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J. KRASINKIEWICZ (Opole)

On homeomorphisms of the Sierpiński curve*

We say that a topological space X is *homogeneous between points* $p, q \in X$ provided there exists a homeomorphism h such that $h(X) = X$ and $h(p) = q$. A simple closed curve is homogeneous between each pair of its points, but this is by no means a property characterizing simple closed curves even among those curves (i.e. 1-dimensional compact connected metric spaces) which are locally connected. It has been proved by Anderson [1] that the universal Menger curve is homogeneous between each pair of points. The Menger curve which lies in the 3-dimensional Euclidean space and is universal in the class of all curves has an analogue on the plane, namely the Sierpiński curve (see [2], p. 202) which is universal in the class of plane curves. The Sierpiński curve is an example of an *S-curve*, i.e. a plane locally connected curve whose complement in the plane consists of components with frontiers being mutually disjoint simple closed curves. The union of all these frontiers of components of the complement will be called the *rational part* of the Sierpiński curve. The rational part does not fill up the Sierpiński curve; the remainder is a G_δ dense set in the Sierpiński curve. This set will be called the *irrational part.* It has been proved by Whyburn [3] that S-curves are homeomorphic to each other; thus they are all homeomorphic to the Sierpiński curve. Moreover, if K_1, K_2 are S-curves and C_1, C_2 are frontiers of unbounded components of complements of K_1, K_2 in the plane, respectively, then each homeomorphism of C_1 onto C_2 can be extended to a homeomorphism of K_1 onto K_2 (see [3], p. 323).

THEOREM. *The Sierpiński curve is homogeneous between points p, q if and only if both points p, q belong either to the rational part or to the irrational part of the Sierpiński curve.*

Proof. Let us denote the Sierpiński curve and its rational part by D

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and W, respectively. The plane will be denoted by E^2. We consider the following three cases:

(i) $\qquad p \in W, \quad q \in W,$

(ii) $\qquad p \in W, \quad q \in D \setminus W,$

(iii) $\qquad p \in D \setminus W, \quad q \in D \setminus W.$

In case (i) there exist simple closed curves $C_1, C_2 \subset D$ such that $p \in C_1$, $q \in C_2$, and C_1, C_2 are frontiers of components of $E^2 \setminus D$. First we find homeomorphisms h_1, h_2 such that $h_i(D) \subset E^2$, h_i is the identity on C_i, and $h_i(C_i)$ is the frontier of the unbounded component of $E^2 \setminus h_i(D)$ $(i = 1, 2)$. Namely, if C_i is the frontier of the unbounded component of $E^2 \setminus D$, then let h_i be the identity on D. If C_i is the frontier of a bounded component of $E^2 \setminus D$, then let h_i be the inversion in the square C_i. Now, since $h_i(D)$ are S-curves $(i = 1, 2)$, a homeomorphism of C_1 onto C_2 which carries p into q can be extended to a homeomorphism h of $h_1(D)$ onto $h_2(D)$ (see [3], p. 323). Thus the homeomorphism $h_2^{-1} h h_1$ maps D onto D, and p onto q. In other words, the Sierpiński curve is homogeneous between points p, q.

In case (ii) there exists a simple closed curve $C \subset D$ such that $p \in C$ and C is the frontier of a component of $E^2 \setminus D$. Notice the set $D \setminus C$ is connected. Suppose on the contrary that D is homogeneous between p, q. Then there exists a homeomorphism h satisfying $h(D) = D$ and $h(p) = q$. Since the set $D \setminus h(C) = h(D \setminus C)$ is connected, it is contained in a component of $E^2 \setminus h(C)$. But $h(C) \subset D$ is a simple closed curve and, by the Jordan theorem, $h(C)$ is the frontier of the component R of $E^2 \setminus h(C)$ that is disjoint with D. Thus $R \subset E^2 \setminus D \subset E^2 \setminus h(C)$ which implies that R also is a component of $E^2 \setminus D$, and therefore $q = h(p) \in h(C) \subset W$. This contradicts (ii), and we have shown that in case (ii) the Sierpiński curve is not homogeneous between points p, q.

In case (iii) there exist, as it will be shown, arcs $L_1, L_2 \subset D$ such that $p \in L_1$, $q \in L_2$, and L_1, L_2 have no points but ends in common with W, while the ends of arcs L_1, L_2 lie on a simple closed curve C_0 which is the frontier of the unbounded component of $E^2 \setminus D$. Such arcs L_1, L_2 might be constructed in the following way. Let us form a sequence $C_1, C_2, \ldots$ of all frontiers of bounded components of $E^2 \setminus D$, and consider the decomposition of the curve D into sets $C_1, C_2, \ldots$ and single points from the set $D \setminus (W \setminus C_0)$. This decomposition is upper semi-continuous and can easily be completed in order to yield a decomposition of the whole disk bounded by C_0 into disks bounded by C_i $(i > 0)$ and single points from the complement. According to the Moore theorem (see [2], p. 380), the decomposition space of the latter decomposition is homeomorphic to the circular disk T. Let $f\colon D \to T$ be the natural mapping generated by the decomposition. Take all chords of the circle T which contain the point $f(p)$. Since every two of these chords have no point but $f(p)$ in common, and there are uncountably many of them, there exists a chord I_1 disjoint with the countable set $f(W \setminus C_0)$. Similarly, we find a chord I_2 which contains $f(q)$ and is disjoint with $f(W \setminus C_0)$. Then the sets $L_i = \overset{-1}{f}(I_i)$ $(i = 1, 2)$ are arcs satisfying the requirements prescribed above. We obtain

$$C_0 = M_1 \cup N_1 = M_2 \cup N_2$$

where M_i, N_i are arcs having common ends with the arc L_i $(i = 1, 2)$. Let

$$g_0: L_1 \cup M_1 \to L_2 \cup M_2, \qquad h_0: L_1 \cup N_1 \to L_2 \cup N_2$$

be homeomorphisms such that $g_0(L_1) = L_2$, $g_0(p) = q$, and $g_0|L_1 = h_0|L_1$. The intersections S_i, S_i' of D with the closures of domains bounded by simple closed curves $L_i \cup M_i$, $L_i \cup N_i$, respectively, are S-curves $(i = 1, 2)$. It follows that the homeomorphisms g_0 and h_0 can be extended to homeomorphisms g and h mapping S_1 onto S_2 and S_1' onto S_2', respectively (see [3], p. 323). But since $S_1 \cap S_1' = L_1$ and $g|L_1 = g_0|L_1 = h_0|L_1 = h|L_1$, the junction of g and h is possible. It yields a homeomorphism of the Sierpiński curve onto itself which carries p into q; hence we get the homogeneity between these points.

COROLLARY. *The pairs of points between which the Sierpiński curve is homogeneous, as well as those between which it is not homogeneous, constitute a dense set in the Cartesian product of the Sierpiński curve by itself.*

References

[1] R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. 67 (1958), pp. 313-324.

[2] C. Kuratowski, *Topologie II*, Warszawa 1961.

[3] G. T. Whyburn, *Topological characterization of the Sierpiński curve*, Fund. Math. 45 (1958), pp. 320-324.