ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XIII (1970)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XIII (1970)

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Concerning biorthogonal systems in $C(S)$*

Introduction. Let X be a Banach space and X^* its conjugate space. If $(x_i)_{i=1}^{\infty}$ is a sequence in X and $(g_i)_{i=1}^{\infty}$ a sequence in X^* with the property that $g_i(x_j) = \delta_{ij}$, the pair (x_i, g_i) is called a *biorthogonal system in* X. In [3] Orlicz proved the following theorem: *If (x_i, g_i) is a biorthogonal system in $C[a, b]$ with (x_i) fundamental in $C[a, b]$ and there exists a point t_0 in $[a, b]$ for which*

$$\sum_{i=1}^{\infty} [g_i(t_0+) - g_i(t_0-)] x_i(t_0) \neq 1,$$

then there is an x in $C[a, b]$, $x \sim \sum_{i=1}^{\infty} g_i(x) x_i$, for which

$$\sum_{i=1}^{\infty} |g_i(x)|\,|x_i(t_0)| = +\infty.$$

A corollary to this theorem is the result of Karlin that $C[a, b]$ has no unconditional basis. In this paper we apply Orlicz's methods to prove an analogous theorem about biorthogonal systems in $C(S)$, where S is a compact metric space. The proof differs from that of Orlicz in that the Tietze extension theorem is employed instead of an explicit construction of certain functions, and general properties of Radon measures are used in place of particular properties of functions of bounded variation. From this theorem we obtain the known result that if S is an uncountable metric space, then $C(S)$ has no unconditional basis.

Notation and preliminary results. Throughout this paper S will denote a compact metric space and $C(S)$ the space of all continuous real valued functions on S with the supremum norm. We will denote by (x_i) a fundamental sequence in $C(S)$ and by (μ_i) a corresponding biorthogonal sequence of Radon measures in $C(S)^*$; i.e. $\mu_i(x_j) = \delta_{ij}$. For x in $C(S)$ a series of the form $x \sim \sum_{i=1}^{\infty} a_i x_i$, where $a_i = \int_S x(t)\, d\mu_i(t)$ for

* This paper is part of the author's research for his doctoral thesis being written at the Louisiana State University under the direction of Professor J. R. Retherford.

$i = 1, 2, \ldots$, will be called the *expansion of* x with respect to the biorthogonal system (x_i, μ_i). If d is the metric on S we will let $\bar{B}_r(t_0) = \{t \in S \mid d(t, t_0) \leqslant r\}$ and $\mathring{B}_r(t_0) =$ interior $\bar{B}_r(t_0)$.

DEFINITION 1. A biorthogonal system (x_i, μ_i) for $C(S)$ is said to have *property* $(*)$ at a point t_0 in S if there exists an $x \sim \sum_{i=1}^{\infty} a_i x_i$ in $C(S)$ for which $\sum_{i=1}^{\infty} |a_i|\,|x_i(t_0)| = +\infty$.

DEFINITION 2. Let X be a linear space on which two norms, $\|\ \|$ and $\|\ \|^*$, are defined. Let $X_S = \{x \in X \mid \|x\| \leqslant 1\}$. Define a metric d on X_S by $d(x_1, x_2) = \|x_1 - x_2\|^*$ for x_1 and x_2 in X_S. If X_S is complete under this metric, then X_S is called a *Saks space* and is denoted by $[X_S; \|\ \|; \|\ \|^*]$.

DEFINITION 3. A Saks space $[X_S; \|\ \|; \|\ \|^*]$ is said to satisfy *condition* (Σ_1) if for every x_0 in X_S and any $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|x\|^* < \delta$ and x is in X_S, then $x = x_1 - x_2$, where x_1 and x_2 are in X_S and $\|x_1 - x_0\|^* < \varepsilon$, $\|x_2 - x_0\|^* < \varepsilon$.

The following lemma is an analogue for Saks spaces of the Banach-Steinhaus theorem for complete metric linear spaces.

LEMMA 1. *Let* $[X_S; \|\ \|; \|\ \|^*]$ *be a Saks space satisfying condition* (Σ_1) *and let* (f_n) *be a sequence of functionals linear on* X *and continuous on* $[X_S, \|\ \|^*]$. *If there is a function* $f(x)$ *defined by* $f(x) = \lim_{n\to\infty} f_n(x)$ *for all* x *in* X, *then* $f(x)$ *is a linear functional on* X, *continuous on* $[X_S; \|\ \|^*]$.

Proof. The functional $f(x)$ is obviously linear. Since $f_n(x)$ converges to $f(x)$ for all x in X_S, by Osgood's theorem (f_n) is equicontinuous on $X_S \setminus Z$, where Z is a first category set in $[X_S; \|\ \|^*]$. Since $[X_S; \|\ \|^*]$ is complete $X_S \setminus Z$ is not empty. Let x_0 in X_S be a point of equicontinuity of the family (f_n). Then given $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|x - x_0\|^* < \delta$, then $|f_n(x) - f_n(x_0)| < \varepsilon/2$ for all $n = 1, 2, \ldots$ Hence $|f(x) - f(x_0)| \leqslant \varepsilon/2 < \varepsilon$ and f is continuous at x_0 in $[X_S, \|\ \|^*]$.

Now for any $\varepsilon > 0$ there is a $\delta_1 > 0$ such that if $\|x - x_0\|^* < \delta_1$ with x in X_S, then $|f(x) - f(x_0)| < \varepsilon/4$. By the (Σ_1) condition there is an ε_1 such that if z is in X_S and $\|z\|^* < \varepsilon_1$, then $z = z_1 - z_2$ with $\|z_1 - x_0\|^* < \delta_1$, $\|z_2 - x_0\|^* < \delta_1$.

Let x_1 be arbitrary in X_S and let y in X_S satisfy $\|(y - x_1)/2\|^* < \varepsilon$. Since $(y - x_1)/2$ is in X_S, $(y - x_1)/2 = z_1 - z_2$, where $\|z_1 - x_0\|^* < \delta_1$, $\|z_2 - x_0\|^* < \delta_1$. Therefore,

$$|f(y) - f(x_1)| = 2\left|f\left(\frac{y - x_1}{2}\right)\right| = 2\,|f(z_1) - f(z_2)|$$

$$\leqslant 2\,[|f(z_1) - f(x_0)| + |f(z_2) - f(x_0)|] < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Hence f is continuous on $[X_S; \| \|^*]$ since x_1 in X_S was arbitrary.

The main result.

THEOREM. *If (x_i, μ_i) is a biorthogonal system in $C(S)$ and if there exists a t_0 in S such that $\sum_{i=1}^{\infty} \mu_i(t_0)\, x_i(t_0) \neq 1$, then (x_i, μ_i) has property $(*)$ at t_0.*

Proof. If (x_i, μ_i) does not have property $(*)$ at t_0, then

$$\sum_1^{\infty} |a_i|\, |x_i(t_0)| < \infty \quad \text{for each } x \sim \sum_1^{\infty} a_i x_i \text{ in } C(S). \tag{1}$$

Let $\varepsilon = (\varepsilon_i)$ denote a sequence such that $\varepsilon_i = \pm 1$ for all i and define $G_n^{\varepsilon} = \sum_{i=1}^{n} \varepsilon_i x_i(t_0) \mu_i$. Note that for $x(t)$ in $C(S)$,

$$\int_S x(t)\, dG_n^{\varepsilon}(t) = \sum_{i=1}^{n} \Big[\int_S x(t)\, d\mu_i(t)\Big] \varepsilon_i x_i(t_0) = \sum_{i=1}^{n} a_i \varepsilon_i x_i(t_0).$$

Hence by condition (1) above,

$$\lim_{n\to\infty} \int_S x(t)\, dG_n^{\varepsilon}(t) \quad \text{exists for all } \varepsilon \text{ and all } x \text{ in } C(S). \tag{2}$$

By the Uniform Boundedness Principle, for every such sequence $\varepsilon = (\varepsilon_i)$, there is an $M_\varepsilon > 0$ for which $\|G_n^{\varepsilon}\|_V \leqslant M_\varepsilon$ for all $n = 1, 2, \ldots$, where $\| \|_V$ denotes the variation norm in $C(S)^*$. Hence, if K^{t_0} denotes the set of all bounded functions on S continuous everywhere except possibly at t_0 and $x(t)$ is in K^{t_0}, then for any $\varepsilon = (\varepsilon_i)$ as above

$$\sup_n \Big|\sum_1^{n} \varepsilon_i x_i(t_0) \int_{S\setminus\{t_0\}} x(t)\, d\mu_i(t)\Big| = \sup_n \Big|\int_{S\setminus\{t_0\}} x(t)\, dG_n^{\varepsilon}(t)\Big|$$
$$\leqslant \sup_{t\in S} |x(t)| \sup_n \|G_n^{\varepsilon}\|_V \leqslant \sup_{t\in S} |x(t)| \cdot M_\varepsilon < +\infty.$$

In particular, this is true for that sequence $\varepsilon_0 = (\varepsilon_i)$ for which

$$\varepsilon_i x_i(t_0) \int_{S\setminus\{t_0\}} x(t)\, d\mu_i(t) \geqslant 0,$$

so we have

$$\sum_1^{\infty} |x_i(t_0)| \Big|\int_{S\setminus\{t_0\}} x(t)\, d\mu_i(t)\Big| < +\infty \quad \text{for all } x(t) \text{ in } K^{t_0},$$

implying

$$\lim_{n\to\infty} \int_{S\setminus\{t_0\}} x(t)\, dG_n^{\varepsilon}(t) \tag{3}$$

exists for all $x(t)$ in K^{t_0} and all $\varepsilon = (\varepsilon_i)$.

Define the following two norms on K^{t_0}:

$$\|x\| = \sup_{t\in S} |x(t)|,$$

$$\|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|_n, \quad \text{where } \|x\|_n = \sup_{S\setminus \mathring{B}_{2^{-n}}(t_0)} |x(t)|.$$

Let $K_S = \{x \in K^{t_0} \mid \|x\| \leqslant 1\}$. We claim that $[K_S; \|x\|; \|\ \|^*]$ is a Saks space satisfying condition (Σ_1). The proof is analogous to that in [2], applying the Tietze extension theorem in place of the construction of the piecewise linear function on the interval given in that proof.

Define functionals f_m^n on K^{t_0} by

$$f_m^n(x) = \int_{S\setminus \mathring{B}_{2^{-m}}(t_0)} x(t)\, dG_n^{\varepsilon_1}(t),$$

where ε_1 is the sequence (ε_i) for which $\varepsilon_i = 1$ for each i. It is easily seen that each f_m^n is continuous on $[K_S, \|\ \|^*]$ and since each is linear,

$$f^n = \lim_{m\to\infty} f_m^n(x) = \lim_{m\to\infty} \int_{S\setminus \mathring{B}_{2^{-m}}(t_0)} x(t)\, dG_n^{\varepsilon_1}(t) = \int_{S\setminus\{t_0\}} x(t)\, dG_n^{\varepsilon_1}(t)$$

is a linear functional. By Lemma 1 each f^n is also continuous on $[K_S, \|\ \|^*]$. Since by (3), $f(x) = \lim f^n(x)$ exists for all $x(t)$ in K^{t_0}, Lemma 1 implies that f is a continuous linear functional on K_S. By (2)

$$\lim_{n\to\infty} \int_S x(t)\, dG_n^{\varepsilon_1}(t) = \sum_{i=1}^{\infty} a_i x_i(t_0) = S_0(x)$$

exists for all $x(t)$ in $C(S)$ and by the Banach-Steinhaus theorem S_0 is in $C(S)^*$. In the remainder of the proof $x(t)$ will denote a function in $C(S)$.

Since

$$\int_{S\setminus\{t_0\}} x(t)\, dG_n^{\varepsilon_1}(t) = \int_S x(t)\, dG_n^{\varepsilon_1}(t) - x(t_0) G_n^{\varepsilon_1}(t_0),$$

$$f^n(x) = \int_S x(t)\, dG_n^{\varepsilon_1}(t) - x(t_0) G_n^{\varepsilon_1}(t_0)$$

for all n. Taking the limit as n tends to infinity we have

$$f(x) = S_0(x) - x(t_0) \sum_1^{\infty} x_i(t_0) \mu_i(t_0). \tag{4}$$

Now $S_0(x_m) = x_m(t_0)$ for all m by the biorthogonality of the pair (x_i, μ_i). Hence, since S_0 is a continuous linear functional on $C(S)$ and (x_i) is fundamental in $C(S)$, $S_0(x) = x(t_0)$ for all x in $C(S)$.

For any $\delta > 0$ define

$$y_{t_0}^{\delta}(t) = \begin{cases} 1 & \text{for } t \in \overline{B}_{\delta/4}(t_0), \\ 0 & \text{for } t \in S \setminus \mathring{B}_{\delta/2}(t_0). \end{cases}$$

Then $y_{t_0}^{\delta}(t)$ is a continuous function on a closed subset of the normal space S and so by the Tietze extension theorem has an extension to a continuous function $x_{t_0}^{\delta}$ on S with $\sup_{t \in S} |x_{t_0}^{\delta}(t)| = 1$. Note then that, for each $\delta > 0$, $x_{t_0}^{\delta}$ is in K^{t_0} and that $S_0(x_{t_0}^{\delta}) = 1$. From the properties of $x_{t_0}^{\delta}$ given above and the definition of $\| \ \|^*$ it is clear that $\lim_{\delta \to 0} \|x_{t_0}^{\delta}\|^* = 0$.

Therefore, $\lim_{\delta \to 0} f(x_{t_0}^{\delta}) = 0$ implying by (4),

$$\begin{aligned} 0 = \lim_{\delta \to 0} S_0(x_{t_0}^{\delta}) - x_{t_0}^{\delta}(t_0) \sum_{i=1}^{\infty} x_i(t_0)\mu_i(t_0) \\ = 1 - x_{t_0}^{\delta}(t_0) \sum_{i=1}^{\infty} x_i(t_0)\mu_i(t_0) = 1 - \sum_{i=1}^{\infty} x_i(t_0)\mu_i(t_0). \end{aligned}$$

Hence, $\sum_{i=1}^{\infty} x_i(t_0)\mu_i(t_0) = 1$.

COROLLARY. *If S is an uncountable compact metric space, then $C(S)$ has no unconditional basis.*

Proof. Using properties of unconditional convergence of a series in a Banach space [1] it is easy to see that if (x_i, μ_i) is an unconditional basis for $C(S)$, then for any $x = \sum_{i=1}^{\infty} a_i x_i$ in $C(S)$ and any t_0 in S, $\sum_{i=1}^{\infty} |a_i| |x_i(t_0)| < \infty$; that is, (x_i, μ_i) does not have property $(*)$ at any point t_0 in S. However, for each Radon measure μ_i there are at most countably many points t in S for which $\mu_i(t) \neq 0$, implying that the set of points t in S for which $\mu_i(t) \neq 0$ for some i is at most countable. Since S is assumed uncountable there is then a point t_0 in S for which $\mu_i(t_0) = 0$ for all i, implying $\sum_{i=1}^{\infty} x_i(t_0)\mu_i(t_0) = 0$ and so by the theorem (x_i, μ_i) has property $(*)$ at t_0, a contradiction to the above assertion.

References

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