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On oscillation behavior of solutions of differential equations generated by delays

It is well known that the delayed arguments have an important influence on the oscillatory properties of solutions of differential equations. In essence, the simple example due to Bielecki [1] demonstrate, that the ordinary differential equation

$$(\mathrm{E}_1) \qquad x''(t)-a^2x(t) = 0, \quad 0 < a = \text{const},$$

has only a non-oscillatory solutions of the form $x(t) = C_1e^{at}+C_2e^{-at}$, however, on the other hand, the delay differential equation

$$(\mathrm{E}_2) \qquad x''(t)-a^2x(t-\pi) = 0$$

has an oscillatory solutions of the form $x(t) = C_1\sin a(t-C_2)$, where C_1 and C_2 are constants. It is clearly, that this difference in the oscillation behaviour of solutions of equations (E_1) and (E_2) is generated by the delay term π.

The purpose of the present paper is to study the oscillatory behaviour of solutions of non-linear delay differential equation

$$(1) \qquad x^{(n)}(t)+(-1)^{n+1}f\big(t, x(g_0(t)), x'(g_1(t)), \ldots, x^{(n-1)}(g_{n-1}(t))\big) = 0, \quad n \geqslant 2,$$

generated by the delayed arguments $g_k(t)$ $(k = 0, 1, \ldots, n-1)$. The theorems of this paper do not hold in the particular case $g_k(t) \equiv t$ $(k = 0, 1, \ldots, n-1)$ of ordinary differential equations.

In the following we shall always suppose that the functions g_k: $\langle t_0, \infty) \to R$ $(k = 0, 1, \ldots, n-1)$ and f: $\langle t_0, \infty) \times R^n \to R$ are continuous and satisfy the assumptions:

$$(2) \qquad g_k(t) \leqslant t \quad \text{and} \quad \lim_{t\to\infty} g_k(t) = \infty \quad (k = 0, 1, \ldots, n-1),$$

$$(3) \qquad x_0f(t, x_0, x_1, \ldots, x_{n-1}) > 0 \quad \text{for } x_0 \neq 0,$$

$$(4) \qquad |f(t, x_0, \ldots, x_{n-1})| \leqslant |f(t, y_0, \ldots, y_{n-1})| \quad \text{for } |x_k| \leqslant |y_k| \quad (k = 0, 1, \ldots, n-1),\ x_0y_0 > 0,$$

$$(5) \qquad \big|f\big(t, a(t)x_0, \ldots, a(t)x_{n-1}\big)\big| \geqslant A\big(a(t)\big)\,|f(t, x_0, \ldots, x_{n-1})| \quad \text{for } x_k \neq 0$$

$(k = 0, 1, \ldots, n-1)$, where the functions $a\colon (0, \infty) \to (0, M)$ and $A\colon (0, M) \to (0, \infty)$ are continuous, and M is a positive constant.

We restrict the attention to non-trivial solutions of (1) which exist on a positive half-line. Such a solution we call oscillatory, if it has an infinite sequence of zeros tending to infinity. Otherwise, we call it non-oscillatory.

For $g_k(t) \leqslant t$ $(k = 0, 1, \ldots, n-1)$ we denote

$$Q_k^m(t) = \frac{[t - g_k(t)]^{m-k}}{(m-k)!}, \qquad 0 \leqslant k \leqslant m.$$

LEMMA 1. *If $u^{(k)}(t)$ $(k = 0, 1, \ldots, n-1)$ are absolutely continuous and of constant sign on the interval $\langle t_0, \infty)$, and*

$$(6) \qquad (-1)^k u(t) u^{(k)}(t) \geqslant 0 \qquad (k = 0, 1, \ldots, n)$$

for $t \geqslant t_0$, then

$$(7) \qquad \left|u^{(k)}\big(g_k(t)\big)\right| \geqslant Q_k^{n-1}(t)\, |u^{(n-1)}(t)| \qquad (k = 0, 1, \ldots, n-1)$$

for $t \geqslant g_k(t) \geqslant t_0$, where g_k satisfies (2).

Proof. From (6) it follows that the functions $|u^{(k)}(t)|$ $(k = 0, 1, \ldots, n-1)$ are non-increasing for $t \geqslant t_0$. Thus, in view of (6), we obtain for $t \geqslant s \geqslant t_0$

$$|u^{(n-1)}(s)| \geqslant |u^{(n-1)}(t)|,$$

$$|u^{(k)}(s)| \geqslant |u^{(k)}(s)| - |u^{(k)}(t)| = \int_s^t |u^{(k+1)}(r)|\, dr$$

$$\geqslant \int_s^t \frac{(t-r)^{n-2-k}}{(n-2-k)!} |u^{(n-1)}(t)|\, dr = \frac{(t-s)^{n-1-k}}{(n-1-k)!} |u^{(n-1)}(t)|$$

$$(k = 0, 1, \ldots, n-2).$$

Therefore from the above inequalities for $t \geqslant g_k(t) \geqslant t_0$ we obtain (7).

LEMMA 2. *If $u^{(k)}(t)$ $(k = 0, 1, \ldots, n-1)$ are absolutely continuous and of constant sign on the interval $\langle t_0, \infty)$ and $u(t) u^{(n)}(t) \leqslant 0$, then there exists an integer l with $0 \leqslant l \leqslant n-1$, $n+l$ odd, such that*

$$(8) \qquad \left|u^{(k)}\big(g_k(t)\big)\right| \geqslant Q_k^l\big(g_k(t)\big) \left|u^{(l)}\big(g_k(t)\big)\right| \qquad (k = 0, 1, \ldots, l),$$

and

$$(9) \qquad \left|u^{(k)}\big(g_k(t)\big)\right| \geqslant Q_k^{n-1}(t)\, |u^{(n-1)}(t)| \qquad (k = l, l+1, \ldots, n-1)$$

for $t \geqslant g_k(t) \geqslant g_k\big(g_k(t)\big) \geqslant t_0$, where g_k satisfies (2).

Proof. From Lemma 1 of Kiguradze [3] it follows that there exists an integer l $(0 \leqslant l \leqslant n-1)$, $n+l$ odd, such that for $t \geqslant t_0$ we have

$$(10) \qquad \begin{aligned} u(t) u^{(k)}(t) &\geqslant 0 \qquad (k = 0, 1, \ldots, l), \\ (-1)^{l+k} u(t) u^{(k)}(t) &\geqslant 0 \qquad (k = l+1, \ldots, n). \end{aligned}$$

Since $|u^{(l)}(t)|$ is non-increasing and $|u^{(k)}(t)|$ $(k = 0, 1, \ldots, l-1)$ are non-decreasing, so for $t \geqslant s \geqslant t_0$ there hold inequalities

$$|u^{(l)}(s)| \geqslant |u^{(l)}(t)|,$$

$$|u^{(k)}(t)| \geqslant |u^{(k)}(t)| - |u^{(k)}(s)| = \int_s^t |u^{(k+1)}(r)|\,dr$$

$$\geqslant \int_s^t \frac{(r-s)^{l-1-k}}{(l-1-k)!}|u^{(l)}(t)|\,dr = \frac{(t-s)^{l-k}}{(l-k)!}|u^{(l)}(t)| \quad (k = 0, 1, \ldots, l-1).$$

Therefore from the above inequalities for $t \geqslant g_k(t) \geqslant g_k(g_k(t)) \geqslant t_0$ we obtain (8).

From (10) it follows that $(-1)^{l+k}u^{(l)}(t)u^{(k)}(t) \geqslant 0$ $(k = l, l+1, \ldots, n)$ for $t \geqslant t_0$. Now applying Lemma 1 we obtain (9).

LEMMA 3. *If $u^{(k)}(t)$ $(k = 0, 1, \ldots, n-1)$ are absolutely continuous and of constant sign on the interval $\langle t_0, \infty)$ and $u(t)u^{(n)}(t) \geqslant 0$, then either*

$$(11) \qquad |u^{(k)}(g_k(t))| \geqslant C[g_k(t)]^{n-1-k} \qquad (k = 0, 1, \ldots, n-1),\ 0 < C = \text{const}$$

for $t \geqslant g_k(t) \geqslant t_0$, or there exists an integer l $(0 \leqslant l \leqslant n-2)$, $n+l$ even, such that inequalities (8) *and* (9) *hold, where g_k is the same as in Lemma* 2.

Proof. From Lemma 2 of Kiguradze [3] it follows that either

$$(12) \qquad u(t)u^{(k)}(t) \geqslant 0 \quad (k = 0, 1, \ldots, n) \text{ for } t \geqslant t_0,$$

or there exists an integer l $(0 \leqslant l \leqslant n-2,\ n+l$ even), so that

$$u(t)u^{(k)}(t) \geqslant 0 \quad (k = 0, 1, \ldots, l),$$

$$(-1)^{l+k}u(t)u^{(k)}(t) \geqslant 0 \quad (k = l+1, \ldots, n),$$

for $t \geqslant t_0$. From (12) follows inequality (11). In the latter case the proof is analogous to the proof of Lemma 2.

THEOREM 1. *If*

$$(13) \qquad \int_0^M \frac{dr}{A(r)} < \infty \quad \textit{for } M > 0,$$

and

$$(14) \qquad \int^{\infty} |f(t, mQ_0^{n-1}(t), Q_1^{n-1}(t), \ldots, Q_{n-1}^{n-1}(t))|\,dt = \infty, \qquad m^2 = 1,$$

then every bounded solution of equation (1) *is oscillatory.*

Proof. Suppose, there exists non-oscillatory bounded solution $x(t)$ of (1) and let $x(t) \neq 0$ for $t \geqslant t_0$. Since $\lim_{t\to\infty} g_0(t) = \infty$, there exists a point $t_1 \geqslant t_0$ such that $x(g_0) \neq 0$ for $t \geqslant t_1$. Then from (1) and (3) we have

$(-1)^n x(t)x^{(n)}(t) \geqslant 0$ for $t \geqslant t_1$. Since $x(t)$ is bounded, there exists a point $t_2 \geqslant t_1$ such that

$$(15) \qquad (-1)^k x(t)x^{(k)}(t) \geqslant 0 \quad (k = 0, 1, \ldots, n) \text{ for } t \geqslant t_2.$$

Choose $T \geqslant t_2$ so large that $g_k(t) \geqslant t_2$ for $t \geqslant T$. Then from Lemma 1 for $t \geqslant T$ we have

$$(16) \qquad |x^{(k)}(g_k)| \geqslant Q_k^{n-1}(t)\,|x^{(n-1)}(t)| \quad (k = 0, 1, \ldots, n-1).$$

Since $\lim_{t\to\infty} |x^{(n-1)}(t)| = 0$, so from (1), (3) and (15) we obtain

$$(17) \qquad |x^{(n-1)}(T)| \geqslant |x^{(n-1)}(t)| = \int_t^\infty \big|f\big(s, x(g_0), \ldots, x^{(n-1)}(g_{n-1})\big)\big|\,ds = a(t)$$

for $t \geqslant T$. Therefore (16), (17) give

$$|x^{(k)}(g_k)| \geqslant a(t)Q_k^{n-1}(t) \quad (k = 0, 1, \ldots, n-1).$$

From this and in view of (4) and (5) we have

$$\begin{aligned} &\big|f\big(t, x(g_0), x'(g_1), \ldots, x^{(n-1)}(g_{n-1})\big)\big| \\ &\qquad \geqslant \big|f\big(t, a(t)Q_0^{n-1}(t)\operatorname{sign} x(g_0), a(t)Q_1^{n-1}(t), \ldots, a(t)Q_{n-1}^{n-1}(t)\big)\big| \\ &\qquad \geqslant A\big(a(t)\big)\big|f\big(t, Q_0^{n-1}(t)\operatorname{sign} x(g_0), Q_1^{n-1}(t), \ldots, Q_{n-1}^{n-1}(t)\big)\big|. \end{aligned}$$

Dividing both sides of the above inequality by $A\big(a(t)\big)$ and integrating the resulting inequality from T to ∞ we obtain

$$\int_T^\infty \big|f\big(t, Q_0^{n-1}(t)\operatorname{sign} x(g_0), Q_1^{n-1}(t), \ldots, Q_{n-1}^{n-1}(t)\big)\big|\,dt \leqslant \int_T^\infty \frac{-a'(t)\,dt}{A\big(a(t)\big)} = \int_0^{a(T)} \frac{dr}{A(r)} < \infty,$$

which contradicts assumption (14).

THEOREM 2. *Let the assumptions of Theorem 1 be satisfied. In addition, suppose that*

$$(18) \qquad \begin{aligned} &\int^\infty \big|f\big(t, mQ_0^l(g_0)Q_l^{n-1}(t), Q_1^l(g_1)Q_l^{n-1}(t), \ldots, Q_l^l(g_l)Q_l^{n-1}(t), Q_{l+1}^{n-1}(t), \ldots \\ &\qquad \ldots, Q_{n-1}^{n-1}(t)\big)\big|\,dt = \infty \quad \textit{for } l = 1, 2, \ldots, n-1;\ m^2 = 1. \end{aligned}$$

Then

(i) *for n odd, every solution of equation* (1) *is oscillatory,*

(ii) *for n even, every solution of equation* (1) *is either oscillatory or tends monotonically to infinity as $t\to\infty$ together with all its derivatives of order up to $(n-1)$ inclusive.*

Proof. Suppose, that equation (1) has a non-oscillatory solution $x(t) \neq 0$ for $t \geqslant t_0$. Since $\lim_{t\to\infty} g_0(t) = \infty$, there exists a point $t_1 \geqslant t_0$ such that $x(g_0) \neq 0$ for $t \geqslant t_1$. Then from (1) we have $(-1)^n x(t)x^{(n)}(t) \geqslant 0$ for $t \geqslant t_1$.

(i) Let n be odd. Then $x(t)x^{(n)}(t) \leqslant 0$ for $t \geqslant t_1$. From Lemma 2 it follows that there exists an even integer l $(0 \leqslant l \leqslant n-1)$ such that for sufficiently large $t \geqslant T \geqslant t_1$ we have

$$\begin{aligned} &|x^{(k)}(g_k)| \geqslant Q_k^l(g_k)|x^{(l)}(g_k)| \qquad (k = 0, 1, \ldots, l), \\ &|x^{(k)}(g_k)| \geqslant Q_k^{n-1}(t)|x^{(n-1)}(t)| \qquad (k = l, \ldots, n-1). \end{aligned} \tag{19}$$

From Theorem 1 it follows that the case $l = 0$ is impossible. We shall prove that the case $2 \leqslant l \leqslant n-1$ is also impossible. From (19) for $2 \leqslant l \leqslant n-1$ we obtain

$$\begin{aligned} &|x^{(k)}(g_k)| \geqslant Q_k^l(g_k)Q_l^{n-1}(t)\,|x^{(n-1)}(t)| \qquad (k = 0, 1, \ldots, l), \\ &|x^{(k)}(g_k)| \geqslant Q_k^{n-1}(t)\,|x^{(n-1)}(t)| \qquad (k = l+1, \ldots, n-1). \end{aligned}$$

Since $|x^{(n-1)}(t)|$ is non-increasing for $t \geqslant T$, by Lemma 2, so from equation (1) we have

$$|x^{(n-1)}(t)| \geqslant \int_t^\infty \big|f\big(s, x(g_0), \ldots, x^{(n-1)}(g_{n-1})\big)\big|\,ds$$

for $t \geqslant T$. Now applying the same method of proof as in Theorem 1 we obtain contradiction to assumption (18).

(ii) Let n be even. Then $x(t)x^{(n)}(t) \geqslant 0$ for $t \geqslant t_1$. From Lemma 3 it follows that either

$$|x^{(k)}(g_k)| \geqslant C\,[g_k(t)]^{n-1-k} \qquad (k = 0, 1, \ldots, n-1), \qquad C > 0, \tag{20}$$

or there exists an even integer l $(0 \leqslant l \leqslant n-2)$ such that condition (19) holds, for sufficiently large $t \geqslant T \geqslant t_1$. In the former case we shall prove that $\lim_{t\to\infty} |x^{(k)}(t)| = \infty$ $(k = 0, 1, \ldots, n-1)$. From (1)–(5) and (20) we obtain

$$\begin{aligned} |x^{(n-1)}(t)| &= |x^{(n-1)}(T)| + \int_T^t \big|f\big(s, x(g_0), \ldots, x^{(n-1)}(g_{n-1})\big)\big|\,ds \\ &\geqslant A(C) \int_T^t \big|f\big(s, g_0^{n-1}(s)\operatorname{sign} x(g_0), g_1^{n-2}(s), \ldots, g_{n-2}(s), 1\big)\big|\,ds \\ &\geqslant A(C) \int_T^t \big|f\big(s, Q_0^{n-1}(g_0)\operatorname{sign} x(g_0), Q_1^{n-1}(g_1), \ldots, Q_{n-1}^{n-1}(g_{n-1})\big)\big|\,ds. \end{aligned}$$

From this and in view of (18) we conclude that $\lim_{t\to\infty} |x^{(k)}(t)| = \infty$ $(k = 0, 1, \ldots, n-1)$. In the latter case the proof is identical as in case (i).

THEOREM 3. *Let the assumptions of Theorem 1 be satisfied. If there exists a constant $\beta > 1$ such that $\beta^{n-1-k} g_k(t) \leqslant t$ $(k = 1, 2, \ldots, n-2)$ and*

$$\int_t^{\infty} \left| f\left(t, m g_0^{n-1}(t), g_1^{n-2}(t), \ldots, g_{n-2}(t), 1\right) \right| dt = \infty, \quad m^2 = 1, \tag{21}$$

then conclusion of Theorem 2 holds.

Proof. Suppose that equation (1) has non-oscillatory solution $x(t) \neq 0$ for $t \geqslant t_0$. Then, like in the proof of Theorem 2, we obtain $(-1)^n x(t) x^{(n)}(t) \geqslant 0$ for $t \geqslant t_1 \geqslant t_0$.

(i) Let n be odd. Then from Lemma 1 of Kiguradze [3] it follows that there exists an even integer l $(0 \leqslant l \leqslant n-1)$ such that

$$\begin{gathered} x(t) x^{(k)}(t) \geqslant 0 \quad (k = 0, 1, \ldots, l), \\ (-1)^k x(t) x^{(k)}(t) \geqslant 0 \quad (k = l+1, \ldots, n), \end{gathered} \tag{22}$$

for sufficiently large $t \geqslant t_2 \geqslant t_1$. The case $l = 0$ has been treated in Theorem 1. Suppose now that $2 \leqslant l \leqslant n-1$. From Lemma 2 of [4] it follows that there exist a positive constants D and T such that

$$\begin{gathered} |x(\beta^{1-n} g_0)| \geqslant D \cdot g_0^{n-1} |x^{(n-1)}(t)|, \\ |x^{(k)}(g_k)| \geqslant D \cdot g_k^{n-1-k} |x^{(n-1)}(t)| \quad (k = 1, 2, \ldots, n-1) \end{gathered} \tag{23}$$

for $t \geqslant T \geqslant t_2$. Since $\beta > 1$ and $|x(t)|$ is non-decreasing and $|x^{(n-1)}(t)|$ is non-increasing, then for $t \geqslant T$ we have

$$\begin{gathered} |x^{(k)}(g_k)| \geqslant D \cdot g_k^{n-1-k} |x^{(n-1)}(t)| \quad (k = 0, 1, \ldots, n-1), \\ |x^{(n-1)}(t)| \geqslant \int_t^{\infty} \left| f\left(s, x(g_0), \ldots, x^{(n-1)}(g_{n-1})\right) \right| ds. \end{gathered} \tag{24}$$

Now using an argument to the one used in Theorem 2 we obtain a contradiction to assumption (21).

(ii) Let n be even. Then from Lemma 2 of Kiguradze [3] it follows that either

$$x(t) x^{(k)}(t) \geqslant 0 \quad (k = 0, 1, \ldots, n) \tag{25}$$

or there exists an even integer l $(0 \leqslant l \leqslant n-2)$ such that (22) holds for sufficiently large $t \geqslant T \geqslant t_1$. In the former case, similarly as in the proof of (ii) of Theorem 2, we obtain $\lim_{t \to \infty} |x^{(k)}(t)| = \infty$ $(k = 0, 1, \ldots, n-1)$. Now, we shall prove that the latter case is impossible. From Theorem 1 it follows that the case $l = 0$ is impossible. If $2 \leqslant l \leqslant n-2$, then in view of Lemma 4 of [4], we obtain inequalities (24). Then from (24), similarly as in case (i), we obtain contradiction to assumption (21).

THEOREM 4. *If $g_0(t)$ is non-decreasing for $t \geqslant t_0$ and*

$$\lim_{v\to 0+} \frac{v}{A(v)} = L < \infty, \quad L = \text{const}, \tag{26}$$

$$\limsup_{t\to\infty} \int_{g_0(t)}^{t} \frac{[r-g_0(t)]^{n-1}}{(n-1)!} |f(r, m, 0, \ldots, 0)| dr \geqslant \begin{cases} L, & \text{when } L \neq 0, \\ \varepsilon > 0, & \text{when } L = 0, \end{cases} \tag{27}$$

$$m^2 = 1,$$

then every bounded solution of equation (1) *is oscillatory.*

Proof. Suppose, that equation (1) has bounded non-oscillatory solution $x(t) \neq 0$ for $t \geqslant t_0$. Like in the proof of Theorem 1 we obtain condition (15), which implies

$$|x(s)| - |x(t)| = \sum_{k=1}^{n-1} \frac{(t-s)^k}{k!} |x^{(k)}(t)| + \int_s^t \frac{(r-s)^{n-1}}{(n-1)!} |x^{(n)}(r)| dr \tag{28}$$

for $t \geqslant s \geqslant T$. Choose $T_1 \geqslant T$ such that $g_0(t) \geqslant T$ for $t \geqslant T_1$. Since $|x(t)|$ is non-increasing and $g_0(t)$ is non-decreasing for $t \geqslant T$, then for $t \geqslant r \geqslant T_1$ we have $|x(g_0(r))| \geqslant |x(g_0(t))| \geqslant |x(g_0(t))| - |x(t)| = b(t)$ and $\lim_{t\to\infty} b(t) = 0$. Therefore, from (1)–(5) and (28) we obtain for $t \geqslant T_1$

$$\begin{aligned} b(t) &\geqslant \int_{g_0(t)}^{t} \frac{[r-g_0(t)]^{n-1}}{(n-1)!} \left|f\left(r, x(g_0(r)), \ldots, x^{(n-1)}(g_{n-1}(r))\right)\right| dr \\ &\geqslant \int_{g_0(t)}^{t} \frac{[r-g_0(t)]^{n-1}}{(n-1)!} \left|f\left(r, b(t)\operatorname{sign} x(g_0), x'(g_1), \ldots, x^{(n-1)}(g_{n-1})\right)\right| dr \\ &\geqslant A\left(b(t)\right) \int_{g_0(t)}^{t} \frac{[r-g_0(t)]^{n-1}}{(n-1)!} \left|f\left(r, \operatorname{sign} x(g_0), 0, \ldots, 0\right)\right| dr, \end{aligned}$$

from where

$$\int_{g_0(t)}^{t} \frac{[r-g_0(t)]^{n-1}}{(n-1)!} \left|f\left(r, \operatorname{sign} x(g_0), 0, \ldots, 0\right)\right| dr \leqslant \frac{b(t)}{A\left(b(t)\right)}.$$

This gives a contradiction as $t \to \infty$. Therefore, $x(t)$ is oscillatory.

Remark. From Theorem 4 in the case $f = p(t)\left[x\left(g_0(t)\right)\right]^a$, $p(t) \geqslant 0$, $0 < a \leqslant 1$, n even, we obtain some results of Gustafson [2].

THEOREM 5. *Let the assumptions of Theorem 4 be satisfied. In addition, suppose that there exists a non-decreasing function $g(t)$ such that $g_k(t) \leqslant g(t) \leqslant t$ $(k = 0, 1, \ldots, n-1)$ for $t \geqslant t_0$ and*

$$(29)\quad \limsup_{t\to\infty} \int_{g(t)}^{t} \frac{[r-g(t)]^{n-1-l}}{(n-1-l)!} \left|f\left(r, mQ_0^l(g_0), Q_1^l(g_1), \ldots, Q_l^l(g_l), 0, \ldots, 0\right)\right| dr \geqslant \begin{cases} L, & L \neq 0, \\ \varepsilon > 0, & L = 0, \end{cases}$$

$$l = 1, 2, \ldots, n-1, \quad m^2 = 1.$$

Then

(i) *for n odd, every solution of equation* (1) *is oscillatory,*

(ii) *for n even, every solution $x(t) = o(t^{n-1})$ $(t \to \infty)$ of* (1) *is oscillatory.*

Proof. Suppose, that equation (1) has non-oscillatory solution $x(t) \neq 0$ for $t \geqslant t_0$. Then, like in the proof of Theorem 2, we obtain $(-1)^n x(t) x^{(n)}(t) \geqslant 0$ for $t \geqslant t_1 \geqslant t_0$.

(i) Let n be odd. Then, by Lemma 2, there exists an even integer l $(0 \leqslant l \leqslant n-1)$ such that

$$(30)\quad \begin{aligned} x(t)x^{(k)}(t) &\geqslant 0 \quad (k = 0, 1, \ldots, l), \\ (-1)^{l+k} x(t) x^{(k)}(t) &\geqslant 0 \quad (k = l+1, \ldots, n) \end{aligned}$$

and

$$(31)\quad \left|x^{(k)}\big(g_k(t)\big)\right| \geqslant Q_k^l\big(g_k(t)\big)\left|x^{(l)}\big(g_k(t)\big)\right| \quad (k = 0, 1, \ldots, l),$$

for sufficiently large $t \geqslant T \geqslant t_1$. From (30) we have for $t \geqslant s \geqslant T$

$$(32)\quad |x^{(l)}(s)| - |x^{(l)}(t)| = \sum_{k=l+1}^{n-1} \frac{(t-s)^{k-l}}{k!} |x^{(k)}(t)| + \int_s^t \frac{(r-s)^{n-1-l}}{(n-1-l)!} |x^{(n)}(r)|\, dr.$$

Since $|x^{(l)}(t)|$ is non-increasing for $t \geqslant T$, then from (30) we get for $t \geqslant r \geqslant T$

$$(33)\quad \begin{aligned} \left|x^{(l)}\big(g_k(r)\big)\right| &\geqslant \left|x^{(l)}\big(g(r)\big)\right| \geqslant \left|x^{(l)}\big(g(t)\big)\right| \\ &\geqslant \left|x^{(l)}\big(g(t)\big)\right| - |x^{(l)}(t)| = b_l(t) \quad (k = 0, 1, \ldots, l) \end{aligned}$$

and $\lim_{t\to\infty} b_l(t) = 0$. Therefore, from (1)–(5) and (31)–(33) we obtain

$$\begin{aligned} b_l(t) &\geqslant \int_{g(t)}^{t} \frac{[r-g(t)]^{n-1-l}}{(n-1-l)!} \left|f\left(r, x\big(g_0(r)\big), \ldots, x^{(n-1)}\big(g_{n-1}(r)\big)\right)\right| dr \\ &\geqslant \int_{g(t)}^{t} \frac{[r-g(t)]^{n-1-l}}{(n-1-l)!} \left|f\left(r, b_l(t) Q_0^l\big(g_0(r)\big) \operatorname{sign} x(g_0), b_l(t) Q_1^l\big(g_1(r)\big), \ldots\right.\right. \end{aligned}$$

$$\ldots, b_l(t)Q_l^l\big(g_l(r)\big), x^{(l+1)}\big(g_{l+1}(r)\big), \ldots, x^{(n-1)}\big(g_{n-1}(r)\big)\big)\Big| dr$$

$$\geqslant A\big(b_l(t)\big) \int\limits_{g(t)}^{t} \frac{[r-g(t)]^{n-1-l}}{(n-1-l)!} \big|f\big(r, Q_0^l(g_0)\operatorname{sign} x(g_0), Q_1^l(g_1), \ldots$$

$$\ldots, Q_l^l(g_l), 0, \ldots, 0\big)\big| dr.$$

From this we obtain a contradiction to assumption (29).

(ii) Let n be even. Then from Lemma 3 it follows that either (20) holds or there exists an even integer l $(0 \leqslant l \leqslant n-2)$ such that conditions (30) and (31) hold. The former case is impossible, since $x(t) = o(t^{n-1})$ $(t \to \infty)$. In the latter case the proof is identical as in case (i).

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