p_i}}};\varepsilon b_{n_{t_{p_i}}}\right).$$

From (6) and (7) it follows that

$$\frac{\left|A\cap\left(0;s_{m_{p_i}}\right)\right|}{s_{m_{p_i}}}\geqslant\frac{\varepsilon b_{n_{t_{p_i}}}-\delta b_{n_{t_{p_i}}}}{2b_{n_{t_{p_i}}}}=\frac{\varepsilon-\delta}{2}>0$$

for almost every i, and so

$$0\notin\Phi^{+}_{\langle s\rangle}(A'). \tag{8}$$

We will show that $0\in\Phi^{+}_{f}(A')$. Let $x\in(a_j;a_{j-1}]$. If $j=k_{n_{t_{p_i}}}$ for some i then

$$\frac{|A\cap(0;x)|}{f(x)}\leqslant\frac{\varepsilon b_{n_{t_{p_i}}}}{r_{t_{p_i}}b_{n_{t_{p_i}}}}=\frac{\varepsilon}{r_{t_{p_i}}}. \tag{9}$$

On the other hand if $k_{m_{n_i}}<j<k_{m_{n_{i+1}}}$ for some i then

$$\frac{|A\cap(0;x)|}{f(x)}\leqslant\frac{\varepsilon b_{n_{t_{p_{i+1}}}}}{a_{j+1}}\leqslant\frac{\varepsilon b_{n_{t_{p_{i+1}}}}}{a_{k_{n_{t_{p_{i+1}}}}}}\leqslant\frac{\varepsilon}{t^2_{p_{i+1}}}\leqslant\frac{\varepsilon}{(i+1)^2}. \tag{10}$$

From (5), (9) and (10) it follows that $0\in\Phi^{+}_{f}(A')=\Phi^{+}_{\langle s\rangle}(A')$, contrary to (8). Hence we conclude that R is bounded.

Let L be a positive integer such that

$$L>\sup R,$$

(p_i) be an increasing sequence of positive integers for which

$$r_{p_i}=L$$

and

$$B=\bigcup_{i=1}^{\infty}\left(\frac{1}{\beta}b_{n_{p_i}};b_{n_{p_i}}\right).$$

Since

$$\frac{\left|B\cap\left(0;b_{n_{p_i}}\right)\right|}{f(b_{n_{p_i}})}\geqslant\frac{b_{n_{p_i}}-\frac{1}{\beta}b_{n_{p_i}}}{r_{p_i}b_{n_{p_i}}}=\frac{1-\frac{1}{\beta}}{L}>0,$$

$$0\notin\Phi^{+}_{f}(B'). \tag{11}$$

We will prove that $0\in\Phi^{+}_{\langle s\rangle}(B')$. By definition of (p_i), $p_i\notin T$ for every i and consequently

$$s_m\notin\left[\beta^3a_{k_{n_{p_i}}+1};p_ib_{n_{p_i}}\right]$$

for $m, i \in \mathbb{N}$. Let us consider any term s_m of $\langle s\rangle$. There is a positive integer j such that $s_m \in (a_j; a_{j-1}]$. If $j = k_{n_{p_i}}$ for some i then

$$s_m < \beta^3 a_{k_{n_{p_i}}+1} \quad \text{or} \quad s_m > p_i b_{n_{p_i}}.$$

In the first case we have

$$\frac{|B \cap (0; s_m)|}{s_m} \leqslant \frac{b_{n_{p_{i+1}}}}{a_{k_{n_{p_i}}+1}} \leqslant \frac{b_{n_{p_{i+1}}}}{a_{k_{n_{p_{i+1}}}}} \leqslant \frac{1}{p_{i+1}^2} \leqslant \frac{1}{(i+1)^2}, \tag{12}$$

and in the second

$$\frac{|B \cap (0; s_m)|}{s_m} \leqslant \frac{b_{n_{p_i}}}{p_i b_{n_{p_i}}} \leqslant \frac{1}{i}. \tag{13}$$

Moreover, if $k_{n_{p_i}} < j < k_{n_{p_{i+1}}}$, then

$$\frac{|B \cap (0; s_m)|}{s_m} \leqslant \frac{b_{n_{p_{i+1}}}}{a_{j+1}} \leqslant \frac{b_{n_{p_{i+1}}}}{a_{k_{n_{p_{i+1}}}}} \leqslant \frac{1}{p_{i+1}^2} \leqslant \frac{1}{(i+1)^2}. \tag{14}$$

From (5) and (12)-(14) it follows that $0 \in \Phi^+_{\langle s\rangle}(B') = \Phi^+_f(B')$, contrary to (11). ■

Corollary 12 *If $\mathcal{T}_{\langle b\rangle} \subsetneq \mathcal{T}_{\langle a\rangle}$ then there is $f \in \mathcal{A}^1$ such that $\mathcal{T}_{\langle b\rangle} \subset \mathcal{T}_f \subset \mathcal{T}_{\langle a\rangle}$ and the topology $\mathcal{T}_f$ is generated by no sequence.*

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