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Cauchy Multiplication of Euler Summable Series in Ultrametric Fields

Abstract. Euler summability method in a complete, non-trivially valued, ultrametric field of the characteristic zero was introduced by Natarajan in [7]. Some properties of the Euler summability method in such fields were studied in [2] and [7]. The purpose of the present note is to continue the study and to prove a pair of theorems on the Cauchy product of Euler summable sequences and series.

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1. Introduction and Preliminaries. In this note, K is a complete, non-trivially valued, ultrametric field of characteristic zero (Q_p, the p-adic field for a prime p, is one such field). Infinite matrices, sequences and series have entries in K. To make the paper self contained, we recall the following. For a given infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \dots$ and a sequence $x = \{x_k\}$, $x_k \in K$, $k = 0, 1, 2, \dots$, by the A-transform of $x = \{x_k\}$, we mean the sequence $Ax = \{(Ax)_n\}$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, 2, \dots,$$

where we suppose that the series on the right converge. If $\{(Ax)_n\}$ converges to s, we say that $x = \{x_k\}$ is summable A or A-summable to s. If $\lim_{n\to\infty} (Ax)_n = s$ whenever $\lim_{k\to\infty} x_k = s$, we say that A is regular. The following theorem, which gives necessary and sufficient conditions for $A = (a_{nk})$ to be regular in terms of the entries of the matrix A, is well-known (see [4] for a proof using the Banach-Steinhaus theorem and [6] for a proof using the 'Sliding hump method').

THEOREM 1.1 *$A = (a_{nk})$ is regular if and only if*
(i) $\sup_{n,k} |a_{nk}| < \infty$;
(ii) $\lim_{n\to\infty} a_{nk} = 0$, $k = 0, 1, 2, \dots$;
and
(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$.

An infinite series $\sum_{k=0}^{\infty} x_k$, $x_k \in K$, $k = 0, 1, 2, \dots$ *is said to be A-summable to s if* $\{s_n\}$ *is A-summable to s, where* $s_n = \sum_{k=0}^{n} x_k$, $n = 0, 1, 2, \dots$.

General references for the study of summability methods in the classical case are [3, 8]. For analysis in ultrametric fields, see [1].

The Euler method of summability in complete, non-trivially valued, ultrametric fields of characteristic zero was earlier introduced by Natarajan in [7] and a good number of properties of the Euler method were studied in [2, 7]. The purpoe of the present note is to continue the study of the Euler method and prove a pair of theorems on the Cauchy product of Euler summable sequences and series.

DEFINITION 1.2 Let $r \in K$ such that $|1 - r| < 1$. The Euler method of order r or the (E, r) method is given by the infinite matrix $(e_{nk}^{(r)})$, which is defined as follows: If $r \neq 1$,

$$e_{nk}^{(r)} = \begin{cases} {}^nC_k r^k (1-r)^{n-k}, & k \leqslant n; \\ 0, & k > n, \end{cases}$$

where ${}^nC_k = \frac{n!}{k!(n-k)!}$, $k \leqslant n$;
If $r = 1$,

$$e_{nk}^{(1)} = \begin{cases} 1, & k = n; \\ 0, & k \neq n. \end{cases}$$

$(e_{nk}^{(r)})$ is called the (E, r) matrix.

REMARK 1.3 Note that $r \neq 0$, since $|1 - r| < 1$.

THEOREM 1.4 ([7, THEOREM 1.2]) *The (E, r) method is regular.*

THEOREM 1.5 ([7, COROLLARY 1.4]) *The (E, r) matrix is invertible and its inverse is the $(E, \frac{1}{r})$ matrix.*

2. Main Results.

THEOREM 2.1 *If* $x_k = o(1)$, $k \to \infty$, *i.e.*, $x_k \to 0$, $k \to \infty$ *and* $\{y_k\}$ *is* (E, r) *summable to* σ, *then* $\{z_k\}$ *is* (E, r) *summable to*

$$\sigma \left[x_0 + \sum_{k=1}^{\infty} x_k r^{k-1} \right],$$

where $z_n = \sum_{k=0}^{n} x_k y_{n-k}$, $n = 0, 1, 2, \ldots$.

PROOF Let $\{\sigma_n\}$ be the (E, r) transform $\{y_k\}$. Then,

$$\sigma_n = \sum_{k=0}^{n} {}^nC_k r^k (1-r)^{n-k} y_k, \quad n = 0, 1, 2, \ldots. \tag{1}$$

By hypothesis, $\lim_{n\to\infty} \sigma_n = \sigma$. Let $\{\tau_n\}$ be the (E, r) transform of $\{z_k\}$ so that

$$\begin{aligned}
\tau_n &= \sum_{k=0}^{n} {}^nC_k r^k (1-r)^{n-k} z_k \\
&= (1-r)^n z_0 + {}^nC_1 r (1-r)^{n-1} z_1 + {}^nC_2 r^2 (1-r)^{n-2} z_2 + \cdots + r^n z_n \\
&= (1-r)^n (x_0 y_0) + {}^nC_1 r (1-r)^{n-1} (x_0 y_1 + x_1 y_0) \\
&\quad + {}^nC_2 r^2 (1-r)^{n-2} (x_0 y_2 + x_1 y_1 + x_2 y_0) + \cdots \\
&\quad + r^n (x_0 y_n + x_1 y_{n-1} + \cdots + x_n y_0) \\
&= x_0 [(1-r)^n y_0 + {}^nC_1 r (1-r)^{n-1} y_1 + {}^nC_2 r^2 (1-r)^{n-2} y_2 + \cdots + r^n y_n] \\
&\quad + x_1 [{}^nC_1 r (1-r)^{n-1} y_0 + {}^nC_2 r^2 (1-r)^{n-2} y_1 + \cdots + r^n y_{n-1}] \\
&\quad + x_2 [{}^nC_2 r^2 (1-r)^{n-2} y_0 + {}^nC_3 r^3 (1-r)^{n-3} y_1 + \cdots + r^n y_{n-2}] \\
&\quad + \cdots + x_n r^n y_0 \\
&= x_0 \left[\sum_{k=0}^{n} {}^nC_k r^k (1-r)^{n-k} y_k \right] + x_1 \left[\sum_{k=1}^{n} {}^nC_k r^k (1-r)^{n-k} y_{k-1} \right] \\
&\quad + x_2 \left[\sum_{k=2}^{n} {}^nC_k r^k (1-r)^{n-k} y_{k-2} \right] + \cdots + x_n r^n y_0
\end{aligned}$$

$$\begin{aligned}
&= x_0 \sigma_n + x_1 \left[\sum_{k=1}^{n} {}^nC_k r^k (1-r)^{n-k} y_{k-1} \right] \\
&\quad + x_2 \left[\sum_{k=2}^{n} {}^nC_k r^k (1-r)^{n-k} y_{k-2} \right] + \cdots + x_n r^n \sigma_0. \qquad (2)
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{k=1}^{n} {}^{n}C_k r^k (1-r)^{n-k} y_{k-1} \\
&\qquad = \sum_{j=0}^{n-1} {}^{n}C_{j+1} r^{j+1} (1-r)^{n-j-1} y_j \\
&\qquad = \sum_{j=0}^{n-1} \Bigg[{}^{n}C_{j+1} r^{j+1} (1-r)^{n-j-1} \\
&\qquad\qquad \times \left\{ \sum_{k=0}^{j} {}^{j}C_k \left(\frac{1}{r}\right)^k \left(1-\frac{1}{r}\right)^{j-k} \sigma_k \right\} \Bigg] \\
&\qquad\qquad \text{using Theorem 1.5 and (1)} \\
&\qquad = \sum_{k=0}^{n-1} \left[r(1-r)^{n-k-1} \sigma_k \left\{ \sum_{j=k}^{n-1} (-1)^{j-k}\, {}^{n}C_{j+1}\, {}^{j}C_k \right\} \right], \qquad (3) \\
&\qquad\qquad \text{interchanging the order of summation.}
\end{aligned}$$

Using the identity

$$\sum_{k=0}^{n-1} \left(\sum_{j=k}^{n-1} (-1)^{j-k}\, {}^{n}C_{j+1}\, {}^{j}C_k \right) z^k = \sum_{k=0}^{n-1} z^k,$$

we have,

$$\sum_{j=k}^{n-1} (-1)^{j-k}\, {}^{n}C_{j+1}\, {}^{j}C_k = 1, \quad 0 \leqslant k \leqslant n-1. \tag{4}$$

Thus, using (3), (4), we have,

$$\sum_{k=1}^{n} {}^{n}C_k r^k (1-r)^{n-k} y_{k-1} = \sum_{k=0}^{n-1} r(1-r)^{n-k-1} \sigma_k. \tag{5}$$

Using (5) and similar results, (2) can now be written as

$$\begin{aligned}
\tau_n &= x_0\sigma_n + x_1\left(\sum_{k=0}^{n-1} r(1-r)^{n-k-1}\sigma_k\right) \\
&\quad + x_2\left(\sum_{k=0}^{n-2} r^2(1-r)^{n-k-2}\sigma_k\right) + \cdots + x_n r^n \sigma_0 \\
&= x_0(\sigma_n - \sigma) + x_1\left\{\sum_{k=0}^{n-1} r(1-r)^{n-k-1}(\sigma_k - \sigma)\right\} \\
&\quad + x_2\left\{\sum_{k=0}^{n-2} r^2(1-r)^{n-k-2}(\sigma_k - \sigma)\right\} + \cdots + x_n r^n(\sigma_0 - \sigma) \\
&\quad + \sigma\left[x_0 + x_1\left\{\sum_{k=0}^{n-1} r(1-r)^{n-k-1}\right\}\right. \\
&\qquad\qquad \left. + x_2\left\{\sum_{k=0}^{n-2} r^2(1-r)^{n-k-2}\right\} + \cdots + x_n r^n\right] \\
&= x_0(\sigma_n - \sigma) + x_1\left\{\sum_{k=0}^{n-1} r(1-r)^{n-k-1}(\sigma_k - \sigma)\right\} \\
&\quad + x_2\left\{\sum_{k=0}^{n-2} r^2(1-r)^{n-k-2}(\sigma_k - \sigma)\right\} + \cdots + x_n r^n(\sigma_0 - \sigma) \\
&\quad + \sigma\left[x_0 + x_1 r\left\{\frac{1-(1-r)^n}{1-(1-r)}\right\} + x_2 r^2\left\{\frac{1-(1-r)^{n-1}}{1-(1-r)}\right\} + \cdots + x_n r^n\right] \\
&= x_0(\sigma_n - \sigma) + x_1\left\{\sum_{k=0}^{n-1} r(1-r)^{n-k-1}(\sigma_k - \sigma)\right\} \\
&\quad + x_2\left\{\sum_{k=0}^{n-2} r^2(1-r)^{n-k-2}(\sigma_k - \sigma)\right\} + \cdots + x_n r^n(\sigma_0 - \sigma) \\
&\quad + \sigma\left[x_0 + x_1\{1-(1-r)^n\} + x_2 r\{1-(1-r)^{n-1}\}\right. \\
&\qquad\qquad \left. + \cdots + x_n r^{n-1}\{1-(1-r)\}\right] \\
&= x_0(\sigma_n - \sigma) + x_1 r\left\{\sum_{k=0}^{n-1} (1-r)^{n-k-1}(\sigma_k - \sigma)\right\} \\
&\quad + x_2 r^2\left\{\sum_{k=0}^{n-2} (1-r)^{n-k-2}(\sigma_k - \sigma)\right\} + \cdots + x_n r^n(\sigma_0 - \sigma) \\
&\quad + \sigma\left[(x_0 + x_1 + x_2 r + \cdots + x_n r^{n-1})\right. \\
&\qquad\qquad \left. - \left\{x_1(1-r)^n + x_2 r(1-r)^{n-1} + \cdots + x_n r^{n-1}(1-r)\right\}\right]. \qquad (6)
\end{aligned}$$

Now,

$$|x_n r^{n-1}| = |x_n|, \quad \text{since} \quad |r| = 1$$

$$\to 0, \quad n \to \infty$$

and

$$|(1-r)^n| = |1-r|^n \to 0, \quad \text{since} \quad |1-r| < 1.$$

Thus $\{x_n r^{n-1}\}$ and $\{(1-r)^n\}$ are null sequences.
Note that the sequence

$$\{x_1(1-r)^n + x_2 r(1-r)^{n-1} + \cdots + x_n r^{n-1}(1-r)\}$$

is the Cauchy product of $\{x_n r^{n-1}\}$ and $\{(1-r)^n\}$.
In view of Theorem 1 of [5],

$$\lim_{n\to\infty} \{x_1(1-r)^n + x_2 r(1-r)^{n-1} + \cdots + x_n r^{n-1}(1-r)\} = 0.$$

Let $\alpha_n = x_n r^n$. Note that $\{\alpha_n\}$ is a null sequence since $|r| = 1$ and $x_n \to 0$, $n \to \infty$. Let

$$\beta_n = \sum_{k=0}^{n-1} (1-r)^{n-k-1}(\sigma_k - \sigma).$$

Now, $\{\beta_n\}$ is the Cauchy product of the null sequences $\{(1-r)^n\}$ and $\{\sigma_n - \sigma\}$. In view of Theorem 1 of [5], $\beta_n \to 0$, $n \to \infty$. We now note that

$$\lim_{n\to\infty} \left[x_1 r \left\{ \sum_{k=0}^{n-1} (1-r)^{n-k-1}(\sigma_k - \sigma) \right\} + x_2 r^2 \left\{ \sum_{k=0}^{n-2} (1-r)^{n-k-2}(\sigma_k - \sigma) \right\} \right.$$
$$\left. + \cdots + x_n r^n (\sigma_0 - \sigma) \right] = 0,$$

since $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$, again appealing to Theorem 1 of [5]. Thus, taking limit as $n \to \infty$ in (6), we have,

$$\lim_{n\to\infty} \tau_n = \sigma \left[x_0 + \sum_{k=1}^{\infty} x_k r^{k-1} \right],$$

noting that the series on the right converges since $|r| = 1$ and $x_k \to 0$, $k \to \infty$. In other words, $\{z_k\}$ is (E, r) summable to

$$\sigma \left[x_0 + \sum_{k=1}^{\infty} x_k r^{k-1} \right],$$

completing the proof of the theorem. ■

The following result can be proved in a similar fashion.

THEOREM 2.2 *If* $\sum_{k=0}^{\infty} x_k$ *converges and* $\sum_{k=0}^{\infty} y_k$ *is* (E, r) *summable to* σ,*then* $\sum_{k=0}^{\infty} z_k$ *is* (E, r) *summable to* $\sigma \left[x_0 + \sum_{k=1}^{\infty} x_k r^{k-1} \right]$.

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