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A note on integrals of distributions

1. Introduction. The notion of a definite integral of a distribution over the n-dimensional space was introduced by L. Schwartz in [5] and [6]. Let $x = (x_1, \ldots, x_n)$ be a point of the n-dimensional Euclidean space R_n and let us write

$$D^p = \frac{\partial^{p_1+\ldots+p_n}}{\partial x_1^{p_1}\ldots\partial x_n^{p_n}},$$

where $p = (p_1, \ldots, p_n)$ and p_i are non-negative integers. L. Schwartz denotes by $\mathscr{B}$ the B_0-space of all infinitely differentiable functions φ such that all derivatives of φ are bounded, with pseudonorms

$$\|\varphi\|_p = \sup_{x \in R_n} |D^p \varphi(x)|,$$

and by $\dot{\mathscr{B}}$ the subspace of all $\varphi \in \mathscr{B}$ such that $D^p\varphi(x) \to 0$ as $|x| \to \infty$ for every p ([4], Chap. VI, § 8). Now, denoting by $\mathscr{D}'_{L^1}$ the strong dual of $\dot{\mathscr{B}}$, it is proved ([5], § 21, Prop. 1) that $\mathscr{B}$ is the strong dual of $\mathscr{D}'_{L^1}$. In particular, the function $\varphi(x) \equiv 1$ defines a linear functional $\langle T, 1\rangle$ continuous over $\mathscr{D}'_{L^1}$, and this functional is called the integral of the distribution T over R_n; distributions belonging to $\mathscr{D}'_{L^1}$ are called summable. The concept of integrating distributions was developed by J. Mikusiński and R. Sikorski for one-dimensional intervals ([1], 19.1) and by R. Sikorski for many-dimensional intervals ([7], § 2); the starting-point of their investigations is the formula for differentiation of indefinite integrals.

The purpose of this note is to define the integral of a distribution $T \in \mathscr{D}'$ as a set function $\int_\Omega T_x dx$ whose values are distributions, where Ω is a bounded measurable set in R_n, and the integrals $\int_{R_n} T_x dx$ and $\int_{R_n} |T_x| dx$ run over the whole space R_n. It is proved that $\int_{R_n} |T_x| dx$ exists if and only if T is summable in Schwartz's sense; in the case when $T \in \mathscr{D}'_{L^1}$, Schwartz's integral coincides with $\int_{R_n} T_x dx$. On the other hand, 2.3 shows that, in the case of a one-dimensional interval, the definition of $\int_\Omega T_x dx$ given here coincides with the definition of J. Mikusiński and R. Sikorski, given in [1], 19.1, formula (1). The definition of R. Sikorski [7] may also be shown to be equivalent to that given here in the case of integration over R_n; hence, the integrals of Schwartz and those of Sikorski over R_n are also equivalent.

2. Integral over a bounded measurable set. Denote by $\chi_\Omega(x)$ the characteristic function of a bounded measurable set $\Omega \subset R_n$, and let $\check{\chi}_\Omega(x) = \chi_\Omega(-x)$.

Let T be any distribution. Then there exists a convolution $T * \check{\chi}_\Omega$ ([4], Chap. VI, Theorem 1).

DEFINITION 1. For an arbitrary distribution T, the distribution

$$\int_\Omega T_x dx = T * \check{\chi}_\Omega ,$$

is called the *integral of the distribution T over the set Ω*. If Ω is an n-dimensional interval $\xi < x < \eta$, we write

$$\int_\Omega T_x dx = \int_\xi^\eta T_x dx .$$

2.1. *Let $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$. Then*

$$\Big(\int_\Omega T_x dx\Big)(\varphi) = T_y\Big(\int_\Omega \varphi(y-x)\,dx\Big) = \int_\Omega T_y\big(\varphi(y-x)\big)\,dx .$$

Proof. We have

$$\Big(\int_\Omega T_x dx\Big)(\varphi) = (T * \check{\chi}_\Omega)(\varphi) = T_y\big[\check{\chi}_\Omega\big(\varphi(x+y)\big)\big]$$

$$= T_y\big[\chi_\Omega\big(\varphi(y-x)\big)\big] = T_y\Big(\int_\Omega \varphi(y-x)\,dx\Big).$$

However, $T_y\big(\varphi(y-x)\big)$ is an infinitely differentiable function of x, whence

$$\Big(\int_\Omega T_x dx\Big)(\varphi) = (\check{\chi}_\Omega * T)(\varphi) = \check{\chi}_\Omega\big[T_y\big(\varphi(x+y)\big)\big]$$

$$= \chi_\Omega\big[T_y\big(\varphi(y-x)\big)\big] = \int_\Omega T_y\big(\varphi(y-x)\big)\,dx .$$

Remark. In particular, it follows from 2.1 that, if T is a locally integrable function $T(x)$, then the distribution $\int_\Omega T_x dx$ is a function

$$\Big(\int_\Omega T_x dx\Big)(x) = \int_{\Omega+x} T(t)\,dt ,$$

where $\Omega+x$ means the set of vectors of the form $t+x$ with $t\in\Omega$. Moreover, $(\int_\Omega T_x dx)(0) = \int_\Omega T(x)dx$ (in the one-dimensional case, cf. [1], 19.1, formula (3), where the authors define an integral of T over the interval (ξ, η) as the value of the distribution $\int_\xi^\eta T_x dx$ at the point 0).

2.2. *Writing* $p = (p_1, \ldots, p_n)$, $|p| = p_1+\ldots+p_n$, *we have*

$$\Big(\int_\Omega D^p T_x dx\Big)(\varphi) = (-1)^{|p|}\Big[\Big(\int_\Omega T_x dx\Big)(D^p\varphi)\Big]$$

for every $\varphi\in\mathscr{D}$.

In 2.3 and 2.4 we restrict ourselves to the case of a one-dimensional interval. This restriction is essential in 2.3; in 2.4 it is made for the sake of brevity.

2.3. *Let* $T\in\mathscr{D}'$, $S' = T$. *Write* $(\tau_h\varphi)(x) = \varphi(x-h)$, $(\tau_h T)(\varphi) = T(\tau_{-h}\varphi)$ *for* $\varphi\in\mathscr{D}$. *Then*

$$\Big(\int_\xi^\eta T_x dx\Big)(\varphi) = (\tau_{-\eta}S - \tau_{-\xi}S)(\varphi).$$

Proof. Take a $\varphi_0\in\mathscr{D}$ with $\int_{-\infty}^{\infty}\varphi_0(x)dx = 1$. Then

$$S(\varphi) = \lambda T(\varphi_0) - T\Big\{\int_{-\infty}^{x}(\varphi-\lambda\varphi_0)(t)dt\Big\},$$

where $\lambda = \int_{-\infty}^{\infty}\varphi(x)dx$. We have

$$(\tau_{-\eta}S - \tau_{-\xi}S)(\varphi) = S_x[\varphi(x-\eta)-\varphi(x-\xi)].$$

However,

$$\int_{-\infty}^{\infty}[\varphi(x-\eta)-\varphi(x-\xi)]dx = 0,$$

whence

$$(\tau_{-\eta}S - \tau_{-\xi}S)(\varphi) = -T\Big\{\int_{-\infty}^{x}[\varphi(t-\eta)-\varphi(t-\xi)]dt\Big\}$$

$$= T_x\Big(\int_\xi^\eta \varphi(y-x)dy\Big) = \Big(\int_\xi^\eta T_x dx\Big)(\varphi).$$

2.4. *If $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$, then*

$$\frac{\partial}{\partial\eta}\Big[\Big(\int_{\xi}^{\eta} T_x dx\Big)(\varphi)\Big] = T_y(\varphi(y-\eta)), \quad \frac{\partial}{\partial\xi}\Big[\Big(\int_{\xi}^{\eta} T_x dx\Big)(\varphi)\Big] = -T_y(\varphi(y-\xi)).$$

Proof. For instance, the first of the above formulae follows from the formula

$$\Big|\frac{1}{h}\Big[\Big(\int_{\xi}^{\eta+h} T_x dx\Big)(\varphi) - \Big(\int_{\xi}^{\eta} T_x dx\Big)(\varphi)\Big] - T_y(\varphi(y-\eta))\Big|$$
$$= \Big|\int_{\eta}^{\eta+h} T_y\Big[\frac{\varphi(y-x)-\varphi(y-\eta)}{h}\Big]dx\Big|,$$

since the set of functions $\psi_{h,x}(y) = \frac{1}{h}[\varphi(y-x)-\varphi(y-\eta)]$ with $0 \leqslant x \leqslant h$ and $0 < h < 1$ is bounded in $\mathcal{D}$.

3. Integral over R_n. We shall investigate the behaviour of $\int_{\xi}^{\eta} T_x dx$ as $\xi \to -\infty$, $\eta \to \infty$ (i.e. $\xi_1, \ldots, \xi_n \to -\infty$, $\eta_1, \ldots, \eta_n \to \infty$). We adopt the following

DEFINITION 2. We shall write

$$\Big(\int_{R_n} T_x dx\Big)(\varphi) = \lim_{\substack{\xi\to-\infty \\ \eta\to\infty}} \Big[\Big(\int_{\xi}^{\eta} T_x dx\Big)(\varphi)\Big],$$

and call $\int_{R_n} T_x dx$ the *integral of the distribution T over R_n* if the above limit exists for all $\varphi \in \mathcal{D}$.

Remark. If $\int_{R_n} T_x dx$ exists, it is a distribution, and the distributions $\int_{\xi}^{\eta} T_x dx$ tend to $\int_{R_n} T_x dx$ as $\xi \to -\infty$, $\eta \to \infty$, where convergence is understood in the sense of strong convergence in $\mathcal{D}'$ ([4], Chap. III, Theorem 13).

3.1. *Let $n = 1$, $T \in \mathcal{D}'$, and let $S' = T$. The integral $\int_{-\infty}^{\infty} T_x dx$ exists if and only if S has limits at infinity, i.e. $\lim_{\xi\to-\infty}(\tau_{-\xi}S)(\varphi)$ and $\lim_{\eta\to\infty}(\tau_{-\eta}S)(\varphi)$ exist for all $\varphi \in \mathcal{D}$. Moreover,*

$$\int_{-\infty}^{\infty} T_x dx = \lim_{\eta\to\infty}(\tau_{-\eta}S) - \lim_{\xi\to-\infty}(\tau_{-\xi}S).$$

3.2. *The formula*

$$\Big(\int_{R_n} T_x dx\Big)(\varphi) = \int_{R_n} T_y\big(\varphi(y-x)\big)\,dx$$

holds for every $\varphi\in\mathscr{D}$, the left side of the formula existing if and only if the right side exists.

3.3. *If $\int_{R_n} T_x dx$ exists, then $\int_{R_n} D^p T_x dx$ exists also and is equal to* 0.

Propositions 3.1, 3.2 and 3.3 are obvious consequences of 2.3, 2.1 and 2.2, respectively.

DEFINITION 3. The integral $\int_{R_n} |T_x|\,dx$ is the *non-linear functional over $\mathscr{D}$* defined for all $\varphi\in\mathscr{D}$ by the formula

$$\Big(\int_{R_n} |T_x|\,dx\Big)(\varphi) = \int_{R_n} \big|T_y\big(\varphi(y-x)\big)\big|\,dx.$$

3.4. *The integral $\int_{R_n} |T_x|\,dx$ exists if and only if $T\in\mathscr{D}'_{L^1}$.*

Proof. $T\in\mathscr{D}'_{L^1}$ is equivalent to $T*\varphi^{\vee}\in L^1(R_n)$ for every $\varphi\in\mathscr{D}$, where $\varphi^{\vee}(x)=\varphi(-x)$ ([4], Chap. VI, Theorem 25, 2°). But

$$(*)\qquad (T*\varphi^{\vee})(x) = T_y\big(\varphi(y-x)\big),$$

and the existence of $\int_{R_n} |T_x|\,dx$ follows from 3.2. Now assume that the integral $\int_{R_n} |T_x|\,dx$ exists. Then, by $(*)$, $T*\varphi^{\vee}\in L^1(R_n)$, whence $T\in\mathscr{D}'_{L^1}$.

In order to formulate the next theorem, let us note that by [4], Chap. VI, Theorem 25, 1° (cf. also [2], 2.3 (d)), T belongs to $\mathscr{D}'_{L^1}$ if and only if there are functions $f_i\in L^1(R_n)$, $i=0,1,\ldots,m$, such that

$$(**)\qquad T = \sum_{i=0}^{m} D^{p^i} f_i,$$

where $|p^0|=0$, $|p^i|>0$ for $i=1,\ldots,m$.

3.5. *If $T\in\mathscr{D}'_{L^1}$, then $\int_{R_n} T_x dx$ is a constant distribution, and*

$$\int_{R_n} T_x dx = \int_{R_n} f_0(x)\,dx,$$

where f_0 is given by $(**)$.

Proof. Formula 3.3 implies

$$\int_{R_n} T_x dx = \int_{R_n} f_{0x} dx.$$

However, $f_0(x)$ is integrable over R_n, whence we have

$$\Big(\int_{R_n} f_{0x}dx\Big)(\varphi) = \int_{R_n} f_{0y}\big(\varphi(y-x)\big)dx$$

$$= \int_{R_n}\Big(\int_{R_n} f_0(y)\varphi(y-x)\,dx\Big)dy = \int_{R_n} f_0(x)\,dx \int_{R_n}\varphi(y)\,dy,$$

and we infer 3.5.

3.6. *If $T \in \mathscr{D}'_{L^1}$, then the integral $\int_{R_n} T_x dx$ is equal to the integral of T in the sense of Schwartz.*

Proof. If $T \in L^1(R_n)$, the theorem follows from 3.5. However, $L^1(R_n)$ is dense in $\mathscr{D}'_{L^1}$ and so it is sufficient to prove that for every $T \in \mathscr{D}'_{L^1}$ there are $T_i \in L^1(R_n)$, $T_i \to T$ in $\mathscr{D}'_{L^1}$, such that

$$\int_{R_n} T_{ix}dx \to \int_{R_n} T_x dx.$$

Now assume T to be of compact support. Then there is a sequence of $T_i \in \mathscr{D}$ such that $T_i \to T$ in $\mathscr{D}'_{L^1}$ and all supports of T_i and T are contained in a compact. If T is of form (**), then for every set A bounded in $\mathscr{B}^{\cdot}$ we have

$$\int_{R_n} T_i(x)\varphi(x)\,dx \to \sum_{j=0}^{m}(-1)^{|p^j|}\int_{R_n} f_j(x)\,D^{p^j}\varphi(x)\,dx$$

uniformly in A. Hence it easily follows that

$$\int_{R_n} T_i(x)\,dx \to \int_{R_n} f_0(x)\,dx,$$

which proves the theorem in the case of T of compact support.

Finally, let $T \in \mathscr{D}'_{L^1}$ be arbitrary. Given an $\varepsilon > 0$, choose a positive number a such that $\int_{|x|>a} |f_0(x)|\,dx < \varepsilon$, f_0 being defined by (**), and a sequence of $a_i \in \mathscr{D}$ bounded in $\dot{\mathscr{B}}$ and such that $\sup_{|x|\leqslant a}|1-a_i(x)| \to 0$ (for the construction of such a sequence, cf. e.g. [3]). Writing $M = \sup_{|x|<\infty}|a_i(x)|$ and choosing $\varphi \in \mathscr{D}$ so that $\int_{R_n}\varphi(x)\,dx = 1$, we have

$$\Big|\int_{R_n}(1-a_i)T_x dx\Big| < \varepsilon\Big(M+1+\int_{R_n}|f_0(x)|\,dx\Big)$$

for i sufficiently large. This implies

$$(\ast\ast\ast) \qquad \int_{R_n} a_i T_x dx \to \int_{R_n} T_x dx.$$

However, $a_i T$ are of compact support, whence the integrals $\int\limits_{R_n} a_i T_x dx$ are equal to integrals of $a_i T$ in the sense of Schwartz. Now, since the sequence $\{a_i\}$ is bounded in $\dot{\mathscr{B}}$, we have $a_i T \to T$ in $\mathscr{D}'_{L^1}$. Hence ($^*_*{}^*$) implies that $\int\limits_{R_n} T_x dx$ is equal to the integral of T in the sense of Schwartz.

References

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