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Notes on binary trees of elements in $C(K)$ spaces with an application to a proof of a theorem of H. P. Rosenthal

Abstract. A Banach space X contains an isomorphic copy of $C([0,1])$, if it contains a binary tree (e_n) with the following properties (1) $e_n = e_{2n} + e_{2n+1}$ and (2) $c \max_{2^n \leqslant k < 2^{n+1}} |a_k| \leqslant \| \sum_{k=2^n}^{2^{n+1}-1} a_k e_k \| \leqslant C \max_{2^n \leqslant k < 2^{n+1}} |a_k|$ for some constants $0 < c \leqslant C$ and every n and any scalars $a_{2^n}, \dots, a_{2^{n+1}-1}$. We present a proof of the following generalization of a Rosenthal result: *if E is a closed subspace of a separable $C(K)$ space with separable annihilator and $S : E \to X$ is a continuous linear operator such that S^* has nonseparable range, then there exists a subspace Y of E isomorphic to $C([0,1])$ such that $S|_Y$ is an isomorphism*, based on the fact.

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Binary trees with some algebraic and geometric properties are sometimes used to describe properties of some special Banach spaces. For example we have the following

Proposition 0.1 *If $p \geqslant 1$ and (e_n) is a sequence in a Banach space X such that*

(1) $e_n = \frac{e_{2n}+e_{2n+1}}{2^p}$ for every n,

(2) there exist constants $0 < c \leqslant C$ such that for every n and scalars $a_{2^n}, \dots, a_{2^{n+1}-1}$

$$c\Big(\sum_{k=2^n}^{2^{n+1}-1} |a_k|^p\Big)^{\frac{1}{p}} \leqslant \Big\|\sum_{k=2^n}^{2^{n+1}-1} a_k e_k\Big\| \leqslant C\Big(\sum_{k=2^n}^{2^{n+1}-1} |a_k|^p\Big)^{\frac{1}{p}},$$

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then the subspace $\overline{\text{lin}}\{e_n : n \in \mathbb{N}\}$ of X is isomorphic to the Lebesgue space $L^p(\lambda)$, where λ is the Lebesgue measure on $[0,1]$. Moreover, if $c = C$, the subspace is isometric to $L^p(\lambda)$.

The proposition is an exercise in [3, p. 211]. A Banach space X contains a subspace isomorphic to the space $C([0,1])$, if there exists a binary tree of elements of X with properties (1) and (2) for the special case $p = 0$ of the fact above (see Proposition 1.1). The author uses the fact in [7] and [8] to study properties of vector valued monotonic functions and continuous linear operators on the space $D(0,1)$ of all bounded scalar (real or complex) functions on the interval $[0,1)$ that are right continuous at each point of $[0,1)$ with left-hand limit at each point of $(0,1]$. The aim of the paper is to show the following generalization of the Rosenthal theorem (see [10, Thm. 1], [11]) for subspaces of separable $C(K)$ spaces with small annihilators:

THEOREM 2.1 *Let K be a compact metric space. Let X be a Banach space. Let E be a closed subspace of $C(K)$ such that $E^{\perp} = \{x^* \in C(K)^* : x^*(E) = 0\}$ is separable. If $S : E \to X$ is a continuous linear operator such that $S^*(X^*)$ is a nonseparable subset of E^*, then there exists a subspace Y of E isomorphic to $C([0,1])$ such that $S|_Y$ is an isomorphism.*

If $E = C(K)$ above, we get the Rosenthal theorem. We should notice that Theorem 2.1 may be deduced from Proposition 3.6 of [2]. The Alspach result, in fact, does not concern operators, it works with nonseparable weak$*$ compact subsets of E^* and subspaces of E that are normed by them. We do not know whether has someone noted the result in the form above. We present a proof of the result based on quite simple and standard ideas. The reader may find another proof of the Rosenthal result in [10] and more information about the subject in [12]. The Rosenthal result was generelized by Lotz and Rosenthal in [6] for continuous linear operators on separable Banach lattices with weakly sequentially completes duals. The author find some problems with the proof of the Rosenthal theorem in [15]. The Rosenthal result is a straightforward consequence of Theorem 2 in [15]. But the proof of Theorem 2 in [15] applied in the part c) on page 178 the fact that the operator $T : C(X) \to C(Y)$ is a difference of two positive operators, i.e. $T = T^+ - T^-$, where X and Y are perfect, compact, metric spaces. Operators that are differences of two positive operators between Banach lattices form usually a small subspace of the space of all operators. It is so also for operators between $C(X)$ and $C(Y)$ (see [1, p. 10] for an example of a continuous linear operator between $C[-1,1]$ and $C[-1,1]$ that is not a difference of two positive operators).

The proof of Theorem 2.1 for $E = C(K)$ is a little bit easier then the one presented in the general case. The proof of the general case involves the Rudin-Carleson theorem (see Proposition 1.2) that may be replaced by the Tietze theorem if $E = C(K)$ (see Remark 2.2).

There are quite many natural examples of subspaces of separable $C(K)$ spaces with small annihilators. For example: the disk algebra $A(\mathbb{D})$ i.e. the closed linear hull of functions $\{z^n : n \in \mathbb{N} \cup \{0\}\}$ in $C(\mathbb{T})$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the closed linear hull of functions $\{z_1^n z_2^m : (n,m) \in \mathbb{Z}^2 \setminus \Pi\}$ in $C(\mathbb{T}^2)$ where Π is a sector in $\mathbb{R}^2$ whose aperture is less then π (it is a consequence of the Bochner theorem (see [5, p. 168])), any kernel of a continuous linear surjection from separable $C(K)$

space onto l_2 (see [14, p. 187] for examples of such operators). Moreover, if E is a closed subspace of a $C(K)$ space with small annihilator, then every closed subspace F of $C(K)$ such that $E \subset F$ has also small annihilator. The reader may find more information about subspaces of $C(K)$ with small annihilators in [9].

1. Preliminaries. Throughout the paper X will be a real or complex Banach space and K will be a compact Hausdorff space. The closed unit ball of X is denoted by B_X. For a given subset A of X its closed linear span is denoted by $\overline{\text{lin}}\, A$. The Banach space of all scalar (real or complex), continuous function on K equipped with the supremum norm is denoted by $C(K)$. We identify the dual space $C(K)^*$ with the Banach space $\text{ca}(K)$ of all scalar countably additive Radon measures on K equipped with the norm $\|x\| = |x|(K)$, where $|x|$ denotes the variation of $x \in \text{ca}(K)$. The subset of $\text{ca}(K)$ of all positive measures is denoted by $\text{ca}^+(K)$. For compact metric spaces the families of Radon and Borel measures coincide. We say that the measures μ and ν are singular and write $\mu \perp \nu$ if there exists Borel subsets A and B of K such that $A \cap B = \emptyset$, $|\mu|(B) = 0 = |\nu|(A)$ and $\mu(C \cap A) = \mu(C)$, $\nu(C \cap B) = \nu(C)$ for every Borel subset C of K. For a closed subspace E of $C(K)$ we denote by $E^\perp = \{\nu \in ca(K) : \langle f, \nu\rangle = \int_K f\, d\nu = 0$ for every $f \in E\}$ the annihilator of E. The subspace of $\text{ca}(K)$ of all absolutely continuous measures with respect to a given $\nu \in \text{ca}^+(K)$ is denoted by $L^1(\nu)$. There are more information about Borel and Radon measures in [13].

We say that x is a condensation point of a subset A of a topological space L, if every neighborhood of x contains uncountable many elements of A. In the sequel we will apply the fact that an uncountable subset A of a separable metric space contains a condensation point which is a member of A.

We start with the following result on copies of the space $C(\{-1,1\}^{\mathbb{N}})$ in Banach spaces. The reader may find its proof in [7, Fact 3]. According to the Milyutin theorem for any compact metric uncountable space K the Banach space $C(K)$ is isomorphic to $C(\{-1,1\}^{\mathbb{N}})$.

Proposition 1.1 *If (e_n) is a sequence in a Banach space X such that*

(1) $e_n = e_{2n} + e_{2n+1}$ for every n,

(2) there exist constants $0 < c \leqslant C$ such that for every n and scalars $a_{2^n}, \dots, a_{2^{n+1}-1}$

$$c \max_{2^n \leqslant k < 2^{n+1}} |a_k| \leqslant \Big\| \sum_{k=2^n}^{2^{n+1}-1} a_k e_k \Big\| \leqslant C \max_{2^n \leqslant k < 2^{n+1}} |a_k|,$$

then the subspace $\overline{\text{lin}}\{e_n : n \in \mathbb{N}\}$ of X is isomorphic to $C(\{-1,1\}^{\mathbb{N}})$. Moreover, if $c = C$, the subspace is isometric to $C(\{-1,1\}^{\mathbb{N}})$.

The idea of the proof of Theorem 2.1 is the following: we construct a sequence $(f_n) \subset E$ with the properties (1) and (2) such that also the sequence $S(f_n)$ has the properties with another constants where S is the operator from Theorem 2.1. But to do this we will need some more tools. First of them is the following version of the Rudin-Carleson theorem (see [5, p. 57-58]).

PROPOSITION 1.2 *Let K be a compact metric space. Let E be a closed subspace of $C(K)$. Let S be a closed subset of K such that $|\nu|(S) = 0$ for every $\nu \in E^\perp$. Then for every $f \in C(S)$ and a strictly positive $\varphi \in C(K)$ with $|f(s)| \leqslant \varphi(s)$ for every $s \in S$ there exists $g \in E$ such that $g|_S = f$ and $|g(t)| \leqslant \varphi(t)$ for every $t \in K$.*

The next tool is the following

PROPOSITION 1.3 *Let K be a compact metric space. Let E be a closed subspace of $C(K)$ such that $E^\perp$ is separable. Let $\lambda \in \mathrm{ca}^+(K)$ be a measure such that $E^\perp \subset L^1(\lambda)$. Let Γ_0 be the set of all countable ordinals. If L is a bounded, absolutely convex, nonseparable subset of $E^* = \mathrm{ca}(K)/E^\perp$ in the norm topology of E^*, then there exists $\varepsilon > 0$ such that for every $0 < \delta < \varepsilon$ there exists a subset $\{x_\gamma : \gamma \in \Gamma_0\}$ of $\mathrm{ca}(K)$ such that*

a) $x_\gamma + E^\perp \in L$ and $\|x_\gamma\| = \varepsilon$ for every $\gamma \in \Gamma_0$,

b) for every $\beta \in \Gamma_0$ there exists a Borel subset A_β of K such that

$$|x_\alpha|(A_\beta) = 0 \quad \text{for every } \alpha < \beta, \quad \|\chi_{A_\beta} x_\beta\| > \varepsilon - \delta, \quad \lambda(A_\beta) = 0.$$

The proof of the fact above is very similar to the proof of [10, Lemma 4], but for the sake of completeness we present it here.

PROOF Since L is a nonseparable subset of E^* in the norm topology of E^*, there exists $\varepsilon > 0$ and $\{y_\gamma : \gamma \in \Gamma_0\} \subset L$ such that

$$\|y_\alpha - y_\beta\| \geqslant 2\varepsilon$$

for every $\alpha, \beta \in \Gamma_0$, $\alpha \neq \beta$. Let $0 < \delta < \varepsilon$. Let $q : \mathrm{ca}(K) \to \mathrm{ca}(K)/L^1(\lambda)$ be the quotient map. For every $\gamma \in \Gamma_0$ there exists a $x_\gamma \in \tilde{L} = (\sup\{\|x\| : x \in L\} + 1)B_{\mathrm{ca}(K)} \cap q^{-1}(L)$ such that $q(x_\gamma) = y_\gamma$. Consequently,

$$\|x_\alpha - x_\beta\| \geqslant 2\varepsilon$$

for every $\alpha, \beta \in \Gamma_0$, $\alpha \neq \beta$. For every measure $\mu \in \mathrm{ca}^+(K)$, the space $L^1(\mu+\lambda)$ is a separable subspace of $\mathrm{ca}(K)$ (by the Lusin theorem (see [13, Thm. 2.23]) measures $\{f(\mu+\lambda) : f \in C(K)\}$ form a dense subset of $L^1(\mu+\lambda)$). For a given $x \in \mathrm{ca}(K)$ and $\nu \in \mathrm{ca}^+(K)$ we denote by $\frac{dx}{d\nu}\nu$ the Radon-Nikodym derivative of x with respect to ν. The set $\left\{\frac{dx_\gamma}{d(\mu+\lambda)}(\mu+\lambda) : \gamma \in \Gamma_0\right\}$ has a condensation point $\frac{dx_{\gamma_0}}{d(\mu+\lambda)}(\mu+\lambda) \in L^1(\mu+\lambda)$ for some $\gamma_0 \in \Gamma_0$. Hence there exists $\alpha \in \Gamma_0$ such that

$$\left\|\frac{d(x_\alpha - x_{\gamma_0})}{d(\mu+\lambda)}(\mu+\lambda)\right\| < \delta.$$

Then

$$\frac{2\varepsilon}{\|x_\alpha - x_{\gamma_0}\|}\frac{x_\alpha - x_{\gamma_0}}{2} \in \tilde{L}, \qquad \left\|\frac{2\varepsilon}{\|x_\alpha - x_{\gamma_0}\|}\frac{x_\alpha - x_{\gamma_0}}{2}\right\| = \varepsilon,$$

$$x_\alpha - x_{\gamma_0} - \frac{d(x_\alpha - x_{\gamma_0})}{d(\mu+\lambda)}(\mu+\lambda) \perp \mu, \quad x_\alpha - x_{\gamma_0} - \frac{d(x_\alpha - x_{\gamma_0})}{d(\mu+\lambda)}(\mu+\lambda) \perp \lambda,$$

$$\left\|\frac{\varepsilon}{\|x_\alpha - x_{\gamma_0}\|}\Big(x_\alpha - x_{\gamma_0} - \frac{d(x_\alpha - x_{\gamma_0})}{d(\mu+\lambda)}(\mu+\lambda)\Big)\right\| > \varepsilon - \delta.$$

Applying the transfinite induction, the fact above and the fact that every separable subset of $\mathrm{ca}(K)$ is contained in $L^1(\nu)$ for some $\nu \in \mathrm{ca}^+(K)$ we show that there exists a subset $\{x_\gamma : \gamma \in \Gamma_0\}$ of $\tilde{L}$ with properties a) and b). ■

Let (K, ρ) be a compact metric space. We denote by $\mathcal{K}(K)$ the space of all nonempty closed subsets of K equipped with the Hausdorff metric i.e.

$$d_H(A, B) = \max\big(\sup_{x\in A}\inf_{y\in B}\rho(x, y), \sup_{y\in B}\inf_{x\in A}\rho(x, y)\big).$$

It is well known that $(\mathcal{K}(K), d_H)$ is a compact metric space. The reader may find more information about $\mathcal{K}(K)$ in [4, Problems 2.7.20, 3.12.26 and 4.5.22]. In the sequel we will need the following fact.

Proposition 1.4 *Let (K, ρ) be a compact metric space. If U is an open subset of K, then $\{A \in \mathcal{K}(K) : A \subset U\}$ is an open subset of $\mathcal{K}(K)$.*

Proof For every closed subset B of U $\delta = \inf_{x\in K\setminus U}\inf_{y\in B}\rho(x, y) > 0$. Consequently, $\{A \in \mathcal{K}(K) : d_H(A, B) < \frac{\delta}{2}\} \subset \{A \in \mathcal{K}(K) : A \subset U\}$. ■

In the proof of Theorem 2.1 we will need the following technical result.

Proposition 1.5 *If (f_n) is a sequence in a Banach space X such that*

$$f_{2k+1} = f_k - f_{2k}$$

for every $k \in \mathbb{N}$, then

a)

$$f_{2k+1} = f_l - \sum_{j=1}^{\log_2 \frac{2k+2}{l+1}} f_{2^j l + 2^j - 2}$$

for every k, where $l \in 2\mathbb{N} \cup \{1\}$ is the smallest natural number such that $(l+1)2^j = 2k + 2$ for some $j \in \mathbb{N} \cup \{0\}$,

b) in the descriptions of all members of the set $\{f_{2^n}, \ldots, f_{2^{n+1}-1}\}$ for $n \in \mathbb{N}$ every function from the set $\{f_j : j = 2, 4, \ldots, 2^{n+1} - 2\}$ appears exactly two times: once with plus and once with minus and the function f_1 appears only once.

Proof a) The formula is clear for $k = 1$ and $k = 2, 3$. For $2^n \leqslant k < 2^{n+1}$ we apply the mathematical induction with step 2^n.

b) Since $\{f_2, f_3\} = \{f_2, f_1 - f_2\}$, the fact is clear for $n = 1$. Applying the mathematical induction and the fact that $\{f_{2k}, f_{2k+1}\} = \{f_{2k}, f_k - f_{2k}\}$ for every $2^n \leqslant k < 2^{n+1}$ we show the fact for each $n \in \mathbb{N}$. ■

2. Main results. Now we are prepared to show our main result.

THEOREM 2.1 *Let K be a compact metric space. Let X be a Banach space. Let E be a closed subspace of $C(K)$ such that $E^\perp = \{x^* \in C(K)^* : x^*(E) = 0\}$ is separable. If $S : E \to X$ is a continuous linear operator such that $S^*(X^*)$ is a nonseparable subset of E^*, then there exists a subspace Y of E isomorphic to $C([0,1])$ such that $S|_Y$ is an isomorphism.*

PROOF We use the mathematical inducton to construct a sequence $(f_n) \subset E$ with the following properties:

(1) $f_n = f_{2n} + f_{2n+1}$,

(2)
$$\frac{c}{\|S\|} \max_{2^n \leqslant k < 2^{n+1}} |a_k| \leqslant \Big\| \sum_{k=2^n}^{2^{n+1}-1} a_k f_k \Big\| \leqslant 4 \max_{2^n \leqslant k < 2^{n+1}} |a_k|,$$

(3)
$$c \max_{2^n \leqslant k < 2^{n+1}} |a_k| \leqslant \Big\| S\Big(\sum_{k=2^n}^{2^{n+1}-1} a_k f_k \Big) \Big\| \leqslant 4 \|S\| \max_{2^n \leqslant k < 2^{n+1}} |a_k|,$$

for every n and scalars $a_{2^n}, \ldots, a_{2^{n+1}-1}$ and some constant $c > 0$. First, let us note that if we find a sequence with property (1), then we only need to show the right-hand side inequality of (2) and the left-hand side inequality of (3). It is a straightforward consequence of the continuity of S;

$$c \max_{2^n \leqslant k < 2^{n+1}} |a_k| \leqslant \Big\| S\Big(\sum_{k=2^n}^{2^{n+1}-1} a_k f_k \Big) \Big\| \leqslant \|S\| \Big\| \sum_{k=2^n}^{2^{n+1}-1} a_k f_k \Big\| \leqslant 4 \|S\| \max_{2^n \leqslant k < 2^{n+1}} |a_k|.$$

Let $L = S^*(B_{X^*})$. It is clear that L is an absolutely convex weak$*$ compact subset of $\mathrm{ca}(K)/E^\perp$. We assume that L is a nonseparable subset of $\mathrm{ca}(K)/E^\perp$ in the norm topology of $\mathrm{ca}(K)/E^\perp$. Let $\lambda \in \mathrm{ca}^+(K)$ be a measure such that $E^\perp \subset L^1(\lambda)$. Let $q : \mathrm{ca}(K) \to \mathrm{ca}(K)/L^1(\lambda)$ be the quotient map. Let $\tilde{L} = q^{-1}(L) \cap \{\mu \in \mathrm{ca}(K) : \|\mu\| \leqslant 1 + \sup\{\|x\| : x \in L\}\}$. Let Γ_0 be the set of all countable ordinals. By Proposition 1.3 there exists $\varepsilon > 0$ such that for every $0 < \delta < \varepsilon$ there exists a subset $\{x_\gamma : \gamma \in \Gamma_0\}$ of $\tilde{L}$ with properties (1) and (2) of Proposition 1.3. Fix $0 < \delta < \min\{\frac{1}{4}, \frac{\varepsilon}{4}\}$. Then there exists a subset $\{x_\gamma : \gamma \in \Gamma_0\}$ of $\tilde{L}$ with the following properties

a) $q(x_\gamma) = x_\gamma + E^\perp \in L$ and $\|x_\gamma\| = \varepsilon$ for every $\gamma \in \Gamma_0$,

b) for every $\beta \in \Gamma_0$ there exists a Borel subset A_β of K such that

$$|x_\alpha|(A_\beta) = 0 \text{ for every } \alpha < \beta, \quad \|\chi_{A_\beta} x_\beta\| > \varepsilon - \frac{\delta}{4^2}, \quad \lambda(A_\beta) = 0.$$

For every $\gamma \in \Gamma_0$ we find a continuous function $g_\gamma \in C(K)$ such that $\|g_\gamma\| \leqslant 1$ and $\|g_\gamma x_\gamma - |x_\gamma|\| < \frac{\delta}{4^2}$ (equivalently, $|g_\gamma x_\gamma|(K) > \|x_\gamma\| - \frac{\delta}{4^2}$). It is a straightforward consequence of the Hahn decomposition theorem (see [13, Thm. 6.12]) and the Lusin theorem.

First step. For every $\gamma \in \Gamma_0$ there exists a compact subset B^1_γ of K such that $B_\gamma \subset A_\gamma$, and $|x_\gamma|(B^1_\gamma) > \varepsilon - \frac{\delta}{4^2}$. Since $\mathcal{K}(K) \times B_{C(K)}$ is a separable metric space, there exists a condensation point $(B^1_{\gamma_1}, g_{\gamma_1})$ of the set $\{(B^1_\gamma, g_\gamma) : \gamma \in \Gamma_0\}$ in $\mathcal{K}(K) \times B_{C(K)}$, where $\gamma_1 \in \Gamma_0$. Let U_1 be an open subset of K such that $B^1_{\gamma_1} \subset U_1$. By Proposition 1.2 there exists $f_1 \in E$ such that $f_1|_{B_{\gamma_1}} = g_{\gamma_1}|_{B_{\gamma_1}}$, $\|f_1\| \leqslant 1$ and $|f_1(t)| < \frac{\delta}{8}$ for every $t \in K \setminus U_1$. Let V_1 be an open subset of K such that $B^1_{\gamma_1} \subset V_1 \subset \overline{V_1} \subset U_1$ and $|f_1(t) - g_{\gamma_1}(t)| < \frac{\delta}{8}$ for every $t \in \overline{V_1}$. Let $\Gamma_1 = \{\gamma \in \Gamma_0 : \gamma > \gamma_1, B^1_\gamma \subset V_1, \|g_\gamma|_{\overline{V_1}} - g_{\gamma_1}|_{\overline{V_1}}\|_{C(\overline{V_1})} < \frac{\delta}{8}\}$. Applying Proposition 1.4 it is easy to see that the set Γ_1 is uncountable.

Second step. Let $\beta(\alpha) = \min\{\gamma \in \Gamma_1 : \gamma > \alpha\}$ for every $\alpha \in \Gamma_1$. Since $\Gamma_1 \subset \Gamma_0$, $\beta(\alpha)$ is well defined member of Γ_1. Since measures $\chi_{A_\alpha} x_\alpha = \chi_{A_\alpha \setminus A_{\beta(\alpha)}} x_\alpha$, $\chi_{A_{\beta(\alpha)}} x_{\beta(\alpha)}$ are singular, we find compact subsets $B^2_\alpha, B^3_{\beta(\alpha)}$ of V_1 such that

$$\begin{aligned} &B^2_\alpha \cap B^3_{\beta(\alpha)} = \emptyset, && B^2_\alpha \subset A_\alpha, && B^3_{\beta(\alpha)} \subset A_{\beta(\alpha)},\\ &|x_\alpha|(B^2_\alpha) > \varepsilon - \frac{\delta}{4^2}, && |x_{\beta(\alpha)}|(B^3_{\beta(\alpha)}) > \varepsilon - \frac{\delta}{4^2}, && |x_\alpha|(B^3_{\beta(\alpha)}) = 0. \end{aligned}$$

Since $\mathcal{K}(K)^2 \times B^2_{C(\overline{V_1})}$ is a separable metric space, there exists a condensation point $(B^2_{\gamma_2}, B^3_{\gamma_3}, g_{\gamma_2}|_{\overline{V_1}}, g_{\gamma_3}|_{\overline{V_1}})$ of the subset $\{(B^2_\alpha, B^3_{\beta(\alpha)}, g_\alpha|_{\overline{V_1}}, g_{\beta(\alpha)}|_{\overline{V_1}}) : \alpha \in \Gamma_1, \alpha > \gamma_1\}$ of $\mathcal{K}(K)^2 \times B^2_{C(\overline{V_1})}$ where $\gamma_2, \gamma_3 \in \Gamma_1$ and $\gamma_1 < \gamma_2 < \gamma_3 = \beta(\gamma_2)$. Let U_2, U_3 be open subsets of K such that $U_2 \cap U_3 = \emptyset$ and $B^j_{\gamma_j} \subset U_j \subset V_1$ for $j = 2, 3$. By Proposition 1.2 there exists $f_2 \in E$ such that $f_2|_{B^2_{\gamma_2}} = g_{\gamma_2}|_{B^2_{\gamma_2}}$, $f_2|_{B^3_{\gamma_3}} = 0$, $\|f_2\| \leqslant 1$ and $|f_2(t)| < \frac{\delta}{4^2 2}$ for every $t \in K \setminus U_2$. Define $f_3 = f_1 - f_2$. Then $|f_3(t)| \leqslant |f_1(t) - g_{\gamma_1}(t)| + |g_{\gamma_1}(t) - g_{\gamma_2}(t)| < \frac{\delta}{4}$ for every $t \in B^2_{\gamma_2}$, $|f_3(t) - g_{\gamma_3}(t)| \leqslant |f_1(t) - g_{\gamma_1}(t)| - |g_{\gamma_1}(t) - g_{\gamma_3}(t)| + |f_2(t)| < \frac{\delta}{4}$ for every $t \in B^3_{\gamma_3}$. Let V_2, V_3 be open subsets of K such that $B^j_{\gamma_j} \subset V_j \subset \overline{V_j} \subset U_j$ for $j = 2, 3$ and $|f_2(t) - g_{\gamma_2}(t)| < \frac{\delta}{4^2 2}$ for every $t \in \overline{V_2}$, $|f_2(t)| < \frac{\delta}{4^2 2}$ for every $t \in \overline{V_3}$, and $|f_3(t) - g_{\gamma_3}(t)| < \frac{\delta}{4} + \frac{\delta}{4^2 2}$ for every $t \in \overline{V_3}$, $|f_3(t)| < \frac{\delta}{4} + \frac{\delta}{4^2 2}$ for every $t \in \overline{V_2}$. Let $\Gamma_j = \{\gamma \in \Gamma_1 : \gamma > \gamma_j, B^j_\gamma \subset V_j, \|g_\gamma|_{\overline{V_j}} - g_{\gamma_j}|_{\overline{V_j}}\|_{C(\overline{V_j})} < \frac{\delta}{4^j 2}\}$ for $j = 2, 3$. According to Proposition 1.4 sets Γ_2, Γ_3 are uncountable.

Next steps. Suppose that we are able to construct sets $\Gamma_1, \dots, \Gamma_{2^n - 1}$ for some $n \in \mathbb{N}$. Applying consideration similar to the one in the second step for every $k = 2^{n-1}, \dots, 2^n - 1$, we are able to construct measures $x_{\gamma_{2k}}, x_{\gamma_{2k+1}} \in \tilde{L}$, $\gamma_{2k}, \gamma_{2k+1} \in \Gamma_k$, $\gamma_{2k-1} < \gamma_{2k} < \gamma_{2k+1}$, nonempty compact subsets $B^{2k}_{\gamma_{2k}}, B^{2k+1}_{\gamma_{2k+1}}$ of K, open subsets U_{2k}, U_{2k+1} of K such that $U_{2k} \cap U_{2k+1} = \emptyset$ and

$$B^j_{\gamma_j} \subset A_{\gamma_j}, \quad B^j_{\gamma_j} \subset U_j \subset \overline{U_j} \subset V_k, \quad |x_{\gamma_j}|(B^j_{\gamma_j}) > \varepsilon - \frac{\delta}{4^2}$$

for $j = 2k, 2k+1$. Then $|g_{\gamma_k}(t) - g_{\gamma_j}(t)| < \frac{\delta}{4^k 2}$ for every $t \in \overline{V_k}$ where $j = 2k, 2k+1$. By Proposition 1.2 for every $k = 2^{n-1}, \dots, 2^n - 1$ there exists $f_{2k} \in E$ such that $f_{2k}|_{B^{2k}_{\gamma_{2k}}} = g_{\gamma_{2k}}|_{B^{2k}_{\gamma_{2k}}}$, $f_{2k}|_{B^{2k+1}_{\gamma_{2k+1}}} = 0$, $\|f_{2k}\| \leqslant 1$ and $|f_{2k}(t)| < \frac{\delta}{4^{2k} 2}$ for every $t \in$

$K \setminus U_{2k}$. Define $f_{2k+1} = f_k - f_{2k}$. According to Proposition 1.5

$$f_{2k+1} = f_l - \sum_{j=1}^{\log_2 \frac{2k+2}{l+1}} f_{2^j l + 2^j - 2}$$

where $l \in 2\mathbb{N} \cup \{1\}$ is the smallest natural number such that $(l+1)2^j = 2k+2$ for some $j \in \mathbb{N}$. It is clear that if $r, s \in \mathbb{N}$ and $r < s$, then $U_s \subset U_r$ if and only if $s \in [2^m r, 2^m r + 2^m - 1]$ where m is the integer part of $\log_2 \frac{s}{r}$. Since $k = 2^{p-1}l + 2^{p-1} - 1$ where $p = \log_2 \frac{2k+2}{l+1}$, $\overline{V_k} \subset V_l$ and $\overline{V_k} \cap U_{2^j l + 2^j - 2} = \emptyset$ for $j = 1, \ldots, p-1$. Hence

$$\begin{aligned} |f_{2k+1}(t)| &\leqslant |f_l(t) - g_{\gamma_l}(t)| + |g_{\gamma_l}(t) - g_{\gamma_{2k}}(t)| + |g_{\gamma_{2k}}(t) - f_{2k}(t)| \\ &+ \sum_{j=1}^{\log_2 \frac{2k+2}{l+1} - 1} |f_{2^j l + 2^j - 2}(t)| < \frac{\delta}{4^l 2} + \frac{\delta}{4^l 2} + \sum_{j=1}^{\log_2 \frac{2k+2}{l+1} - 1} \frac{\delta}{4^{2^j l + 2^j - 2} 2} \end{aligned}$$

for every $t \in B^{2k}_{\gamma_{2k}}$ and

$$\begin{aligned} |f_{2k+1}(t) - g_{\gamma_{2k+1}}(t)| &\leqslant |f_l(t) - g_{\gamma_l}(t)| + |g_{\gamma_l}(t) - g_{\gamma_{2k+1}}(t)| + \sum_{j=1}^{\log_2 \frac{2k+2}{l+1}} |f_{2^j l + 2^j - 2}(t)| \\ &< \frac{\delta}{4^l 2} + \frac{\delta}{4^l 2} + \sum_{j=1}^{\log_2 \frac{2k+2}{l+1} - 1} \frac{\delta}{4^{2^j l + 2^j - 2} 2} \end{aligned}$$

for every $t \in B^{2k+1}_{\gamma_{2k+1}}$. Let V_{2k}, V_{2k+1} be open subsets of K such that $B^j_{\gamma_j} \subset V_j \subset \overline{V_j} \subset U_j$ for $j = 2k, 2k+1$ and

$$|f_{2k}(t) - g_{\gamma_{2k}}(t)| < \tfrac{\delta}{4^{2k} 2} \text{ and } |f_{2k+1}(t)| < \tfrac{\delta}{4^l} + \sum_{j=1}^{\log_2 \frac{2k+2}{l+1}} \frac{\delta}{4^{2^j l + 2^j - 2} 2} \quad \text{for every } t \in \overline{V_{2k}},$$

$$|f_{2k}(t)| < \tfrac{\delta}{4^{2k} 2} \text{ and } |f_{2k+1}(t) - g_{\gamma_{2k+1}}(t)| < \tfrac{\delta}{4^l} + \sum_{j=1}^{\log_2 \frac{2k+2}{l+1}} \frac{\delta}{4^{2^j l + 2^j - 2} 2} \quad \text{for every } t \in \overline{V_{2k+1}}.$$

Let $\Gamma_j = \{\gamma \in \Gamma_1 : \gamma > \gamma_j, B^j_{\gamma_j} \subset V_j, \|g_\gamma - g_{\gamma_j}\|_{C(\overline{V_j})} < \frac{\delta}{4^j 2}\}$ for $j = 2k, 2k+1$. According to Proposition 1.4 sets Γ_{2k}, Γ_{2k+1} are uncountable.

Now we show that the right-hand side inequality of (2) holds. Let $h_n = \sum_{k=2^n}^{2^{n+1}-1} |f_k|$. It is enough to show that $h_n \leqslant 3 + \frac{7\delta}{20}$. Let us note that

$$h_2(t) = |f_2(t)| + |f_1(t) - f_2(t)| \leqslant \begin{cases} 1 + \frac{\delta}{4} + \frac{\delta}{4^2} & \text{if } t \in V_2 \cup V_3 \\ 3 & \text{if } t \notin V_2 \cup V_3. \end{cases}$$

First we show that

$$h_n(t) \leqslant 1 + \frac{\delta}{4} + \sum_{k=1}^{2^n - 1} \frac{3\delta}{4^{2k} 2}$$

for every $t \in \bigcup_{k=2^n}^{2^{n+1}-1} V_k$. If $f_{2k+1} = f_l - \sum_{j=1}^{p} f_{2^j l+2^j-2}$ where $p = \log_2 \frac{2k+2}{l+1}$ and $l \in 2\mathbb{N} \cup \{1\}$ is the smallest natural number such that $(l+1)2^j = 2k+2$ for some $j \in \mathbb{N} \cup \{0\}$, then

$$|f_{2k+1}(t)| \leqslant 1 + \frac{\delta}{4^l} + \sum_{j=1}^{\log_2 \frac{2k+2}{l+1}} \frac{\delta}{4^{2^j l+2^j-2}2}$$

for every $t \in \overline{V_{2k+1}}$. Let us note that $\{2k+1\} \cup \bigcup_{j=1}^{p} [2^{p-j}(2^j l + 2^j - 2), 2^{p-j}(2^j l + 2^j - 2) + 2^{p-j} - 1] = [2^p l, 2^p l + 2^p - 1]$. If $j \in [2^m(2^r l + 2^r - 2), 2^m(2^r l + 2^r - 2) + 2^m - 1]$ where $r = 1, \ldots, p$ and $m = p - r$, then $V_j \subset V_{2^r l+2^r-2}$. Moreover, $U_{2^{j_1} l+2^{j_1}-2} \cap U_{2^{j_2} l+2^{j_2}-2} = \emptyset$ for every $j_1, j_2 \in \{1, 2, \ldots, p\}$, $j_1 \neq j_2$. Hence

$$\begin{aligned} |f_{2k+1}(t)| &\leqslant |f_l(t) - g_{\gamma_l}(t)| + |g_{\gamma_l}(t) - g_{\gamma_{2^r l+2^r-2}}(t)| + |g_{\gamma_{2^r l+2^r-2}}(t) - f_{2^r l+2^r-2}(t)| \\ &+ \sum_{j=1, j\neq r}^{\log_2 \frac{2k+2}{l+1}} |f_{2^j l+2^j-2}(t)| \leqslant \frac{\delta}{4^l} + \sum_{j=1}^{\log_2 \frac{2k+2}{l+1}} \frac{\delta}{4^{2^j l+2^j-2}2} \end{aligned}$$

for $j \in [2^m(2^r l + 2^r - 2), 2^m(2^r l + 2^r - 2) + 2^m - 1]$ and every $t \in \overline{V_j}$, and

$$|f_{2k+1}(t)| \leqslant \frac{\delta}{4^l 2} + \sum_{j=1}^{\log_2 \frac{2k+2}{l+1}} \frac{\delta}{4^{2^j l+2^j-2}2}$$

for $j \in [2^n, 2^{n+1} - 1] \setminus [2^p l, 2^p l + 2^p - 1]$ and every $t \in \overline{V_j}$. When we gather together the estimations above and we apply Proposition 1.5 b), we get

$$h_n(t) \leqslant 1 + \frac{\delta}{4} + \sum_{l=1}^{2^n-1} \frac{3\delta}{4^{2l}2} < 1 + \frac{7\delta}{20}$$

for every $t \in \bigcup_{j=2^n}^{2^{n+1}-1} V_j$. The estimations show also that

$$h_n(t) - |f_k(t)| \leqslant \frac{\delta}{4} + \sum_{l=1}^{2^n-1} \frac{3\delta}{4^{2l}2} < \frac{7\delta}{20}.$$

for every $k \in [2^n, 2^{n+1} - 1]$ and $t \in \bigcup_{j=2^n, j\neq k}^{2^{n+1}-1} V_j$. We show now applying the mathematical induction that

$$h_n(t) \leqslant 3 + \frac{\delta}{4} + \sum_{k=1}^{2^n-1} \frac{3\delta}{4^{2k}2}$$

for every $t \in K$ and $n \in \mathbb{N}$. It is clear that $h_{n+1} = \sum_{k=2^n}^{2^{n+1}-1} |f_{2k}| + |f_k - f_{2k}|$. Applying the following facts: $U_{2k} \subset V_k$, $|f_{2k}(t)| \leqslant \frac{\delta}{4^{2k}2}$ for every $t \notin U_{2k}$ and

$k = 2^n, \ldots, 2^{n+1}-1$ and $V_l \cap V_j = \emptyset$ for every $l, j \in \{2^n, \ldots, 2^{n+1}-1\}$, $l \neq j$, we obtain

$$h_{n+1}(t) \leqslant h_n(t) + 2\sum_{k=2^n}^{2^{n+1}-1} |f_{2k}(t)| \leqslant 3 + \frac{\delta}{4} + \sum_{k=1}^{2^n-1} \frac{3\delta}{4^{2k}2} + 2\sum_{k=2^n}^{2^{n+1}-1} \frac{\delta}{4^{2k}2} \leqslant 3 + \frac{7\delta}{20}$$

for every $t \notin \bigcup_{k=2^n}^{2^{n+1}-1} V_k$. Applying the inequality proved above we obtain

$$h_{n+1}(t) \leqslant 2|f_{2j}(t)| + h_n(t) + 2\sum_{k=2^n, k\neq j}^{2^{n+1}-1} |f_{2k}(t)| \leqslant 2+1+\frac{\delta}{4} + \sum_{k=1}^{2^n-1} \frac{3\delta}{4^{2k}2} + 2\sum_{k=2^n}^{2^{n+1}-1} \frac{\delta}{4^{2k}2}$$

for each $j \in \{2^n, \ldots, 2^{n+1}-1\}$ and $t \in V_j$.

It remains to show the left-hand side inequality of (3). Applying the estimations above for every $j \in \mathbb{N}$ we get

$$\begin{aligned}|\langle f_j, \chi_{B^j_{\gamma_j}} x_{\gamma_j}\rangle| &\geqslant |\langle f_j - g_{\gamma_j} + g_{\gamma_j}, \chi_{B^j_{\gamma_j}} x_{\gamma_j}\rangle| \geqslant \int_{B^j_{\gamma_j}} g_{\gamma_j}\, dx_{\gamma_j} - \varepsilon\Big(\frac{\delta}{4} + \sum_{k=1}^{\infty} \frac{\delta}{4^{2k}2}\Big) \\ &\geqslant \int_{B^j_{\gamma_j}} d|x_{\gamma_j}| - \frac{\delta}{4^2} - \varepsilon\frac{17\delta}{60} \geqslant \varepsilon - \frac{\delta}{4^2} - \frac{\delta}{4^2} - \varepsilon\frac{17\delta}{60}.\end{aligned}$$

For every n and $2^n \leqslant j < 2^{n+1}$ and scalars $a_{2^n}, \ldots, a_{2^{n+1}-1}$ such that $|a_j| = \max_{2^n \leqslant i < 2^{n+1}} |a_i|$ we have

$$\begin{aligned}\Big\|S\Big(\sum_{k=2^n}^{2^{n+1}-1} a_k f_k\Big)\Big\| &\geqslant \Big|\Big\langle \sum_{k=2^n}^{2^{n+1}-1} a_k f_k, x_{\gamma_j}\Big\rangle\Big| \\ &\geqslant \Big|\Big\langle \sum_{k=2^n}^{2^{n+1}-1} a_k f_k, \chi_{B^j_{\gamma_j}} x_{\gamma_j}\Big\rangle\Big| - |a_j||\langle h_n, \chi_{K\setminus B^j_{\gamma_j}} |x_{\gamma_j}|\rangle| \\ &\geqslant |a_j|\big(|\langle f_j, \chi_{B^j_{\gamma_j}} x_{\gamma_j}\rangle| - |\langle h_n - |f_j|, \chi_{B^j_{\gamma_j}} |x_{\gamma_j}|\rangle| - 4\frac{\delta}{4^2}\big) \\ &\geqslant |a_j|\Big(\varepsilon - \frac{2\delta}{4^2} - \varepsilon\frac{17\delta}{60} - \varepsilon\frac{7\delta}{20} - \frac{\delta}{4}\Big) \geqslant \frac{\varepsilon}{2} \max_{2^n \leqslant i < 2^{n+1}} |a_i|.\end{aligned}$$

Thus we show the left-hand side inequality of (3). ∎

Remark 2.2 If $E = C(K)$, we may use the Tietze theorem to construct functions $\varphi_j \in C(K)$ such that $\varphi_j(K) \subset [0,1]$, $\varphi_j|_{\overline{V_j}} = 1$ and $\varphi_j|_{K\setminus U_j} = 0$ for every j where U_j and V_j are sets defined in the proof above. Next we apply the following simple fact:

Proposition 2.3 *Let K be a compact Hausdorff space. If (φ_n) is a sequence in $C(K)$ such that*

1) $\varphi_n \geqslant 0$ and $\|\varphi_n\| = 1$ for every n and

2) $\varphi_{2n} + \varphi_{2n+1} \leqslant \varphi_n$ *for every* n,

then the sequence (ψ_n) *given by* $\psi_1 = \varphi_1$ *and*

$$\psi_n = \begin{cases} \varphi_n & \text{if } n = 2k \\ \psi_k - \varphi_{2k} & \text{if } n = 2k+1 \end{cases}$$

satisfies condition (1) and (2) of Proposition 1.1 with constants $c = C = 1$*. Consequently, the subspace* $\overline{\text{lin}}\{\varphi_n : n \in 2\mathbb{N} \cup \{1\}\}$ *of* $C(K)$ *is isometric to* $C(\{-1, 1\}^{\mathbb{N}})$*.*

Since we may select g_{γ_1} in such a way that $|g_{\gamma_1}| = 1$, the sequence (f_n) given by $f_n = \psi_n g_{\gamma_1}$ has properties (1) and (2) of Proposition 1.1 with constants $c = C = 1$. Applying the fact that $f_j(t) = g_{\gamma_1}(t)$ and $|g_{\gamma_j} - g_{\gamma_1}(t)| \leqslant \sum_{k=1}^{j} \frac{\delta}{4^k} < \frac{\delta}{3}$ for every $t \in \overline{V_j}$ and $j \in \mathbb{N}$ similar as in the proof above we show that the sequence (f_n) satisfies the left-hand side inequality of (3).

Corollary 2.4 *Let* K *be a compact metric space. Let* X *be a Banach space. Let* E *be a closed subspace of* $C(K)$ *such that* $E^{\perp}$ *is separable. If* $S : C(K) \to X$ *is a continuous linear operator such that* $S^*(X^*)$ *is a nonseparable subset of* $\text{ca}(K)$*, then there exists a subspace* Y *of* E *isomorphic to* $C([0, 1])$ *such that* $S|_Y$ *is an isomorphism.*

Proof It is clear that $(S|_E)^*(X^*)$ is nonseparable subset of $E^* = \text{ca}(K)/E^{\perp}$. ■

As a straightforward consequence of Theorem 2.1 we get the following fact noted in [9, p. 52].

Corollary 2.5 *Let* K *be a compact metric space. Let* X *be a Banach space. Let* E *be a closed subspace of* $C(K)$ *such that* $E^{\perp}$ *is separable. If* Y *is a quotient space of* E *with nonseparable dual, then* Y *contains an isomorphic copy of* $C([0, 1])$*.*

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