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On the Krull property in topological algebras

Abstract. We introduce Krull topological algebras. In particular, we characterize the Krull property in some special classes of topological algebras. Connections with the theory of semisimple annihilator Q'-algebras are given. Relative to this, an investigation on the relationship between Krull and (weakly) regular (viz. modular) annihilator algebras is considered. Subalgebras of certain Krull algebras are also presented. Moreover, conditions are supplied under which the Krull (resp. Q'-) property is preserved via algebra morphisms. As an application, we show that the quotient of a Krull Q'-algebra, modulo a 2-sided ideal, is a topological algebra of the same type. Finally, we study the Krull property in a certain algebra-valued function topological algebra.

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Introduction. In [14: p. 3732, Theorem 4.7] we considered the relationship between "duality" and "complementation" for (Hausdorff) locally C^*-algebras, the latter being Krull algebras in the sense of Definition 2.1 below, which that motivated the present study of this sort of topological algebras (see also [15: p. 198, Corollary 2.2 and Theorem 2.4]). So, we characterize the Krull property in a topologically semiprime, weakly regular annihilator (much more precomplemented) (D)-algebra (Theorem 3.2) or even in a semisimple regular annihilator topological algebra (Proposition 3.7). Conditions under which the Krull (resp. Q'-)property is preserved, via algebra morphisms, are also considered (Lemma 3.10). The quotient of a Krull Q'- algebra, modulo a 2-sided ideal, is an algebra of the same type (Theorem 3.11). Examples of Krull and semisimple annihilator as well as $(D)Q'$-algebras are given. We also exhibit an example of a Q'- algebra which is not a Q-algebra. In Theorem 3.14 we present certain algebraic-topological properties of a semisimple annihilator Q_l' (or Q_r')-algebra. Using this, we get information about the factors (being minimal closed 2-sided ideals) of a decomposition of the algebra in question (see Remark 4.4). Furthermore, conditions are given that a locally convex H^*-algebra be Krull (Proposition 4.1). We also study special subalgebras of certain Krull algebras. Finally, the Krull property is studied in a certain algebra-valued function topological algebra (see Section 5).

1. Notation and preliminaries. All algebras considered are taken over the field $\mathbb{C}$ of complexes. Notation and definitions not given here are taken from Rickart's book [24].

Let E be an algebra. If $(\emptyset \neq) S \subseteq E$, $\mathcal{A}_l(S)$ (resp. $\mathcal{A}_r(S)$) denotes the left (right) annihilator of S. $\mathcal{A}_l(S)$ (resp. $\mathcal{A}_r(S)$) is a left (right) ideal of E, which in particular, is a $2-$sided ideal, if S is a left (right) ideal. In case of a *topological algebra* (separately continuous multiplication) the previous ideals are closed. We denote by $\mathcal{L}_l(E) \equiv \mathcal{L}_l$ ($\mathcal{L}_r(E) \equiv \mathcal{L}_r$, $\mathcal{L}$) the set of all closed left (right, $2-$sided) ideals in a topological algebra E, while $\mathfrak{M}_l(E)$ (resp. $\mathfrak{M}_r(E)$) stands for the set of all closed maximal regular left (right) ideals of E. An algebra E is called *left* (resp. *right*) *preannihilator,* if $\mathcal{A}_l(E) = (0)$ (resp. $\mathcal{A}_r(E) = (0)$). If $\mathcal{A}_l(E) = \mathcal{A}_r(E) = (0)$, E is called *preannihilator.* In particular, a topological algebra E is said to be an *annihilator algebra,* if it is preannihilator with $\mathcal{A}_r(I) \neq (0)$ for every $I \in \mathcal{L}_l$, $I \neq E$ and $\mathcal{A}_l(J) \neq (0)$ for every $J \in \mathcal{L}_r$, $J \neq E$. A topological algebra E is called an $(\mathcal{M}_r')-$*algebra* (alias, a *right annihilator algebra* [26]), if E is the only closed left ideal having a trivial right annihilator; viz. one has $I = E$ for every $I \in \mathcal{L}_l$ with $\mathcal{A}_r(I) = (0)$. An $(\mathcal{M}_l')-$*algebra* is defined analogously, by interchanging "left" by "right". Obviously, an annihilator algebra is an $(\mathcal{M}_l')$ and $(\mathcal{M}_r')-$algebra. A topological algebra E satisfying $\mathcal{A}_l(\mathcal{A}_r(I)) = I$ for all $I \in \mathcal{L}_l$ and $\mathcal{A}_r(\mathcal{A}_l(J)) = J$ for all $J \in \mathcal{L}_r$ is called a *dual algebra.* A topological algebra E such that $I \in \mathcal{L}$ and $I^2 = (0)$ implies $I = (0)$ is called *topologically semiprime.* A topological algebra E is named a Q_l' (resp. Q_r')$-$*algebra,* if every maximal regular left (resp. right) ideal is closed. In particular, E is a $Q'-$*algebra,* if it is both a Q_l' and Q_r'-algebra. *A $Q-$algebra is a Q'-algebra* (see, for instance, [22: p. 43, Definition 6.3 and p. 67, Theorem 6.1]). In this respect, we note that the conditions involved in the definition of a Q'-algebra have also been considered by Smiley and Yood for topological rings (see [25: p. 698], [31: p. 33]) and by Barnes, Husain and Wong for topological algebras ([3: p. 571, Theorem 5.2 and p. 578, Example 8.1], [18: p. 142, Lemma 2.1, p. 143, Lemma 3.3 and p. 144, Theorem 3.6]). Yet, A. Mallios too has explicitly remarked [22: p. 73, Scholium 7.1] the significance of $Q'-$*property,* singling out, for instance, *continuous characters.* However, the terminology Q'-algebra was introduced in [12]. Now, by a *deep* (or shortly (D))-*algebra,* is meant an algebra whose every (non-zero) left (resp. right) ideal contains a minimal left (resp. right) ideal. We denote by $\mathcal{I}d(E)$ the set of all non-zero idempotent elements of an algebra E, namely, the set of all $x \in E$ with $0 \neq x = x^2$. A *minimal* element of an algebra E, is a non-zero idempotent x such that xEx is a division algebra. A non-zero element of E is called *primitive,* if it can not be expressed as the sum of two orthogonal idempotents; viz. of some $y, z \in \mathcal{I}d(E)$ with $yz = zy = 0$. The Jacobson radical of E is denoted by $R(E)$; E is *semisimple,* if $R(E) = (0)$. A topological algebra E is called *left* (resp. *right) pre-*

complemented, if for every $I \in \mathcal{L}_l$ (resp. in $\mathcal{L}_r$) there exists $I' \in \mathcal{L}_l$ (resp. in $\mathcal{L}_r$) with $E = I \oplus I'$; E is called *precomplemented*, if it is both left and right precomplemented. A *locally convex H*-algebra* is an algebra E equipped with a family $(p_\alpha)_{\alpha\in\Lambda}$ of *Ambrose seminorms* in the sense that p_α , $\alpha \in \Lambda$ arises from a positive semi-definite (pseudo-) inner product (denoted by $< , >_\alpha$) such that the induced topology makes E into a locally convex (topological) algebra. Moreover, the following conditions are satisfied: For any $x \in E$, there is an $x^* \in E$, such that

$$<xy, z>_\alpha = <y, x^*z>_\alpha \tag{1}$$

$$<yx, z>_\alpha = <y, zx^*>_\alpha \tag{2}$$

for any $y, z \in E$ and $\alpha \in \Lambda$.

A locally m-convex H^*-algebra is also considered.

Given a locally convex H^*-algebra E and $I \in \mathcal{L}_l$, the *orthogonal* of I , denoted by $I^\perp$ is

$$I^\perp = \{x \in E : <x, y>_\alpha = 0 \ \text{ for every } \ y \in I, \alpha \in \Lambda\}, \tag{3}$$

which obviously, is a *closed left ideal* (cf. [10: p. 456, Lemma 3.2]). In this context, E is called an *orthocomplemented algebra,* if for every $I \in \mathcal{L}_l$ and every $J \in \mathcal{L}_r$, $E = I \oplus^\perp I^\perp$ and $E = J \oplus^\perp J^\perp$ (here $\oplus^\perp$denotes orthogonal direct sum). This kind of complementation leads to a complemented algebra in the sense of [14: p. 3723, Definition 2.1]; see also [10: p. 457, Lemma 3.3]. A *locally C^*-algebra* is an involutive complete locally (m-)convex algebra $(E, (p_\alpha)_{\alpha\in A})$ such that $p_\alpha, \alpha \in A$ is a C^*-seminorm (viz. $p_\alpha, \alpha \in A$ is a seminorm with $p_\alpha(x^*x) = p_\alpha(x)^2$ for every $x \in E$ [19: p. 198, Definition 2.2]).

The next lemma is frequently used in the sequel (cf. [12: p. 150, Lemma 3.3]).

LEMMA 1.1 *A semisimple (topological) algebra E is (topologically) semiprime and thus preannihilator. Moreover, E has no nilpotent left or right ideals.*

2. Krull algebras.

DEFINITION 2.1 A *left Krull algebra* is a topological algebra satisfying the condition:

Every (proper) closed left ideal is contained
in a closed maximal regular left ideal.

A *right Krull algebra* is analogously defined, by considering right ideals. A left and right Krull algebra is simply called a *Krull algebra.*

EXAMPLES 2.2 1) *The group algebra $L^1(G)$ with G a locally compact abelian group is a Krull algebra.* In fact, $L^1(G)$ as a Banach algebra, has the $Q'-$property (see

Section 1). Besides, every proper closed ideal in $L^1(G)$ is contained in a maximal regular ideal (cf. for instance, [24: p. 326]), closed by "Q'". See also Example 3) below.

2) The algebra given by B.A. Barnes in [2] is a normed (non-complete) semisimple annihilator $Q'-$algebra and hence (Theorem 3.14 below) a Krull algebra.

The following examples are referred to non-normed topological algebras.

3) We note that *every unital Q'-algebra is a Krull algebra.* In fact, if I is a proper closed left ideal, then it is regular and hence (Zorn) it is contained in a maximal (regular) left ideal, closed by the Q'-property. Similarly, for right ideals. An instance of this type of algebra is provided in Examples 3.6, 6).

4) *Every locally C^*-algebra is a Krull algebra* (see [15: p. 198, Corollary 2.2]). A fortiori; *every C^*-algebra E is a Krull algebra.* In this case, every closed (left) ideal of E is the intersection of the (closed) maximal regular (left) ideals containing it (see, for instance, [6: p. 56, Theorem 2.9.5]).

5) In [22: p. 350, Theorem 6.1], Mallios proves that for a (commutative) *Pták-Šilov-algebra with a locally equicontinuous spectrum the notion "Wiener-Tauber algebra" is equivalent to the property "every proper closed ideal has a non-empty hull", equivalently, it is contained in a closed maximal regular ideal.* Thus, in this case, *the notions "Wiener-Tauber algebra" and "Krull algebra" coincide.* See also [ibid. Corollary 6.1].

6) *Let X be a (Hausdorff) completely regular space and $\mathcal{C}_c(X)$ the (locally m-convex) algebra of all continuous $\mathbb{C}$-valued functions on X in the compact open topology* (see, for instance, [22: p. 19, Example 3.1]). *Then $\mathcal{C}_c(X)$ is a Krull algebra.* Apply Theorem 3.1 in (loc.cit., p. 337; see also p. 338, (3.11)). We note that $\mathcal{C}_c(X)$ is not complete, unless X is a k-space. In this case, and under the standard involution, $\mathcal{C}_c(X)$ is a locally C^*-algebra (see [21: p. 231, Theorem 12] and/or [30: p. 267]). Thus, by Example 4) above, we get another justification that this algebra is a Krull one.

7) *Every semisimple Hausdorff orthocomplemented (locally convex H^*-)algebra E being moreover, a Q'-algebra is Krull.* In fact, E is dual and it has a dense socle (see Lemma 1.1, [10: p. 458, Theorem 3.9] and [13: p. 149, Theorem 3.10]). Moreover, by [31: p. 41, Lemma 3.11], E is a regular annihilator algebra. The assertion now follows from Theorem 3.14 below.

Based on Example 7) and the fact that a proper (Banach) H^*-algebra is orthocomplemented [1] and semisimple [20: p. 696] we get that *a proper Hausdorff H^*-algebra is Krull.* See also the comments after Proposition 4.1.

8) *A proper complete locally m-convex H^*- algebra $(E,(p_\alpha)_{\alpha\in A})$ with continuous involution and a unit e such that $p_\alpha(e)=1, \alpha\in A$ is a Krull algebra,* (see Example 4) above and [15: p. 198, Proposition 2.5]).

9) (See also [33: p. 292]). Let E be a unital commutative complete locally $m-$convex algebra. For any closed ideal I of E the quotient algebra is a locally $m-$convex algebra as well (see [22]); thus every closed ideal is contained in a closed maximal (regular) ideal and so E is a Krull algebra.

For a non-Krull algebra, see Scholium 3.15 below.

Remark 2.3 - *If a precomplemented Krull algebra E has no divisors of zero, then it is left (resp. right) topologically simple*: If I is a proper closed left ideal in E, then (Krull) $I \subseteq M$ for some closed maximal regular left ideal of the form $M = E(1 - x_0) \equiv \{y - yx_0 : y \in E\}$ for some $x_0 \in \mathcal{I}d(E)$ (cf. also [14: p. 3729, Theorem 3.9 and Remarks 3.10]). Now, if $x \in M$, then $x \in \mathcal{A}_l(x_0E)$ and hence $xx_0 = 0$. So, $x = 0$; namely, $M = (0)$, a f o r t i o r i, $I = 0$. Similarly, for closed right ideals.

In this connection, I am indebted to the referee whose relevant remarks led to the above.

For convenience we also use the following terminology: Let E be a (topological) algebra. A (closed) maximal regular left (resp. right) ideal is called *Peirce decomposable,* if it has the form $E(1 - x)$ (resp. $(1 - x)E \equiv \{y - xy \ : \ y \in E\}$) for some $x \in \mathcal{I}d(E)$. Thus, *a topological algebra in which every maximal regular left (right) ideal is Peirce decomposable is a Q'-algebra.* So (see also the justification in Remark 2.3) *a precomplemented algebra is a Q'-algebra if and only if its maximal regular left (right) ideals are Peirce decomposable.*

3. (Weakly) regular annihilator, quotient, Krull and Q'- algebras. We are mainly concerned here with relations between Krull, weakly regular annihilator or even regular annihilator topological algebras, annihilator algebras and Q'-algebras.

In the sequel by a *left (* resp. *right) regular annihilator algebra* we shall mean an algebra in which $\mathcal{A}_l(M) \neq (0)$ (resp. $\mathcal{A}_r(M) \neq (0)$) for every maximal regular right (left) ideal M; if both conditions hold, it is called a *regular annihilator algebra.* The definition given by Barnes in [3: p. 568] includes, moreover, the condition $\mathcal{A}_l(E) = (0)$ (resp. $\mathcal{A}_r(E) = (0)$). However, Yood still employs regular annihilator algebras with no non-trivial nilpotent 1-sided ideals (see also [31: p. 37]).

By looking at the analogous topological notions, we set the next.

Definition 3.1 A topological algebra E is called a *weakly left* (resp. *right*) *regular annihilator algebra,* if $\mathcal{A}_l(M) \neq (0)$ (resp. $\mathcal{A}_r(K) \neq (0)$) for all $M \in \mathfrak{M}_r(E)$ (resp. $K \in \mathfrak{M}_l(E)$). If both conditions hold, E is said to be a *weakly regular annihilator algebra.*

Remarks - The class of weakly (right) regular annihilator algebras includes that of modular annihilator (Banach) algebras, given by Duncan in [8: p. 89], since in this case, every maximal regular (left) ideal is closed.

The following are direct consequences of the very definitions:

i) An annihilator algebra is weakly regular annihilator. The reverse implication holds e.g. for preannihilator Krull algebras (cf. Remarks after Theorem 3.2 below). Notice that, in general, an annihilator algebra is not a regular annihilator algebra (see [31: p. 39, the comments preceding Theorem 3.5]).

ii) A topological algebra E with $\mathcal{A}_l(E) \neq (0)$ (resp. $\mathcal{A}_r(E) \neq (0)$) is weakly left (resp. right) regular annihilator. But, one has weakly regular annihilator algebras even if $\mathcal{A}_l(E) = \mathcal{A}_r(E) = (0)$ (take e.g. an annihilator algebra, see i)).

Left regular annihilator topological algebras coincide with weakly left regular annihilator algebras for the following sorts of (topological) algebras:

1) Q'_r−algebras or equivalently topological algebras that have no proper dense regular right ideals (cf. [12: p. 149, Proposition 1.3]). In particular,

2) topological algebras, say E, in which every maximal regular right ideal M has an algebraic complement (namely, $E = M \oplus N$ for some right ideal N of E), since in that case, the only maximal regular right ideals of E are Peirce decomposable (see [14: p. 3729, Theorem 3.9 and p. 3730 Definition 3.13]). Thus, E is a Q'_r-algebra (see the comments at the end of Section 2). In this context, we also note that *a maximal regular (left) ideal of an algebra is Peirce decomposable if and only if it has an algebraic complement.*

Now, we show that the Krull property characterizes annihilator algebras over all topologically semiprime, (D)-algebras which moreover, are weakly regular annihilator (a f o r t i o r i precomplemented) algebras. Notice that (D)-property is used only in 1)⇒2) while the assumption that E is a weakly regular annihilator algebra or yet precomplemented is used only in 2) ⇒ 1). Moreover, 1) ⇒ 2) improves and generalizes Theorem 5.2 in [3: p. 571]; see Remarks.-2) below, as well as Remarks after Definition 3.1.

THEOREM 3.2 *Let E be a topologically semiprime, (D)-algebra which moreover, is a weakly regular annihilator (a f o r t i o r i, a precomplemented) algebra. Then the following are equivalent:*

1) E is an annihilator algebra.

2) E is a Krull algebra.

PROOF 1) ⇒ 2) : If $I \in \mathcal{L}_l$ with $I \neq E$, then $\mathcal{A}_r(I) \neq (0)$. Thus, in virtue of (D)-property (see Section 1), there exists a minimal right ideal of the form xE with x a minimal primitive (idempotent) element, such that $xE \subseteq \mathcal{A}_r(I)$ (cf. also [12: p. 154, Corollary 3.7]). Therefore, $I \subseteq \mathcal{A}_l(\mathcal{A}_r(I)) \subseteq E(1-x)$. Since the closed right ideal xE is minimal, it is minimal closed and hence $E(1-x)$ is a maximal (regular) left ideal (see Lemma 1.1 and [12: p. 151, Lemma 2.6]), which obviously, is closed. That is, E is a left Krull algebra. Similarly, for closed right ideals.

2) ⇒ 1) : We first note that E, as topologically semiprime, is preannihilator (see Lemma 1.1). Suppose now that E is a weakly regular annihilator algebra. Consider a proper closed left ideal I in E. Then (Krull), there exists some closed maximal regular left ideal, say M, with $I \subseteq M$. Thus, $\mathcal{A}_r(M) \subseteq \mathcal{A}_r(I)$ and hence $\mathcal{A}_r(I) \neq (0)$. Similarly, $\mathcal{A}_l(J) \neq (0)$ for every proper closed right ideal J in E. Namely, E is an annihilator algebra.

Claim that *a precomplemented algebra is a weakly regular annihilator algebra.* In fact, if M is a closed maximal regular left ideal, then $E = M \oplus M'$ with $M' \in \mathcal{L}_l$. Thus, by [14: p. 3729, Theorem 3.9 and Remarks 3.10], M has the form $M = E(1-x)$ for some $x \in \mathcal{I}d(E)$ and hence $\mathcal{A}_r(M) = xE \neq (0)$. Similarly, E is weakly regular annihilator on the right and this completes the proof. ∎

REMARKS - As a byproduct of the previous proof one gets that *for a Krull algebra the notions "$(\mathcal{M}'_l)$* and *$(\mathcal{M}'_r)$−algebra" and "weakly regular annihilator algebra" coincide.*

1) Concerning 2) $\Rightarrow$ 1) in the previous theorem, something more general is actually true. More precisely, *every preannihilator weakly regular annihilator Krull algebra is an annihilator one.* (Note that a topologically semiprime algebra is preannihilator; see Lemma 1.1).

2) In [3: p. 571, Theorem 5.2], Barnes proves that *a semisimple annihilator Q'_l-algebra is regular annihilator.* Using this, he proves that the algebra, in question, has (in our terminology) the Krull property (on the left). Since a semisimple regular annihilator topological algebra is topologically semiprime, $(D)Q'$-algebra (see also [ibid. p. 568, Theorem 4.1 and p. 569, Theorem 4.2]), we get that the information, of being the algebra a Krull one, is a consequence of our Theorem 3.2 above (see also the comments preceding it), where *the algebra is not necessarily assumed to be a Q'_l-algebra.*

3) *Equivalence of statements* 1) and 2), *in Theorem 3.2, is still valid, if we replace "precomplementation" of the algebra, by the property "every closed maximal regular left (right) ideal is Peirce decomposable".*

4) *The implication* 2) $\Rightarrow$ 1) *of the same theorem is valid, as well for any preannihilator left precomplemented *-algebra.* This is immediate from Corollary 4.6 in [14: p. 3732], where a left Krull algebra suffices instead of a Krull algebra.

Since every unital algebra is preannihilator and its ideals are regular, we get that *every unital weakly regular annihilator Q'- algebra is an annihilator one.* In fact, for $I \in \mathcal{L}_r$ with $I \neq E$ there exists a maximal (regular) right ideal, say M, with $I \subseteq M$, which is closed. Thus, $\mathcal{A}_l(M) \subseteq \mathcal{A}_l(I)$ and hence $\mathcal{A}_l(I) \neq (0)$.

Since Q'-property is frequently used in this paper, our objective is to give some examples of Q'-algebras. For this, we make use of the following result taken from [3: p. 569, Theorem 4.3]. *Let E be an algebra with no nilpotent left or right ideals* (take e.g. E semisimple; see Lemma 1.1). *Then E is a right regular annihilator algebra if and only if every maximal regular left ideal has the form E(1-x) with x a minimal (idempotent) element.* Obviously, this leads to the next.

Proposition 3.3 *Every regular annihilator topological algebra with no nilpotent left or right ideals is a $Q'-$ algebra.*

Proposition 3.4 *Let E be a topologically semiprime $(\mathcal{M}'_r)-$ algebra. Then the following are equivalent:*

1) E is a Q'_l-algebra.

2) $\mathcal{A}_r(M)$ is not contained in $R(E)$ for every maximal regular left ideal M.

Proof 1) $\Rightarrow$ 2) : If $\mathcal{A}_r(M) \subseteq R(E)$ for some maximal regular left ideal M, then $\mathcal{A}_r(M) \subseteq M$ and $\mathcal{A}_r(M) \subseteq \mathcal{A}_r(\mathcal{A}_r(M))$. Thus, $\mathcal{A}_r(M)^2 = (0)$ and hence $\mathcal{A}_r(M) = (0)$ (cf. [12: p. 149, Theorem 2.1] as well). Therefore, $M = \overline{M} = E$, a contradiction (see also [ibid. p. 149, Proposition 1.3]).

2) $\Rightarrow$ 1) : By [3: p. 568, Theorem 3.4], $M = E(1-x)$ with x a minimal (idempotent) element. Thus, M is closed. Notice that 2) $\Rightarrow$ 1) *holds, more generally, for any right preannihilator topological algebra.* ■

By the above proof, one gets $\mathcal{A}_r(M) = xE \neq (0)$. Thus, *a right preannihilator topological algebra satisfying 2) of* Proposition 3.4 *is a (weakly) right regular annihilator algebra* (see also [3: p. 569, Theorem 4.3 and its proof]).

The following result provides a sort of a converse to Proposition 3.3, pertaining to 2)⇒ 1) in Corollary 3.5 (see also [3: p. 571, Theorem 5.2, (i)]; a similar remark is valid for Example 8.1 in [3: p. 578]. For 1)⇒ 2) of the same corollary, apply Lemma 1.1 and Proposition 3.3.

Corollary 3.5 *Let E be a semisimple topological algebra. Consider the assertions:*
1) E is a regular annihilator algebra.
2) E is a Q'_l-algebra.
Then 1) ⇒ 2). (In effect 1) ⇒ 2′) E is a Q'-algebra. Besides, 2) ⇒ 1), if E is an annihilator algebra.

As a consequence of the above Corollary 3.5, (see also Remarks, 1) after Definition 3.1, and [12: p. 153, Theorem 3.6]), *Q'_l-algebras, algebras with no proper dense regular left ideals, regular annihilator algebras, Q'-algebras, and (D)-algebras agree for semisimple annihilator algebras.* (See also Theorem 3.14 below). Further, based on Corollary 3.5, the proof of Theorem 3.2, [4: p. 659, C7] and [5: p. 126, Corollary 20], it is easily seen that *the 2-sided ideals of a semisimple regular annihilator topological algebra E are semisimple regular annihilator $(D)Q'$-algebras.* Besides, *E is a $(D)Q'$-algebra* too. Thus, the aforementioned ideals are algebras of the same type as E. Here we get an example of a $(D)Q'$-algebra that *indirectly* entails the properties (D) and Q' on some special class of subalgebras.

Now, we come to the promised examples of Q'-algebras, some of which are $(D)-$algebras, as well.

Examples 3.6 1) Let E be a non-unital commutative semisimple topological algebra with discrete space of maximal regular ideals. Then by [3: p. 578, Example 8.4] E is a regular annihilator algebra. Hence (Lemma 1.1, Proposition 3.3 and [3: p. 569, Theorem 4.2]) E is a *a $(D)Q'$-algebra,* as well. In the unital case, there are topological algebras as before, which are not Q'-algebras. Take, for instance, the (semisimple) locally C^*-algebra $\mathcal{C}_c(X)$ with X a discrete space; see also Examples 2.2, 6) and Scholium 3.15 below.

2) According to [3: p. 576], a semisimple finite-dimensional algebra is regular annihilator. Therefore, *every semisimple finite-dimensional topological algebra is a $(D)Q'$-algebra.*

3) By Lemma 1.1, Proposition 3.3 and [3: p. 569, Theorem 4.2 and p. 576, Theorem 7.2], *every semisimple normed algebra, whose left regular representations are completely continuous* (take e.g. the algebra completely continuous) *is a $(D)Q'$-algebra.*

4) *If E is a semisimple normed regular annihilator algebra* (thus, a Q'-algebra; see Lemma 1.1 and Proposition 3.3) *and I a maximal regular 2-sided ideal of E, then E/I is an algebra of the same type as E* (with an identity and finite-dimensional), *which in particular, is a (D)-algebra.* Apply the argument of Example 2) above, and [3: p. 574, Theorem 6.4, see also its proof]. Here we also use the fact that a maximal

regular 2−sided ideal is primitive (cf. [24: p. 54, Theorem 2.2.9]). Concerning the Q'-property of E/I, see also Theorem 3.11 below.

5) Consider the normed algebra $\mathcal{A}P(G)$ of all almost periodic functions on a topological group G the underlying linear space being a pre-Hilbert space. Then *every (not necessarily closed) 2-sided ideal in $\mathcal{A}P(G)$ is a regular annihilator dual Q'-algebra* (cf. [32: pp. 261 and 266]). Here we also get *an example of a (not-complete) normed algebra which is dual* (a f o r t i o r i an annihilator one).

6) In [34] Żelazko provides *an example of a topological Q'-algebra, which is not a Q-algebra.* More precisely, the free algebra in countably many variables, provided with the maximal locally convex topology, is a unital non-commutative complete locally convex Q'- algebra (all linear subspaces and so all ideals are closed), which is not a Q-algebra (the only invertible elements are the scalar multiples of the identity).

In this respect, I am greatly indebted to Professor W. Żelazko, who drew my attention to this example.

Notice that the same example is also a Krull algebra, since it is a unital Q'-algebra (see Examples 2.2, 3)).

In the next proposition, we further characterize the Krull property in a semisimple, regular annihilator topological algebra.

PROPOSITION 3.7 *Let E be a semisimple, regular annihilator topological algebra. Then the following are equivalent:*

1) E is a Krull algebra.

2) Every proper closed left (resp. right) ideal is contained in a proper left (resp. right) annihilator ideal.

PROOF 1) $\Rightarrow$ 2) : If I is a proper closed left ideal in E, then by assumption, $I\subseteq M$ for some closed maximal regular left ideal of the form $M = E(1 - x)$, with x a minimal (idempotent) element in E (see also the comments before Proposition 3.3). Thus $I\subseteq\mathcal{A}_l(xE)$. A similar proof holds for "right".

2) $\Rightarrow$ 1) : Let I be a non-zero proper closed left ideal in E. Then, $I\subseteq\mathcal{A}_l(J)$, where J is a (non-empty) subset of E. Obviously, $J\neq(0)$ and $J\neq E$. So, since $J\subseteq\mathcal{A}_r(\mathcal{A}_l(J))\subseteq\mathcal{A}_r(I)$, $\mathcal{A}_r(I)\subsetneq R(E)$. Hence, by [3: p. 569, the proof of Theorem 4.2, (3)], $I\subseteq\mathcal{A}_l(\mathcal{A}_r(I))\subseteq E(1-x)$ with $E(1-x)$ a maximal regular left ideal, which is closed, since x is idempotent. Similarly, for right ideals. ■

Given an algebra E, $r_E(x)$ denotes the *spectral radius* of an element $x \in E$.

COROLLARY 3.8 *Let $(E, (p)_\alpha)_{\alpha\in A})$be a Hausdorff regular annihilator locally convex algebra, such that*

$$p_\alpha(x) \leq r_E(x); for \;\; every \;\; x \in E, \alpha \in A. \tag{4}$$

Suppose that every proper closed left (right) ideal is contained in a proper left (right) annihilator ideal. Then E is a semisimple annihilator Krull (D)Q'-algebra (with dense socle).

Proof For $x \in R(E)$, $r_E(x) = 0$, provided that $R(E) \subseteq \{x \in E : r_E(x) = 0\}$ (see [5: p. 126, Proposition 1]; this is stated for Banach algebras, but its proof is purely algebraic). Thus, $x = 0$ and hence E is semisimple. By Corollary 3.5 and Proposition 3.7, E is a Krull Q'-algebra. Moreover, E as a regular annihilator Q'-algebra, is a weakly regular annihilator algebra (see Remarks after Definition 3.1) and hence it is an annihilator algebra, as well (see Remarks after Theorem 3.2). Theorem 3.14 below, completes the proof (see also [3: p. 571, Theorem 5.2]). ∎

Scholium 3.9 By the above proof, the assertion of Corollary 3.8 remains valid, if (4) holds true for every $x \in R(E)$, and $\alpha \in A$.

We are interested now in conditions under which the Krull (resp. Q'-)property is preserved via algebra morphisms. In what follows, analogous results hold for the left Krull (right Krull, Q'_l or Q'_r-)properties.

Lemma 3.10 *Let E, F be topological algebras and $\varphi : E \to F$ an algebra epimorphism. Then the following hold:*

i) If φ is a continuous closed map and E is a Krull algebra, then F is a Krull algebra.

ii) Suppose that F is a Krull algebra and φ a continuous closed map, such that $ker(\varphi) \subsetneqq I$ for every proper closed left or right ideal I in E. Then E is a Krull algebra.

iii) If φ is a closed map and E is a Q'-algebra, F is a $Q'-$algebra.

iv) Suppose that F is a Q'-algebra and φ continuous such that $ker(\varphi) \subseteq M$for every maximal regular (left) ideal M in E. Then E is a Q'-algebra.

Proof i) If I is a proper closed left ideal in F, then $\varphi^{-1}(I)$ is a closed left ideal in E with $ker(\varphi) \subseteq \varphi^{-1}(I)$ (see, for instance, [7: p. 316, Proposition B.5.4]) so that $\varphi^{-1}(I) \subseteq M$ for some closed maximal regular left ideal M in E. Hence $ker(\varphi) \subseteq M$ and $I = \varphi(\varphi^{-1}(I)) \subseteq \varphi(M)$, so that, by the assumption for φ, $\varphi(M)$ is a closed maximal regular left ideal in F.

ii) If $J \subset E$ is a proper closed left ideal, then $\varphi(J)$ is a proper closed left ideal in F (see also [ibid.]). Hence $\varphi(J) \subseteq N$ for some closed maximal regular left ideal in F. Therefore, $J = \varphi^{-1}(\varphi(J)) \subseteq \varphi^{-1}(N)$, with $\varphi^{-1}(N)$ a closed maximal regular left ideal in E.

iii) If M is a maximal regular (left) ideal in F, then [ibid.], $\varphi^{-1}(M)$ is a maximal regular (left) ideal in E, such that $ker(\varphi) \subseteq \varphi^{-1}(M)$. By hypothesis, $\varphi^{-1}(M)$ is closed. Thus, $M = \varphi(\varphi^{-1}(M))$ is closed.

iv) If N is a maximal regular (left) ideal in E, $\varphi(N)$ is a maximal regular (left) ideal in F [ibid.], hence closed. Therefore, $N = \varphi^{-1}(\varphi(N))$ is closed, as well. ∎

Theorem 3.11 *Let E be a topological algebra and I a 2-sided ideal in E. Then, the following hold.*

i) If E is a Q'-algebra, E/I is also.

ii) If $I \subseteq R(E)$ and E/I is a Q'-algebra, the same is true for E.

iii) If E is a Krull algebra so that E/I is a Q'-algebra, then E/I is a Krull algebra.

iv) If E is a Krull Q'-algebra, E/I is an algebra of the same type with E.

PROOF Notice first that E/I is equipped with the quotient topology.

i) Consider the (canonical) continuous epimorphism $\pi : E \longrightarrow E/I$ and M a maximal regular (left) ideal in E/I. Then $\pi^{-1}(M)$ is a maximal regular (left) ideal in E (cf. e.g. [7: p. 316, Proposition B.5.4]), hence closed by hypothesis, so that M is closed, as well.

ii) Let N be a maximal regular (left) ideal in E. By assumption, $I \subseteq N$ and hence $ker(\pi) \subseteq N$. Now, by the continuity of π and Lemma 3.10, E is a Q'-algebra.

iii) Since π is a continuous epimorphism, E/I is Krull (see [15: p. 197, Proposition 1.2]).

iv) Immediate from i) and iii). ∎

COROLLARY 3.12 *A topological algebra E is a Q'-algebra if and only if $E/R(E)$ is a Q'-algebra.*

Thus, there is *no restriction by looking at semisimple $Q'-$algebras.*

If the cartesian product and the direct sum of topological algebras with the usual topologies are Q'-algebras, then the same holds for the factor algebras, whenever the corresponding projection maps are closed. For the converse it suffices only one of the factors to be a Q'-algebra having also the property that the kernel of the respective projection map is contained in the (Jacobson) radical of the product or direct sum algebra. An analogous statement is valid for a projective limit (topological) algebra, provided the restrictions of projection maps on the projective limit algebra be onto (cf. also [15: p. 199, Definition 2.7]).

Now, we also note that *the inverse limit of Banach algebras may even be not a Q'-algebra,* as the following example shows. Let E be a unital commutative complete (locally) m-convex algebra with non-compact spectrum (provided with the weak star topology; take for instance, the algebra $\mathcal{C}_c(X)$ with X a k-space, see Examples 2.2). Consider the Arens-Michael decomposition $E = \varprojlim \hat{E}_\alpha$ of E (cf. [22: p. 91, relation (3.32)]). The $\hat{E}_\alpha$'s, being Banach algebras, have the Q'-property, while E is not a Q'-algebra, since it contains a dense maximal (regular) ideal (see [12: p. 149, Proposition 1.3] and [33: p. 296, Proposition 1]).

SCHOLIUM 3.13 Concerning Theorem 4.3 in [3: p. 569], Barnes gives some equivalent aspects for an algebra to be right regular annihilator. In this context, we remark that, *if E is a unital advertibly complete locally m-convex algebra with spectrum $\mathfrak{M}(E)$, having no nilpotent left or right ideals and its socle $\mathfrak{S}$ contains invertible elements, then the hull $h(\mathfrak{S})$ of $\mathfrak{S}$ is empty.* In fact, if $x \in \mathfrak{S}$ is an invertible element, then $f(x) \neq 0$ for every $f \in \mathfrak{M}(E)$ (see, for instance, [22: p. 98, Corollary 5.2, see also its proof]). Thus,

$$h(\mathfrak{S}) = \{f \in \mathfrak{M}(E) : ker(f) \supseteq \mathfrak{S}\} = \emptyset$$

(see also [ibid.: p. 330, Definition 1.1]). Therefore, by the above argument and [3: p. 569, Theorem 4.3 and p. 570, (4.2)], *E is a regular annihilator Q'-algebra.* Now, based on [ibid. p. 571, Theorem 5.2. (1)] it is easily seen that *a topological algebra, as before, has either a dense socle or $h(I)=\emptyset$ for some proper closed 2-sided ideal I.* Finally, from [29: p. 7, Theorem 6], see also [22: p. 106, Theorem 6.3] *for a commutative Hausdorff locally m-convex $\mathbb{C}$-algebra the statements "Q-algebra", "advertibly complete algebra with equicontinuous spectrum" and "Q'-algebra with equicontinuous spectrum" are equivalent.*

i) and v) of the next theorem are included in [3: p. 571, Theorem 5.2]. Here, they are reworded in our terminology. The same theorem deals with further properties of a *semisimple annihilator* Q'_l (or Q'_r)*−algebra.* For more information concerning this type of algebras see also [12]. We note that such topological algebras are of interest for instance, in *Quantization Theory* (see e.g. [17]. My thanks are due to Prof. A. Mallios for drawing my attention to that paper). In this context, we also remark that M.A. Hennings proves in [ibid. p. 185, Proposition 4] that, *for any maximal closed left ideal M in a semisimple annihilator Q-*-algebra, the right annihilator $\mathcal{A}_r(M)$ is a (non-zero) minimal right ideal.* This is still valid for more general topological algebras; take, for instance, *a semisimple annihilator Q'_l-algebra* (see the next result and [12: p. 152, Theorem 3.4]).

THEOREM 3.14 *Let E be a semisimple annihilator Q'_l (or Q'_r)-algebra. Then*
i) E is a regular annihilator algebra.
ii) E is a Q'-algebra.
iii) E is a (D)-algebra.
iv) E has a dense socle.
v) E is a Krull algebra.

PROOF By [3: p. 571, Theorem 5.2], E is regular annihilator and hence a $(D)Q'$-algebra. (The properties i)-iii) are actually equivalent; see the comments following Corollary 3.5). Besides, from [3: p. 571, Theorem 5.2], and the Q'-property, E is a Krull algebra, as well; in this concern, we also note that *in any topological algebra E every closed maximal regular (left) ideal (* equivalently, *every closed regular maximal (left) ideal) is a maximal regular closed (left) ideal. The converse is true, if E is,* for instance*, a $Q'-$algebra.* Moreover, *an annihilator Q'-algebra is semisimple if and only if, it has a dense socle* (see [12: p. 158, Theorem 4.3]) and this completes the proof. ■

SCHOLIUM 3.15 In this context, we remark that *the hypothesis* in Theorem 3.14 *of being the algebra Q'_l (or Q'_r) is necessary:* Indeed, Yood remarks that the (Arens) algebra L^ω (cf. e.g. [22]) is an example of *a real unital commutative semisimple annihilator (topological) algebra without closed maximal ideals, hence the algebra in question is not a Q'-algebra neither a Krull one.* Moreover, *L^ω is not a regular annihilator algebra and its socle is trivial* [31: p. 30, Example 2.5 and p. 39, the comments preceding Theorem 3.5]. Moreover, we note that *there are semisimple (normed) Q'-algebras, which are not annihilator algebras and they have not dense* *socle.* Such an example is provided by [2: p. 573, see also p. 575, Lemma 3 and Theorem 1 as well as, its proof]. This same example is, in particular, a closed 2-sided ideal in a semisimple normed annihilator Q'-algebra. But, in case of a *minimal* closed 2$-$sided ideal in a more general topological algebra, i.e., in a semisimple annihilator Q'-algebra, the aforementioned properties i)-v) of Theorem 3.14 remain valid (see Remark 4.4 below). Finally, we note that *a proper H^*-algebra* [1] *is a (regular) annihilator Krull (D)-algebra with dense socle.* This follows from Theorem 3.14 and the fact that *the algebra in question is semisimple and dual* (cf. e.g. [23: p. 334 and p. 335, II]). Concerning semisimplicity, see the comments following Proposition 4.1 below. For an analogous, more general situation, see also Proposition 4.1, i) below, where semisimplicity is assumed. Concerning the density of the socle in a topological algebra, we also note that *a regular annihilator Pták-Šilov-Wiener-Tauber algebra E with minimal ideals and equicontinuous spectrum has a dense socle.* In fact, by semisimlicity of E, the socle $\mathfrak{S}$ is defined (see Lemma 1.1, [22: p. 334, Definition 2.3] and [24: p. 46, Lemma 2.1.12]). Now, if $E \neq \overline{\mathfrak{S}}$, $h(\overline{\mathfrak{S}}) \neq \emptyset$ (see, for instance, [22: p. 350, Theorem 6.1]), a f o r t i o r i, $h(\mathfrak{S}) \neq \emptyset$). Therefore, from [3: p. 569, Theorem 4.3]) E is not regular annihilator, a contradiction.

In this concern we also note that *there are Krull locally m-convex algebras which are not Q'-algebras.* In fact, if X is a (Hausdorff completely regular non-compact) $k-$space and $\mathcal{C}_c(X)$ the complete locally m-convex algebra of all continuous $\mathbb{C}$-valued functions on X, in the compact open topology, then $\mathcal{C}_c(X)$ is a Krull algebra (see Examples 2.2). Besides, the same algebra can not be a Q'-algebra. In fact, $\mathfrak{M}(\mathcal{C}_c(X)) = X$ (within a homeomorphism) is not compact (see [22 : p. 223, Theorem 2.1)]. Hence $\mathcal{C}_c(X)$ has a dense maximal ideal (see [33: p. 296, Proposition 1]) and therefore it is not a Q'-algebra (see [12: p. 149, Proposition 1.3]).

Following [27: p. 50] a (non-zero) idempotent element x of a complemented algebra E (see [14: p. 3723, Definition 2.1]) is called a *left* (resp. *right*) *projection,* if the complement of xE (resp. Ex), in E, is $(xE)^{\perp} = (1-x)E$ (resp. $(Ex)^{\perp} = E(1-x)$). If, moreover, x is minimal, it is called a *minimal left (right) projection.* We quote now a result from [27: p. 50, Lemma 1.1] stated for semisimple annihilator complemented Banach algebras.

Lemma 3.16 *Let E be a semisimple annihilator left complemented Q'_l-algebra. Then, every non-zero left ideal I contains a minimal (primitive) right projection. If, in particular, $(0)\neq I \in \mathcal{L}_l$ and $(x_i)_{i\in\Lambda}$ is the family of all minimal (primitive) right projections in I, then $I=\overline{\sum_i Ex_i}$.*

Proof By Theorem 3.14, E is a (D)-algebra. Thus, I contains a minimal left ideal, say J, of the form Ex with x a minimal primitive (idempotent) element and thus closed (see the proof of Theorem 3.6 in [12]). Claim that the complement $J^{\perp}$ of J, in E, is a closed maximal left ideal. In fact, if M is a proper closed left ideal in E with $J^{\perp} \subseteq M$, then $M^{\perp} \subseteq J$. If $M^{\perp} = (0)$, we derive a contradiction. So, since the ideal J is minimal closed, $M^{\perp} = J$ and hence $J^{\perp} = M$. Besides, M is regular (see Theorem 3.14 and [12: p. 152, Theorem 3.4]; see also the proof of Theorem 3.14). Hence $J^{\perp} = E(1-z)$ and $J = J^{\perp\perp} = Ez,\ z \in \mathcal{I}d(E)$ (cf. [14: p. 3729, Theorem 3.9 and Remark 3.10]). By [12: p. 154, Theorem 3.9 and p. 155, Corollary

3.10] z is a minimal (primitive) right projection and this proves the first part of the assertion. Thus, we can consider the family $(x_i)_{i\in\Lambda}$ of all minimal (primitive) right projections in I. Obviously, $\overline{\sum_i Ex_i} \subseteq I$. Put $K = \overline{\sum_i Ex_i}$. If $I \neq K$, there exists some $x \in I$ with $x \notin K$. Since $E = K \oplus K^{\perp}$, $x = y + z$ with $y \in K$, $z \in K^{\perp}$. Thus, $0 \neq z = x - y \in I \cap K^{\perp}$. So, there exists some minimal right projection, say x_0, in $I \cap K^{\perp}$(being a non-zero left ideal) and hence in I, such that $x_0 \notin K$, a contradiction. ■

The next lemma provides conditions for the uniqueness of a complement.

Lemma 3.17 *Let E be a linear space and A, A', B linear subspaces of E, such that the following hold true:*
1) $E = A \oplus A'$.
2) $A' \subseteq B$.
3) $B \cap A = (0)$.
Then $B = A'$.

Proof If $b \in B$, there exist $a \in A$, $a' \in A'$ with $b = a + a'$. Thus, $b - a' = a \in B \cap A$. Therefore, $b = a' \in A'$. ■

In the next result, we collect a number of equivalent expressions of the Krull property in a certain class of topological algebras; see also [27: p. 50, Theorem 3.3].

Theorem 3.18 *Let E be a semisimple complemented $Q_l{}'$-algebra consider the assertions:*
1) E is a Krull algebra.
2) E is an annihilator algebra.
3) Every non-zero left (right) ideal contains a minimal (idempotent) primitive element.
4) Every non-zero left (right) ideal contains a minimal (primitive) right (left) projection.
5) Every closed maximal left (right) ideal is regular.
6) Every closed maximal left (right) ideal has a non-zero right (left) annihilator.
Then 1) $\Leftrightarrow$ 2), 2) $\Rightarrow$ 3),4),5) and 6), 3) $\Rightarrow$ 5), 4) $\Rightarrow$ 5) $\Rightarrow$ 6). If E is moreover, unital, then 6) $\Rightarrow$ 1). Namely, the above assertions are equivalent.

Proof 1) $\Leftrightarrow$ 2): Apply Lemma 1.1, Theorem 3.2 (see also the remarks preceding it), and Theorem 3.14.

2) $\Rightarrow$ 3): Let $I \neq (0)$ be a left ideal in E. By Theorem 3.14, E is a $(D)-$algebra. Thus, I contains a minimal left ideal of the form Ex with x a minimal primitive element of E (see also [12: p. 153, the proof of Theorem 3.6]).

2) $\Rightarrow$ 4): See Lemma 3.16.

4) $\Rightarrow$ 2): It is trivial.

2) $\Rightarrow$ 5): From [12: p. 152, Theorem 3.4], every maximal closed left ideal M has the form $E(1-x)$ with x a primitive idempotent. Hence M is regular.

2) $\Rightarrow$ 6): It is obvious.

4) $\Rightarrow$ 5): Let M be a closed maximal left ideal. Then the left ideal $M^{\perp}$ is minimal. In fact, if I is a non-zero left ideal, such that $I \subseteq M^{\perp}$, there exists a minimal (primitive) right projection, say y, with $Ey \subseteq I$. Since $I^{\perp} \subseteq (Ey)^{\perp} = E(1-y)$, $M \subseteq E(1-y)$. Moreover, $E(1-y) \neq E$, otherwise, $yE = (0)$. In particular, $y = 0$, a contradiction. Thus, $M = E(1-y)$.

3) $\Rightarrow$ 5): Repeat the reasoning following in the proof of 4) $\Rightarrow$ 5). Here, we use the fact that $(Ez)^{\perp} = E(1-z)$ with z a minimal primitive (idempotent). Indeed, by Peirce decomposition and complementation in E, we get $E = Ez \oplus E(1-z) = Ez \oplus (Ez)^{\perp}$. Thus, $(Ez)^{\perp} \subseteq E(1-z)$ and since $Ez \cap E(1-z) = (0)$, we get (Lemma 3.17), $(Ez)^{\perp} = E(1-z)$.

5) $\Rightarrow$ 6): Let M be a closed maximal left ideal. M, as regular, is Peirce decomposable (cf. [14: p. 3729, Theorem 3.9 and Remarks 3.10]). Therefore $\mathcal{A}_r(M) \neq (0)$.

A similar argument establishes the analogous results for right ideals. Now, suppose that E is moreover, unital. Then we prove that 6) $\Rightarrow$ 1). Indeed, by assumption, every closed maximal (regular) left ideal has non-zero right annihilator. Thus, E is a (weakly) regular annihilator algebra on the left (see also the remarks preceding Theorem 3.2). Analogously, for right ideals. Namely, E is a $(D)Q'$-algebra (see Lemma 1.1, Proposition 3.3, as well as [3: p. 569, Theorem 4.2]). Hence, according to Examples 2.2, 3), E is a Krull algebra. ∎

In the next proposition by a *pre-C*-algebra* we mean a ($\mathbb{C}$-)normed algebra with an involution satisfying the C^*-condition. For its proof apply Corollary 3.5, Theorem 3.14 as well as [24: p. 188, Theorem 4.1.19] and [28: p. 263, Corollary 4.3].

Proposition 3.19 *Let E be a regular annihilator pre-C*-algebra, in which every minimal left ideal is complete. Then E is a semisimple dual Krull $(D)Q'-$algebra (with dense socle).*

We end this section by giving a certain class of topological algebras in which the notions "annihilator algebra" and "dual algebra" coincide.

Theorem 3.20 *Let E be a topological algebra with the following properties:*

i) $x^2 = 0$ if and only if x=0, for every $x \in E$.

ii) $\mathcal{A}_l(I) = \mathcal{A}_r(I)$ for every $I \in \mathcal{L}_l \cup \mathcal{L}_r$ (namely, E is isotropic).

iii) The ideal $I + \mathcal{A}_l(I)$ $(= I + \mathcal{A}_r(I))$ is closed for every $I \in \mathcal{L}_l \cup \mathcal{L}_r$.

Then the following assertions are equivalent:

1) E is an annihilator algebra.

2) E is a dual algebra.

3) E is complemented with complementor the annihilator operator $\mathcal{A}_l (= \mathcal{A}_r)$.

Proof 1) $\Rightarrow$ 2) : Claim that

$$E = I \oplus \mathcal{A}_l(I),\ I \in \mathcal{L}_l. \tag{5}$$

Indeed, if $x \in I \cap \mathcal{A}_l(I)$, then $x^2 = 0$ and hence $x = 0$. Thus, $I \cap \mathcal{A}_l(I) = I \cap \mathcal{A}_r(I) = (0)$ (see also ii)). Consider the closed left ideal $K = I \oplus \mathcal{A}_l(I) = I \oplus \mathcal{A}_r(I)$. If

$y \in \mathcal{A}_r(K)$, $Ky = (0)$ and thus, $Iy = (0)$, $\mathcal{A}_r(I)y = (0)$. Therefore, $y^2 = 0$ and thus, $y = 0$. So, since E is an annihilator algebra, $K = E$. Now, by ii) and (5), we get $E = \mathcal{A}_r(I) \oplus \mathcal{A}_l(\mathcal{A}_r(I))$. Consequently, by $I \subseteq \mathcal{A}_l(\mathcal{A}_r(I))$ and $\mathcal{A}_l(I) \cap \mathcal{A}_l(\mathcal{A}_r(I)) = \mathcal{A}_r(I) \cap \mathcal{A}_l(\mathcal{A}_r(I)) = (0)$, we get that $I = \mathcal{A}_l(\mathcal{A}_r(I))$ (see Lemma 3.17). In the same way, $J = \mathcal{A}_r(\mathcal{A}_l(J))$ for every $J \in \mathcal{L}_r$. The above argument shows that E is a dual algebra.

1) $\Rightarrow$ 3) : The mapping

$$\perp := \mathcal{A}_l : \mathcal{L}_l \longrightarrow \mathcal{L}_l : I \mapsto I^{\perp} := \mathcal{A}_l(I)(= \mathcal{A}_r(I))$$

defines a left complementor on E (see [15], see also the proof of 1) $\Rightarrow$ 2)). Similarly for the "right".

Since a dual algebra is an annihilator one, 2) $\Rightarrow$ 3). Clearly, 3) $\Rightarrow$ 2) $\Rightarrow$ 1) and this completes the proof. ∎

The previous theorem looks rather restricted, but notice that properties i)-iii) are for instance, fulfilled for any commutative locally C^*-algebra E. Indeed, E as topologically semiprime, satisfies i). For condition iii) see the proof of Theorem 3.1 in [16: p. 226; here, we use the fact that the closed ideals in E are self-adjoint]; see also [19: p. 209, Theorem 2.7]. Besides, if X is a discrete space, then $\mathcal{C}_c(X)$ is a commutative annihilator locally C^*-algebra (see Examples 2.2, 6) and [14: p. 3724, Example 2.6]).

4. Orthocomplemented and Krull algebras. Subalgebras of Krull algebras.

ii) of the following result concerns a special class of *locally convex H^*-algebras.* For the terminology applied, we refer to [13].

Proposition 4.1 *Let E be an orthocomplemented algebra. Then the following hold.*

i) If E is a semisimple Hausdorff Q'_l-algebra, then it is a dual (regular annihilator) Krull (D)Q'-algebra with dense socle.

ii) If E is a non-radical (bs) Ambrose Q'-algebra, then it is a semisimple dual ((regular) annihilator) Krull (D)-algebra with dense socle.

Proof i) This is immediate from Lemma 1.1, Theorem 3.14 and [10: p. 458, Theorem 3.9].

ii) By [13: p. 144, Theorem 3.1 and p. 145 Theorem 3.3, see also the Scholium following it][1], E contains a family (of axes), say $(x_i)_{i\in\Lambda}$, such that

$$E = \overline{\oplus^{\perp}_{i\in\Lambda} Ex_i} = \overline{\oplus^{\perp}_{i\in\Lambda} x_i E}. \tag{6}$$

That is, E is the topological orthogonal direct sum of the minimal closed ideals Ex_i, x_iE, $i \in \Lambda$ respectively. Moreover, E contains minimal left ideals (see [9: p. 965, the comments after Theorem 2.3]). Let $(L_k)_{k\in K}$ be the family of all minimal left ideals of E. Since $x_i \in \mathcal{I}d(E)$, $i \in \Lambda$, the minimal closed left ideal Ex_i is also a minimal left ideal (cf. [11: p. 1176, Lemma 1.1]). Thus, the Ex_i's belong to the

[1] The quoted results from [13] are actually valid *without the condition (PH) left or right.*

family $(L_k)_{k\in K}$ and hence $\sum_{i\in\Lambda} Ex_i \subseteq \mathfrak{S}_l$. Similarly, $\sum_{i\in\Lambda} x_iE \subseteq \mathfrak{S}_r$; (Here $\mathfrak{S}_l$, $\mathfrak{S}_r$ denote the left and right socles of E). Thus, (see also (6)), E has a dense socle. Besides, by [10: p. 458, Theorem 3.9], E is a dual algebra *a fortiori* an annihilator one. Therefore (see the proof of Theorem 3.14), E is semisimple, and hence, by the same theorem (see also Examples 2.2, 7)), we get the assertion. ∎

A proper H^*-(Banach) algebra E [1] is a (bs) Ambrose orthocomplemented Q'-algebra (see [11: p. 1182, the comments following Definition 4.1] and [13: p. 143, the comments before Lemma 2.5]). So, E considered as a non-radical algebra, has all the properties referred in ii) of Proposition 4.1. In this respect, we also note that E is, in particular, semisimple getting thus another proof (avoiding "regular representations" as in [23: p. 334]).

We deal now with appropriate subalgebras of certain Krull algebras.

It is obvious that *any subalgebra F (equipped with the relative topology) of a left Krull algebra E, so that $\mathcal{L}_l(F) \subseteq \mathcal{L}_l(E)$ and $\mathfrak{M}_l(E)\cap F \subseteq \mathfrak{M}_l(F)$ is a left Krull algebra. Similarly, on the right.* As a realization, we get the next.

Proposition 4.2 *Let E be a topologically semiprime, precomplemented, Krull algebra. Then every $I \in \mathcal{L}$ is a Krull algebra.*

Proof Our first task is to show that the ideals $I\oplus\mathcal{A}_l(I)$ and $I\oplus\mathcal{A}_r(I)$ are dense in E. Then it will follow that $\mathcal{L}_l(I)\subseteq\mathcal{L}_l(E)$ and $\mathcal{L}_r(I)\subseteq\mathcal{L}_r(E)$. In fact, it is clear that $(I\cap\mathcal{A}_l(I))^2=(0)$. Thus, since E is topologically semiprime, it follows $I\cap\mathcal{A}_l(I)=(0)$. Inasmuch as the ideals I and $\mathcal{A}_l(I)$ are 2-sided, $I\mathcal{A}_l(I)\subseteq I\cap\mathcal{A}_l(I)$ and hence $\mathcal{A}_l(I)\subseteq\mathcal{A}_r(I)$. In the same way, $\mathcal{A}_r(I)\subseteq\mathcal{A}_l(I)$. Therefore $\mathcal{A}_l(I)=\mathcal{A}_r(I)$. Consider the 2-sided ideal $K=I\oplus\mathcal{A}_l(I)$; clearly $\mathcal{A}_l(K)\subseteq\mathcal{A}_l(I)\subseteq K$ and thus $\mathcal{A}_l(K)\subseteq\mathcal{A}_l(\mathcal{A}_l(K))$. It follows $\mathcal{A}_l(K)^2=(0)$ and hence $\mathcal{A}_l(K)=(0)$. This implies $\mathcal{A}_l(\overline{K})=(0)$ with $\overline{K}\in\mathcal{L}$. If $\overline{K}'$ is a complement of $\overline{K}$ in E, then $\overline{K}\,\overline{K}'\subseteq\overline{K}\cap\overline{K}'$ and hence $\overline{K}'\subseteq\mathcal{A}_r(\overline{K})$. Besides, by applying an argument as above, we get $\mathcal{A}_r(\overline{K})\cap\overline{K}=\mathcal{A}_l(\overline{K})\cap\overline{K}=(0)$. It follows from Lemma 3.17, $\overline{K}'=\mathcal{A}_r(\overline{K})$. Therefore

$$E=\overline{I\oplus\mathcal{A}_l(I)}=\overline{I\oplus\mathcal{A}_r(I)}.$$

So, if $J\in\mathcal{L}_l(I)$, we get

$$EJ\subseteq\overline{I\oplus\mathcal{A}_l(I)}\,\overline{J}\subseteq\overline{IJ}\subset J.$$

Thus, $\mathcal{L}_l(I)\subseteq\mathcal{L}_l(E)$. Similarly, $\mathcal{L}_r(I)\subseteq\mathcal{L}_r(E)$.

Now, let J be a proper closed left ideal in I. Since J is also a proper closed left ideal in E, the Krull property implies that $J\subseteq M$ for some $M\in\mathfrak{M}_l(E)$. Thus, $J\subseteq M\cap I$. Claim that $M\cap I\in\mathfrak{M}_l(I)$. Obviously, $M\cap I$ is a regular left ideal in I. Suppose that $M\cap I\subsetneq K\subseteq I$ with $K\in\mathcal{L}_l(I)$. But, $K\in\mathcal{L}_l(E)$, as well. So, since $E=I\oplus I'$, we get $K\cap I'\subseteq I\cap I'=(0)$. As a consequence of Lemma 3.17, we take $K=I$. Therefore I is a left Krull algebra. A similar argument is applied on the right, and this completes the proof. ∎

ii) in the next proposition shows that the minimal closed 2-sided ideals (basic in structure theory) of a semisimple annihilator Q'_l -algebra, are rich enough in algebraic-topological properties. But, see also the comments following Corollary 3.5.

PROPOSITION 4.3 *Let E be a semisimple regular annihilator Krull algebra and $I \in \mathcal{L}$. Then, I is an algebra of the same type as E in each one of the following cases:*

i) I satisfies the relation $\overline{EI} = I$.

ii) I is a minimal closed 2-sided ideal.

PROOF Let us first observe that *a semisimple topological algebra is an annihilator Q'_l (or Q'_r)-algebra if and only if, it is a regular annihilator, Krull algebra* (see [3: p. 571, Theorem 5.2], as well as Lemma 1.1, Theorem 3.2, Corollary 3.5, the comments after it and the proof of Theorem 3.14).

i) Since $R(I) = I \cap R(E)$ (cf. e.g. [5: p. 126, Corollary 20]), I is semisimple. Moreover, *an annihilator Q'-algebra is semisimple if and only if, it has a dense socle* (see [12: p. 158, Theorem 4.3]), so by [28: p. 258, Theorem 3.3], $\overline{EI} = I$ is equivalent to $\overline{IE} = I$. Thus, from Theorem 4.4 in [12: p. 158], I is an annihilator algebra, being by [4: p. 659, C7] regular annihilator and hence (Corollary 3.5) a Q'-algebra. Theorem 3.14 implies that I is a Krull algebra, as well.

ii) By [12: p. 160, Corollary 4.10], I is a semisimple annihilator algebra. For the Q'-property, see the proof of i) above. ∎

REMARK 4.4 It follows from Theorem 3.14 that an algebra E as in Proposition 4.3 is, in fact, a $(D)Q'$-algebra with dense socle. Thus the ideal I shares the same properties with E. Furthermore, by [12: p. 161, Theorem 4.12] (see also Theorem 3.14) *a semisimple annihilator Q'_l (or a Q'_r)$-$algebra is the topological direct sum of its minimal closed 2-sided ideals which are semisimple topologically simple (regular) annihilator Krull $(D)Q'$-algebras with dense socle.* On the other hand, *if a preannihilator topological algebra E is the topological sum of a given family of closed 2-sided ideals, which are annihilator algebras, then E is an annihilator algebra, as well.* This result is stated in [24: p. 106, Theorem 2.8.29], for semisimple Banach algebras, but the proof is still valid in the more general case as above.

5. The Krull property in tensor product topological algebras. Our intention, in this section, is to give conditions under which the Krull property of a topological tensor product algebra is inherited to its tensor factors. In particular, for certain X and E the algebra-valued function topological algebra $\mathcal{C}_c(X, E)$, is a Krull algebra.

Let E, F be topological algebras and $E \underset{\tau}{\otimes} F$ the resulted tensor product topological algebra with respect to a compatible topology τ (see [22: p. 382, Definition 4.1]). Consider the spectrum $\mathfrak{M}(E)$; namely the set of all non-zero continuous multiplicative linear functionals on E. Under this notation, we have the next.

Proposition 5.1 *Let E be a unital topological algebra with spectrum $\mathfrak{M}(E)$ and F a $Q'-$algebra. Suppose that $E \underset{\tau}{\otimes} F$ is a Krull algebra. Then F is a Krull algebra.*

Proof Let f be an element in $\mathfrak{M}(E)$. Consider the continuous algebra homomorphism

$$\theta_f : E \underset{\tau}{\otimes} F \to F, \theta_f(z) = \sum\nolimits_{i=1}^{n} f(x_i)y_i, \; with\ z = \sum\nolimits_{i=1}^{n} x_i \otimes y_i \tag{7}$$

(see [22: p. 361, Lemma 1.4, p. 441, Lemma 4.1 and the comments preceding it]). If 1_E is the unit element in E, then for $y \in F$, $1_E \otimes y \mapsto \theta_f(1_E \otimes y) = y$. Namely, θ_f is finally, an epimorphism. The assertion now follows from [15 : p. 197, Proposition 1.2, (ii)]. ■

Notice that *if E and F are unital Q'-algebras, then they are Krull* (see Examples 2.2, 3)). So, *if $E \underset{\tau}{\otimes} F$ is a Q'-algebra, then it is a Krull algebra.*

An analogous result to Proposition 5.1 can also be considered for the completion $E \hat{\underset{\tau}{\otimes}} F$ of $E \underset{\tau}{\otimes} F$, by taking locally convex algebras E and F with continuous multiplication. $E \hat{\underset{\tau}{\otimes}} F$ is actually, a (complete) locally convex algebra with continuous multiplication, as well. By [22: p. 447, Lemma 5.1] for $f \in \mathfrak{M}(E)$, we consider the continuous homomorphism $\bar{\theta_f} : E \hat{\underset{\tau}{\otimes}} F \to \hat{F}$, (the extension of θ_f; see (7)), which is onto, if E is unital. Thus, we get the next.

Proposition 5.2 *Let E be a unital locally convex algebra with continuous multiplication and spectrum $\mathfrak{M}(E)$ and F a locally convex Q'-algebra with continuous multiplication. Then $\hat{F}$ is a Krull algebra, if $E \hat{\underset{\tau}{\otimes}} F$ is so. Thus, F is Krull if, in particular, it is complete.*

Let X be a topological space and $(E, (p_\alpha)_{\alpha \in A})$ a locally m-convex *-algebra. Consider the locally m-convex *-algebra $\mathcal{C}_c(X, E)$ of all E-valued continuous functions on X (pointwise defined operations, involution given by $f^*(x) = f(x)^*, f \in \mathcal{C}_c(X, E), x \in X$ and the topology "c" of compact convergence in X, defined by the seminorms

$$q_{\alpha,K}(f) := \sup_{x \in K} \{p_\alpha(f(x))\}, f \in \mathcal{C}(X, E), \alpha \in A, K \subset X \ compact \tag{8}$$

see also [22: p. 387]).

We give now conditions under which $\mathcal{C}_c(X, E)$ is a Krull algebra. Moreover, we see when E inherits the Krull property from $\mathcal{C}_c(X, E)$.

Proposition 5.3 *Let X be a completely regular k-space.*

1) If $(E, (p_\alpha)_{\alpha \in A})$ is a locally C^*-algebra, then $\mathcal{C}_c(X, E)$ is Krull.

2) If $(E, (p_\alpha)_{\alpha \in A})$ is a complete locally m-convex Q'-algebra and the algebra $\mathcal{C}_c(X, E)$ is Krull, then E is Krull.

PROOF 1) By Theorem 1.1 in [22: p. 391], $\mathcal{C}_c(X,E)$ is a complete locally m-convex algebra. Besides, for any $f \in \mathcal{C}(X,E)$, $\alpha{\in}A$ and $K \subset X$ compact, we get (see also (8))

$$q_{\alpha,K}(f^*f) = \sup_{x\in K}\{p_\alpha(f^*f)(x)\} = \sup_{x\in K}\{p_\alpha(f(x))^2\} = q_{\alpha,K}(f)^2.$$

Namely, $\mathcal{C}_c(X,E)$ is a locally C^*-algebra and hence a Krull one (see Examples 2.2, 4)).
2) $\mathcal{C}_c(X,E) = \mathcal{C}_c(X)\hat{\underset{\varepsilon}{\otimes}}E$ within an isomorphism of topological algebras (here ε denotes the biprojective tensor product topology; see, for instance [22: p. 371, Definition 2.3 and p. 391, Theorem 1.1]). By assumption and Corollary 1.3 in [15: p. 197], $\mathcal{C}_c(X)\hat{\underset{\varepsilon}{\otimes}}E$ is a Krull algebra. Since $\mathcal{C}_c(X)$ is a unital commutative complete locally m-convex algebra (see Examples 2.2), $\mathfrak{M}(\mathcal{C}_c(X))$ is non-empty (actually, $\mathfrak{M}(\mathcal{C}_c(X)) = X$, within a homeomorphism of topological spaces; see [22: p. 223, Theorem 1.2]). Thus, the assertion follows from Proposition 5.2. ■

We consider now similar statements as in 2) of the previous proposition, for two more algebra-valued function topological algebras:

i) Let X be a $\mathcal{C}^\infty$-manifold, U an open subset of X and E a complete locally convex Q'-algebra. Consider the complete locally convex algebra $\mathcal{C}^\infty_c(U,E)$ of all E-valued $\mathcal{C}^\infty$-maps on U, with respect to the topology of compact convergence of the functions and all of their derivatives. Then

$$\mathcal{C}^\infty_c(U,E) = \mathcal{C}^\infty_c(U)\hat{\underset{\varepsilon}{\otimes}}E (= \mathcal{C}^\infty_c(U)\hat{\underset{\pi}{\otimes}}E)$$

within a topological algebra isomorphism (: π denotes the *projective tensorial topology*; see [22: p. 366 and p. 394, Theorem 2.1]). Besides, $\mathcal{C}^\infty_c(U)$ is the unital commutative complete locally m-convex algebra of all complex $\mathcal{C}^\infty$-functions on U [ibid. p. 129, 4(2)]. Thus, if $\mathcal{C}^\infty_c(U,E)$ is a Krull algebra, then E is a Krull algebra, as well.

ii) Let G be an abelian discrete locally compact group and E a complete locally m-convex Q'-algebra. Consider *the generalized group algebra* $L^1(G,E)$ being in particular, a (commutative) complete locally m-convex algebra [ibid. p. 405, Theorem 5.1]. Suppose $L^1(G,E)$ is a Krull algebra. Since

$$L^1(G,E) = L^1(G)\hat{\underset{\pi}{\otimes}}E$$

within an isomorphism of topological algebras [ibid. p. 406], we get that E is a Krull algebra. Notice that $L^1(G)$, as a commutative unital Banach algebra, has a non-empty spectrum and it is a Krull algebra (see Examples 2.2).

Relative to the above study, the following questions came to light:

1. Is the tensor product $E \underset{\tau}{\otimes} F$ of Krull algebras a Krull algebra?
2. Is the completion of a Krull algebra, Krull?

3. Under what conditions the aforementioned algebra-valued function topological algebras are Krull? An instance is given in Proposition 5.3.

4. Is the tensor product of Q'-algebras a Q'-algebra?

Other results concerning the Krull property in tensor product topological algebras hope to be given elsewhere.

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