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Distance Fibonacci numbers, their interpretations and matrix generators

Abstract. In this paper we define a distance Fibonacci numbers, also for negative integers, which generalize the classical Fibonacci numbers and Padovan numbers, simultaneously. We give different interpretations of these numbers with respect to special partitions and compositions, also in graphs. We show a construction of the sequence of distance Fibonacci numbers using the Pascal's triangle. Moreover, we give matrix generators of these numbers, for negative integers, too.

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1. Introduction. In general we use the standard notation, see [1, 2]. The nth Fibonacci number F_n is defined recursively in the following way $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$. The nth Padovan number $Pv(n)$ is defined by $Pv(0) = Pv(1) = Pv(2) = 1$ and $Pv(n) = Pv(n-2) + Pv(n-3)$, $n \geqslant 3$. There are many interesting generalizations of the Fibonacci numbers, see for example [3, 4]. Some of these generalizations are directly related to studying the concept of k-independent sets in graphs [9, 12, 13, 14]. It is worth mentioning that k-independent sets (and also k-kernels in digraphs) are intensively studied by H. Galeana-Sánchez and C. Hernández-Cruz, see for example their last interesting papers [5, 6, 7]. In this paper we introduce a new generalization of the Fibonacci numbers in the distance sense, which generalize the classical Fibonacci numbers and the Padovan numbers, simultaneously. We give a number of interesting interpretations of these numbers. Firstly we use this generalization to the counting of special families of subsets of the set of n integers and next we use these numbers for counting the number of compositions of an integer n into special parts. Moreover, we give a graph interpretation of these numbers with respect to the number of all $\{P_k, P_{k-1}\}$-matching of the graph P_n with additional restrictions. In the last section of this paper we give matrix representations of generalized Fibonacci numbers, also for negative integers.

2. Distance Fibonacci numbers $Fd(k,n)$ and their interpretations. Let $k \geqslant 2$, $n \geqslant 0$ be integers. The distance Fibonacci numbers $Fd(k,n)$ are defined recursively in the following way

$$(1) \qquad Fd(k,n) = Fd(k,n-k+1) + Fd(k,n-k) \text{ for } n \geqslant k$$

and $Fd(k,n) = 1$ for $0 \leqslant n \leqslant k-1$.
The Table 1 includes first words of the distance Fibonacci numbers defined in (1) for special values of k and n.

Tab.1. The distance Fibonacci numbers $Fd(k,n)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$Fd(3,n)$	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37
$Fd(4,n)$	1	1	1	1	2	2	2	3	4	4	5	7	8	9	12
$Fd(5,n)$	1	1	1	1	1	2	2	2	2	3	4	4	4	5	7
$Fd(6,n)$	1	1	1	1	1	1	2	2	2	2	2	3	4	4	4

It is easy to see that $F(2,n) = F_n$ for $n \geqslant 0$ are the well-known Fibonacci numbers. Moreover for $k = 3$ and $n \geqslant 0$ we obtain that $Fd(3,n) = Pv(n)$ are the well-known Padovan numbers. Among others F_n and $Pv(n)$ have the graph interpretation also with respect to the number of independent sets in graphs.
In [10] Prodinger and Tichy studied the graph parameter $NI(G)$ defined as the number of all independent sets in a graph. They shoved that $NI(P_n) = F_{n+1}$. The parameter $NI(G)$ has many applications in combinatorial chemistry and it is named for the historical reasons as the Merrifield-Simmons index. It is study intensively in the last years, see the survey [8] and its references.
Let $NMI_L(G)$ be the total number of maximal independent sets in a graph G including the set of leaves. It was proved, see [11], that for an arbitrary n-vertex tree T_n holds $NMI_L(T_n) \leqslant Pv(n-3)$.
A special generalization of the Padovan numbers having a graph interpretation with respect to the number of the maximal k-independent sets has been obtained recently, see [13].

Now we give a combinatorial representations of the distance Fibonacci numbers $Fd(k,n)$.

At the beginning we show that for $n \geqslant k-1$ the distance Fibonacci numbers $Fd(k,n)$ can be applied for counting of special families of subsets of n-element set of integers.

Assume that $k \geqslant 2$, $n \geqslant k-1$ are integers. Let $X = \{1, 2, \ldots, n\}$ be the set of n integers. Let $\mathcal{Y} = \{\mathcal{Y}_t; t \in T\}$ be the family of subsets of the set X such that subsets $\mathcal{Y}_t$ contain consecutive integers and satisfy the following conditions
(i) $|\mathcal{Y}_t| \in \{k-1, k\}$ for $t \in T$,
(ii) $\mathcal{Y}_t \cap \mathcal{Y}_s = \emptyset$ for $t, s \in T$ and $t \neq s$,

(iii) $0 \leqslant |X \setminus \bigcup_{t \in T} \mathcal{Y}_t| \leqslant k-2$,
(iv) for each $m \in (X \setminus \bigcup_{t \in T} \mathcal{Y}_t)$ either $m = n$ or $m+1 \in (X \setminus \bigcup_{t \in T} \mathcal{Y}_t)$.
Such a family $\mathcal{Y}$ will be called a quasi k-decomposition of the set X. Clearly, if $X \setminus \bigcup_{t \in T} \mathcal{Y}_t = \emptyset$, then a quasi k-decomposition of the set X is its decomposition into subsets of cardinality k or $k-1$. We can observe that for $k = 2$ each quasi 2-decomposition of the set X is decomposition of X into one-element or two-element subsets.

Theorem 2.1 *Let $k \geqslant 2$, $n \geqslant k-1$ be integers. Then the number of all quasi k-decompositions of the set X is equal to $Fd(k,n)$.*

Proof Let $k \geqslant 2, n \geqslant k-1$ be integers and $X = \{1,2,\ldots,n\}$. Denote by $q(k,n)$ the number of all quasi k-decomposition of the set X. If $n = k-1$, then $\{\{1,2,\ldots,k-1\}\}$ is the unique quasi k-decomposition of the set X. Hence, $q(k,k-1) = 1 = Fd(k,k-1)$. Assume now that $n \geqslant k$ and suppose that the equality $q(k,n) = Fd(k,n)$ is true for an arbitrary n. Let $q_{k-1}(k,n+1)$ be the number of all quasi k-decompositions $\mathcal{Y}$ of the set $X = \{1,2,\ldots,n+1\}$ such that $\{1,2,\ldots,k-1\} \in \mathcal{Y}$ and let $q_k(k,n+1)$ be the number of all quasi k-decompositions $\mathcal{Y}$ of the set X such that $\{1,2,\ldots,k\} \in \mathcal{Y}$. It is clear that $q(k,n+1) = q_{k-1}(k,n+1) + q_k(k,n+1)$. Moreover $q_{k-1}(k,n+1) = q(k,n+1-(k-1))$, because $|X \setminus \{1,2,\ldots,k-1\}| = n+1-(k-1)$ and $q_k(k,k+1) = q(k,n+1-k)$ because $|X \setminus \{1,2,\ldots,k\}| = n+1-k$. Using induction's hypothesis and recurrence relation (1) we obtain $q(k,n+1) = q(k,n-k+2) + q(k,n-k+1) = Fd(k,n-k+2) + Fd(k,n-k+1) = Fd(k,n+1)$.
Thus the theorem is proved. ■

Instead of quasi k-decompositions of n-element set of integers we may consider a special ordered quasi-compositions of an integer n into parts $k-1$ and k. We say that a sum $\sum_{t \in T} n_t + n_r$ is an ordered quasi k-composition of an integer n into parts $k-1$ and k if
(v) $n_t \in \{k-1, k\}$ for $t \in T$,
(vi) $0 \leqslant n_r \leqslant k-2$,
(vii) $\sum_{t \in T} n_t + n_r = n$ and n_r is always the last part of the quasi k-composition.
For example for $n = 8$, $k = 3$ ordered quasi 3-compositions $3+2+2+1$ and $2+3+2+1$ are different. Since the cardinality of X equals n, so every $\mathcal{Y}_t \in \mathcal{Y}, t \in T$ corresponds to a summand n_t of the sum $\sum_{t \in T} n_t$ of an ordered quasi k-composition. This immediately gives

Theorem 2.2 *Let $k \geqslant 2$, $n \geqslant k-1$ be integers. The number of all ordered quasi k-compositions of the integer n into parts $k-1$ and k is equal to $Fd(k,n)$.*

Note that the ordered quasi k-composition corresponds to a p-tuple $(x_1, x_2, \ldots, x_p)$, where $p \geqslant \lfloor \frac{n}{k} \rfloor$ such that $x_1 \in \{k-1, k\}$ and for $i = 2, \ldots, p$ we have possibilities:

(viii) if $\sum_{j=1}^{i-1} x_j \leqslant n-k$, then $x_i \in \{k-1, k\}$,
(ix) if $\sum_{j=1}^{i-1} x_j = n-(k-1)$, then $x_i = k-1$,
(x) if $\sum_{j=1}^{i-1} x_j \geqslant n-(k-2)$, then $x_i = 0$.
From the above we obtain

Theorem 2.3 *Let $k \geqslant 2$, $n \geqslant k-1$ be integers and $p \geqslant \lfloor \frac{n}{k} \rfloor$. The number of all p-tuples $(x_1, x_2, \ldots, x_p)$ satisfying conditions (viii), (ix), (x) is equal to $Fd(k, n)$.*

Using the above considerations we give the graph interpretation of the number $Fd(k, n)$.
Let $\mathcal{H} = \{H_1, \ldots, H_m\}$, $m \geqslant 1$ be a collection of m graphs such that each graph H_i from $\mathcal{H}$ is connected. A subgraph $M \subseteq G$ is an $\mathcal{H}$-matching of G if each connected component of M is isomorphic to some $H_i, 1 \leqslant i \leqslant m$. If an $\mathcal{H}$-matching M meets every vertex of a graph G, then M is a perfect $\mathcal{H}$-matching.
Clearly if $H_i = H$ for all $i = 1, \ldots, m$, then we obtain the well-known definition of an H-matching. In particular if $H = K_2$, then we have the classical definition of a matching. Among $\mathcal{H}$-matchings seems to be the most interesting to study the case when $H_i, i = 1, \ldots, m$ belong to the same class of graphs.
If $G \setminus M = \emptyset$ or no graph H_i, $i = 1, ..., m$, is a subgraph of the graph $G \setminus M$, then we say that M is a quasi perfect $\mathcal{H}$-matching of the graph G. Clearly if $G \setminus M = \emptyset$, then we obtain a perfect $\mathcal{H}$-matching of a graph G. The Figure 1 gives an example of a $\{K_3, K_2\}$-matching of a graph G.

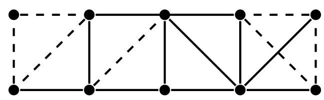

Fig.1. $\{K_3, K_2\}$-matching of the graph G

Using this terminology we have that if the set X corresponds to the vertex set of the graph $P_n, n \geqslant k-1$, then each $\mathcal{Y}_t, t \in T$ corresponds to a subgraph P_l, where $l \in \{k, k-1\}$. Then the quasi k-decomposition $\mathcal{Y}$ corresponds to a quasi perfect $\{P_k, P_{k-1}\}$-matching of the graph P_n i.e. at most a subgraph $P_r, 1 \leqslant r \leqslant k-2$ such that $x_n \in V(P_r)$ do not belong to a $\{P_k, P_{k-1}\}$-matching of the graph G.

Now we give a method generating all quasi k-decompositions of the set $X = \{1, ..., n\}$, quasi k-compositions of an integer n and consequently all quasi perfect $\{P_k, P_{k-1}\}$-matchings of P_n.

By α_m we denote a binary m-tuple. For m-tuples we define a procedure $ADD1(\alpha_m)$.

Procedure ADD1
$d := 1$, $u := 1$,
while $d = 1$ and $u \leqslant m$ **do**
if $\alpha_m(u) = 0$ **then** $\alpha_m(u) := 1$, $d := 0$ **else** $\alpha_m(u) := 0$, $u := u + 1$.

For example if $\alpha_5 = (1, 1, 0, 0, 0)$, then $ADD1(\alpha_5) = (0, 0, 1, 0, 0)$.

ALGORITHM ALL QUASI k-COMPOSITIONS
Input: n, k
Output: all binary m-tuples corresponding to all quasi k-compositions of n, the number of all quasi k-compositions of n
$p := \lfloor \frac{n}{k-1} \rfloor$, $r := \lfloor \frac{n}{k} \rfloor$, $\# := 1$,
for $z := r$ **to** p **step** 1 **do**
  $\alpha_z := (0, 0, \ldots, 0)$, $t := 0$,
  while $t < z$ **do**
    $ADD1(\alpha_z)$, $t = \sum_{i=1}^{z} \alpha_z(i)$,
    $x := tk + (z - t)(k - 1)$,
    if $n - k + 2 \leqslant x \leqslant n$ **then**
      return α_z, $\# := \# + 1$,
return $\#$.

On the output we obtain the set of all binary m-tuples which determine all quasi k-decompositions of integer n and all quasi k-compositions of the set $X = \{1, ..., n\}$. For example for given n, k a 5-tuple $(0, 1, 1, 0, 1)$ corresponds to the quasi k-composition $(k - 1) + k + k + (k - 1) + k + r$. Having the quasi k-composition we obtain quasi k-decomposition of the set $X = \{1, ..., n\}$ on the form $\{1, ..., k - 1\}, \{k, ..., 2k - 1\}, \{2k, ..., 3k - 1\}, \{3k, ..., 4k - 2\}, \{4k - 1, ..., 5k - 2\}, \{5k - 1, ..., n\}$. If $r = 0$, then the last set in this decomposition is empty. Defined subsets directly give the quasi perfect $\{P_k, P_{k-1}\}$-matching of the graph P_n with the numbering of $V(P_n)$ with the natural fashion.

The above algorithm choose only m-tuples which give quasi k-compositions. Clearly the number of all m-tuples is 2^m. If we subtract the number of all m-tuples which do not give quasi k-compositions and quasi k-decompositions, then we obtain a direct formula for distance Fibonacci numbers $Fd(k, n)$. Based on this idea we can prove:

THEOREM 2.4 *Let $n \geqslant 2, k \geqslant 2$ be integers. Then*

$$Fd(k, n) = \sum_{t=\lfloor \frac{n}{k} \rfloor}^{\lfloor \frac{n}{k-1} \rfloor} \sum_{i=tk-n}^{(t+1)k-n-2} \binom{t}{i}. \tag{2}$$

PROOF Consider a quasi k-composition of the integer n. Clearly it satisfies conditions (v), (vi), (vii) so $n = n_1 + ... + n_t + n_r$. Instead of $n_1 + ... + n_t + n_r$ we can equivalently consider a t-tuple $(n_1, ..., n_t)$, because the last part of this sum is uniquely determined by the previous parts. Then $\lfloor \frac{n}{k} \rfloor \leqslant t \leqslant \lfloor \frac{n}{k-1} \rfloor$ and each word of this t-tuple is equal to k or $k - 1$. Evidently we have 2^t such t-tuples. Let a t-tuple has i words $k-1$ and $t-i$ words k. Then $\sum_{i=0}^{t} n_i = i(k-1)+(t-i)k = tk-i$. If $tk-i > n$ or $tk - i < n - (k - 2)$, then the t-tuple does not correspond to quasi k-composition of the integer n. Consequently we have $2^t - \sum_{i=0}^{tk-n-1} \binom{t}{i} - \sum_{i=(t+1)k-n-1}^{t} \binom{t}{i}$ such t-tuples which determine quasi k-compositions. Because $\lfloor \frac{n}{k} \rfloor \leqslant t \leqslant \lfloor \frac{n}{k-1} \rfloor$, so

$Fd(k,n) = \sum\limits_{t=\lfloor \frac{n}{k} \rfloor}^{\lfloor \frac{n}{k-1} \rfloor} \left(2^t - \sum\limits_{i=0}^{tk-n-1} \binom{t}{i} - \sum\limits_{i=(t+1)k-n-1}^{t} \binom{t}{i} \right)$. By simple reformulation we obtain $Fd(k,n) = \sum\limits_{t=\lfloor \frac{n}{k} \rfloor}^{\lfloor \frac{n}{k-1} \rfloor} \sum\limits_{i=tk-n}^{(t+1)k-n-2} \binom{t}{i}$.

Thus the theorem is proved. ■

For $k = 2$ the formula (2) gives identity $F_n = \sum\limits_{t=\lfloor \frac{n}{2} \rfloor}^{n} \binom{t}{2t-n}$ which is equivalent to known $F_n = \sum\limits_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$.

Now we give an another form of the direct formula for the distance Fibonacci numbers $Fd(k,n)$.

THEOREM 2.5 *Let $n \geqslant 2, k \geqslant 2$ be integers. Then*

$$Fd(k,n) = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{j + \lfloor \frac{n-jk}{k-1} \rfloor}{j}. \tag{3}$$

PROOF Let $X = \{1, 2, \ldots, n\}$. If $n \leqslant k-1$, then $\lfloor \frac{n}{k} \rfloor = 0$ and $\sum\limits_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{j+\lfloor \frac{n-jk}{k-1} \rfloor}{j} = 1 = Fd(k,n)$. Suppose that $n \geqslant k$. By Theorem 2.1 the number of all quasi k-decompositions of the set X is equal to $Fd(k,n)$. Each quasi k-decomposition of the set X consists of i sets of cardinality $k-1$ and j sets of cardinality k, where $0 \leqslant i \leqslant \lfloor \frac{n}{k-1} \rfloor$, $0 \leqslant j \leqslant \lfloor \frac{n}{k} \rfloor$. Moreover for fixed j we have $i = \lfloor \frac{n-jk}{k-1} \rfloor$. Thus the number of all quasi k-decompositions of X with j sets having k-element is equal to $\binom{j+i}{j} = \binom{j+\lfloor \frac{n-jk}{k-1} \rfloor}{j}$ and $Fd(k,n) = \sum\limits_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{j+\lfloor \frac{n-jk}{k-1} \rfloor}{j}$.

Thus the theorem is proved. ■

If $k = 2$, then the formula (3) gives well-known formula for the Fibonacci numbers $F_n = \sum\limits_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j}$.

It is easy to see that from the direct formula (3) we can count the total number of all k-element sets of all quasi k-decompositions of the set X by $\sum\limits_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{j+\lfloor \frac{n-jk}{k-1} \rfloor}{j} j$ and the total number of all $k-1$-element sets of all quasi k-decompositions of the set X by $\sum\limits_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{j+\lfloor \frac{n-jk}{k-1} \rfloor}{j} \lfloor \frac{n-jk}{k-1} \rfloor$.

The following Figures shows the rule of obtaining $Fd(2,n)$ and $Fd(3,n)$ using the Pascal's triangle, see Figures 2-4.

$Fd(2,0)$	1							
$Fd(2,1)$	1	1						
$Fd(2,2)$	1	2	1					
$Fd(2,3)$	1	3	3	1				
$Fd(2,4)$	1	4	6	4	1			
$Fd(2,5)$	1	5	10	10	5	1		
$Fd(2,6)$	1	6	15	20	15	6	1	
$Fd(2,7)$	1	7	21	35	35	21	7	1

Fig.2. The Fibonacci numbers, $F_n = Fd(2,n)$.

$Fd(3,0)$	1							
$Fd(3,2)$	1	1						
$Fd(3,4)$	1	2	1					
$Fd(3,6)$	1	3	3	1				
$Fd(3,8)$	1	4	6	4	1			
$Fd(3,10)$	1	5	10	10	5	1		
$Fd(3,12)$	1	6	15	20	15	6	1	
$Fd(3,14)$	1	7	21	35	35	21	7	1

Fig.3. $Fd(3,n)$ for even n.

$Fd(3,1)$	1							
$Fd(3,3)$	1	1						
$Fd(3,5)$	1	2	1					
$Fd(3,7)$	1	3	3	1				
$Fd(3,9)$	1	4	6	4	1			
$Fd(3,11)$	1	5	10	10	5	1		
$Fd(3,13)$	1	6	15	20	15	6	1	
$Fd(3,15)$	1	7	21	35	35	21	7	1

Fig.4. $Fd(3,n)$ for odd n.

Using this idea we can obtain $Fd(k,n)$ for an arbitrary k.

Analogously as the classical Fibonacci numbers the distance Fibonacci numbers can be extended to the negative integers n. Let $k \geqslant 2$, $n \geqslant 0$ be integers. Then

$$Fd(k,-n) = Fd(k,-n+k) - Fd(k,-n+1) \quad \text{for } n \geqslant 0 \tag{4}$$

with initial conditions $Fd(k,n) = 1$ for $0 \leqslant n \leqslant k-1$.

The Table 2 includes first words of the distance Fibonacci numbers for negative integers n.

Tab. 2. The distance Fibonacci numbers for negative integers.

n	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_n	34	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8
$Fd(3,n)$	2	-1	0	1	-1	1	0	0	1	0	1	1	1	2	2	3
$Fd(4,n)$	6	-4	3	-2	2	-1	1	0	1	0	1	1	1	1	2	2
$Fd(5,n)$	-2	2	-1	1	0	0	1	0	1	0	1	1	1	1	1	2

THEOREM 2.6 *Let $k \geqslant 2, n \geqslant 0$ be integers. Then*
(xi) $\sum\limits_{i=0}^{n} Fd(k,-ki) = -Fd(k,-nk-1)+1$,
(xii) $\sum\limits_{i=0}^{n} Fd(k,-ki+1) = -Fd(k,-nk)+2$.

PROOF (xi) For $n=0$ we have $Fd(k,0) = -Fd(k,-1)+1 = 1$. Assume that the equality (xi) holds for an arbitrary n. We shall prove that it is true for integer $n+1$, clearly, that $\sum\limits_{i=0}^{n+1} Fd(k,-ki) = -Fd(k,-nk-k-1)+1$. Using the induction's assumption and the relation (4) we obtain that $\sum\limits_{i=0}^{n+1} Fd(k,-ki) = \sum\limits_{i=0}^{n} Fd(k,-ki) + Fd(k,-nk-k) = -Fd(k,-nk-1)+1+Fd(k,-nk-k) = -Fd(k,-nk-k-1)+1$.
The equality (xii) we prove analogously as (xi), which ends the proof. ∎

3. A matrix representation. In the last decades the theory of the Fibonacci numbers and the like was complemented by the theory of matrix generators. For the classical Fibonacci numbers the matrix generator has the following form $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and it is well-known that $Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$. This generator immediately gives the Cassini formula for the Fibonacci numbers, namely $detQ^n = (-1)^n = F_{n+1}F_{n-1} - F_n^2$. It is worth to mention that this identity is called "the Cassini formula" in honour of the 17th century astronomer Giovanni Cassini who derived this formula. The main purpose of this section is to describe the theory of matrix generators for the distance Fibonacci numbers.

We recall that $Fd(k,n) = Fd(k,n-k+1) + Fd(k,n-k)$ for $n \geqslant k$ and $Fd(k,n) = 1$ for $0 \leqslant n \leqslant k-1$.

Let $Q_k = [q_{ij}]_{k\times k}$. For a fixed $1 \leqslant i \leqslant k$ an element q_{i1} is equal to the coefficient of $Fd(k,n-i)$ in the recurrence formula for the distance Fibonacci numbers. Moreover for $j \geqslant 2$ we have

$$q_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}.$$

This definition gives the following matrices for $k = 2,3,4,...,k$, respectively.

$$Q_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, Q_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \dots$$

$$Q_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The matrix Q_k will be named as the generator of the distance Fibonacci numbers or the distance Fibonacci matrix. Moreover we define the matrix A_k of order k as the matrix of initial conditions

$$A_k = \begin{bmatrix} Fd(k,2k-2) & Fd(k,2k-3) & \cdots & Fd(k,k) & Fd(k,k-1) \\ Fd(k,2k-3) & Fd(k,2k-4) & \cdots & Fd(k,k-1) & Fd(k,k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Fd(k,k) & Fd(k,k-1) & \cdots & Fd(k,2) & Fd(k,1) \\ Fd(k,k-1) & Fd(k,k-2) & \cdots & Fd(k,1) & Fd(k,0) \end{bmatrix}.$$

THEOREM 3.1 *Let $k \geqslant 2$, $n \geqslant 1$ be integers. Then*

$$(5) \quad A_k Q_k^n = \begin{bmatrix} Fd(k,n+2k-2) & \cdots & Fd(k,n+k) & Fd(k,n+k-1) \\ Fd(k,n+2k-3) & \cdots & Fd(k,n+k-1) & Fd(k,n+k-2) \\ \vdots & \ddots & \vdots & \vdots \\ Fd(k,n+k) & \cdots & Fd(k,n+2) & Fd(k,n+1) \\ Fd(k,n+k-1) & \cdots & Fd(k,n+1) & Fd(k,n+0) \end{bmatrix}.$$

PROOF Let $k \geqslant 2$ be a fixed integer. If $n = 1$, then by (1) and simple calculations the result immediately follows. Assume that the formula is true for all integers $1, ..., n$. We shall show that Theorem is true for integer $n+1$. Since $A_k Q_k^{n+1} = (A_k Q_k^n) Q_k$, so by our assumption and from the recurrence form (1) we obtain that

$$A_k Q_k^{n+1} =$$
$$\begin{bmatrix} Fd(k,n+2k-2) & Fd(k,n+2k-3) & \cdots & Fd(k,n+k-1) \\ Fd(k,n+2k-3) & Fd(k,n+2k-4) & \cdots & Fd(k,n+k-2) \\ \vdots & \vdots & \ddots & \vdots \\ Fd(k,n+k) & Fd(k,n+k-1) & \cdots & Fd(k,n+1) \\ Fd(k,n+k-1) & Fd(k,n+k-2) & \cdots & Fd(k,n+0) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} =$$
$$\begin{bmatrix} Fd(k,n+2k-1) & Fd(k,n+2k-2) & \cdots & Fd(k,n+k) \\ Fd(k,n+2k-2) & Fd(k,n+2k-3) & \cdots & Fd(k,n+k-1) \\ \vdots & \vdots & \ddots & \vdots \\ Fd(k,n+k+1) & Fd(k,n+k) & \cdots & Fd(k,n+2) \\ Fd(k,n+k) & Fd(k,n+k-1) & \cdots & Fd(k,n+1) \end{bmatrix},$$

which ends the proof. ■

Using the same idea we can define the matrix generator R_k of the distance Fibonacci numbers for negative integers.

Let $R_k = [q_{ij}]_{k \times k}$. For a fixed $1 \leqslant i \leqslant k$ an element q_{i1} is equal to the coefficient of $Fd(k, -n+i)$ in the recurrence formula (4). For $j \geqslant 2$ we have

$$q_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}.$$

This definition gives the following matrices for $k = 2, 3, 4, ..., k$, respectively.

$$R_2 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, R_3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, R_4 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, ...$$

$$R_k = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Analogously we define the matrix B_k of initial conditions

$$B_k = \begin{bmatrix} Fd(k,0) & Fd(k,1) & \cdots & Fd(k,k-2) & Fd(k,k-1) \\ Fd(k,1) & Fd(k,2) & \cdots & Fd(k,k-1) & Fd(k,k) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Fd(k,k-2) & Fd(k,k-1) & \cdots & Fd(k,2k-4) & Fd(k,2k-3) \\ Fd(k,k-1) & Fd(k,k) & \cdots & Fd(k,2k-3) & Fd(k,2k-2) \end{bmatrix}.$$

Theorem 3.2 *Let $k \geqslant 2$, $n \geqslant 1$ be integers. Then*

$$B_k R_k^n = \begin{bmatrix} Fd(k,-n+0) & \cdots & Fd(k,-n+k-2) & Fd(k,-n+k-1) \\ Fd(k,-n+1) & \cdots & Fd(k,-n+k-1) & Fd(k,-n+k) \\ \vdots & \ddots & \vdots & \vdots \\ Fd(k,-n+k-2) & \cdots & Fd(k,-n+2k-4) & Fd(k,-n+2k-3) \\ Fd(k,-n+k-1) & \cdots & Fd(k,-n+2k-3) & Fd(k,-n+2k-2) \end{bmatrix}.$$

Theorem 3.3 *Let $k \geqslant 2$ be integer. Then*

$$detQ_k = detR_k = (-1)^{k+1}. \tag{6}$$

$$detA_k = detB_k = (-1)^{\frac{(k+1)k-2}{2}}. \tag{7}$$

The next theorem gives the generalization of the Cassini formula for distance Fibonacci numbers.

Theorem 3.4 *Let $k \geqslant 2$, $n \geqslant 1$ be integers. Then*

$$det(A_k Q_k^n) = det(B_k R_k^n) = (-1)^{\frac{(k+1)(k+2n)-2}{2}}. \tag{8}$$

For given k, matrices Q_k^{-1} and R_k^{-1} have the following form.

$$Q_k^{-1} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, R_k^{-1} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 1 & \cdots & 0 & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Using the same method as in Theorem 3.1 we can prove that Q_k^{-1} and R_k^{-1} also are matrix generators of distance Fibonacci numbers for negative and non-negative integers, respectively.

THEOREM 3.5 *Let $k \geqslant 2$, $n \geqslant 0$ be integers. Then*

$$A_k(Q_k^{-1})^n = \begin{bmatrix} Fd(k,-n+2k-2) & \cdots & Fd(k,-n+k) & Fd(k,-n+k-1) \\ Fd(k,-n+2k-3) & \cdots & Fd(k,-n+k-1) & Fd(k,-n+k-2) \\ \vdots & \ddots & \vdots & \vdots \\ Fd(k,-n+k) & \cdots & Fd(k,-n+2) & Fd(k,-n+1) \\ Fd(k,-n+k-1) & \cdots & Fd(k,-n+1) & Fd(k,-n+0) \end{bmatrix}$$

$$B_k(R_k^{-1})^n = \begin{bmatrix} Fd(k,n+0) & \cdots & Fd(k,n+k-2) & Fd(k,n+k-1) \\ Fd(k,n+1) & \cdots & Fd(k,n+k-1) & Fd(k,n+k) \\ \vdots & \ddots & \vdots & \vdots \\ Fd(k,n+k-2) & \cdots & Fd(k,n+2k-4) & Fd(k,n+2k-3) \\ Fd(k,n+k-1) & \cdots & Fd(k,n+2k-3) & Fd(k,n+2k-2) \end{bmatrix}.$$

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