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Brown's theorem for cohomology theories on categories of chain complexes

Introduction. Let $E = \{E_n, h_n\}$ be a spectrum; that is a sequence of topological spaces E_n with base point and base points preserving maps $h_n\colon E_n \to \Omega E_{n+1}$. A spectrum E is called an *Ω-spectrum* if h_n are homotopy equivalences. From Brown's results in [1] and [2] it follows that

1. *If $E = \{E_n, h_n\}$ is an Ω-spectrum, then there exists a generalized cohomology theory H_E^* defined on the category C of pairs of CW-complexes such that*

$$H_E^n(X, \emptyset) = [X \cup p, E_n] \qquad (p \notin X)$$

for all n. ($[X\cup p, E_n]$ is the set of homotopy classes of maps from $X \cup p$ to E_n.

2. *For any generalized cohomology theory H^* on C satisfying certain conditions one can find an Ω-spectrum E such that H^* and H_E^* are naturally equivalent.*

In this paper the notion of spectrum is defined in the category of complexes of modules and it is shown that Brown's results hold for cohomology theories on some subcategories of this category.

1. Normal sequences. Let R be a fixed ring with identity and let ${}_RM$ be the category of all left R-modulus. Denote by K (K^-) the category of all (left) chain complexes over ${}_RM$ and recall that complex maps $f, g\colon X \to Y$ are called *homotopic* ($f \sim g$) if there exist module homomorphisms $s_n\colon X_n \to Y_{n+1}$ such that $f_n - g_n = d_{n+1}s_n + s_{n-1}d_n$. The cone and suspension functors are defined as follows:

$$(CX)_n = X_n \oplus X_{n-1}, \qquad d_n(x_n, x_{n-1}) = (dx_n - x_{n-1}, -d_{n-1}x_{n-1}),$$
$$(SX)_n = X_{n-1}, \qquad d^{SX} = -d^X.$$

If $f\colon X \to Y$ is a complex map, then $(Cf)_n = f_n \oplus f_{n-1}$, $(Sf)_n = f_{n-1}$. CX is a contractible complex (i.e. $1_{CX} \sim 0$) and $f \sim 0\colon X \to Y$ if and only if f can be factored through the natural complex map $j\colon X \to CX$.

DEFINITION. An exact sequence of complexes

$$\boldsymbol{X}\colon\ 0 \to X' \xrightarrow{i} X \xrightarrow{p} X'' \to 0$$

is said to be *normal* if the sequence

$$0 \to X'_n \to X_n \to X''_n \to 0$$

splits for all $n \in Z$. A complex map $f\colon X \to Y$ is called a *normal monomorphism* if $0 \to X \xrightarrow{f} Y \to \mathrm{Coker}(f) \to 0$ is the normal sequence.

In what follows dealing with a normal sequence X we assume that $X_n = X'_n \oplus X''_n$ and that i, p are the natural injections and projections respectively.

If X is a normal sequence, then it is easy to see that the maps $\theta_n\colon X''_n \to X'_{n-1}$ given by the equality

$$d_n(0, x''_n) = \big(\theta_n(x''_n), d''_n(x'_n)\big)$$

define a complex map $\theta(X)\colon X'' \to SX'$. Moreover, for any commutative diagram with normal rows:

$$\begin{array}{ccccccccc} 0 & \to & X' & \to & X & \to & X'' & \to & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \to & Y' & \to & Y & \to & Y'' & \to & 0 \end{array}$$

the diagram

$$\begin{array}{ccc} X'' & \xrightarrow{\theta(X)} & SX' \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle Sf'} \\ Y'' & \xrightarrow{\theta(Y)} & SY' \end{array}$$

is homotopy commutative.

(1.1) LEMMA (Homotopy Extension Property). *If*

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "i"] \arrow[d, "f'"] & X \arrow[r, "p"] \arrow[dl, "f"] & X'' \arrow[r] & 0 \\ & Y & & & \end{tikzcd}$$

is a commutative diagram with the normal row and $g' \sim f'$, then there exists a complex map $g\colon X \to Y$ such that $gi = g'$ and $g \sim f$.

Proof. Since $g' \sim f'$ then

$$(*) \qquad g'_n - f'_n = d^Y_{n+1} s'_n + s'_{n-1} d'_n$$

for some module homomorphisms $s'_n\colon X'_n \to Y_{n+1}$. Put

$$g_n(x'_n, x''_n) = f_n(0, x''_n) + g'_n(x'_n) + s'_{n-1}\theta_n(x''_n).$$

A straightforward computation shows that the maps $g_n\colon X_n \to Y_n$ define a complex map $g\colon X \to Y$ such that $gi = g'$. Furthermore setting $s_n(x'_n, x''_n) = s'_n(x'_n)$ ($s_n\colon X_n \to Y_{n+1}$) and using equality $(*)$ we have

$$\begin{aligned} (d^Y_{n+1}s_n + s_{n-1}d_n)(x'_n, x''_n) &= d^Y_{n+1}s'_n(x'_n) + s_{n-1}\big(d'_n x'_n + \theta_n(x''_n), d''_n x''_n\big) \\ &= g'_n(x'_n) - f'_n(x'_n) - s'_{n-1}d'_n(x'_n) + s'_{n-1}d'_n(x'_n) + s'_{n-1}\theta_n(x''_n) \\ &= (g_n - f_n)(x'_n, x''_n). \end{aligned}$$

Therefore $g \sim f$ and the lemma is proved.

Similarly as Lemma 1.1 one can prove the following

(1.2) LEMMA (Homotopy Lifting Property). *If*

$$
\begin{tikzcd}
 & & Y \arrow[d, "f"] \arrow[dr, "f'"] & & \\
0 \arrow[r] & X' \arrow[r, "i"] & X \arrow[r, "p"] & X'' \arrow[r] & 0
\end{tikzcd}
$$

is a commutative diagram with the normal row and $f' \sim g'$, then there is a complex map $g\colon Y \to X$ such that $pg = g'$ and $f \sim g$.

(1.3) COROLLARY. *If $(f', f, f'')\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of normal sequences and $f' \sim g'$, then there exists a map of normal sequences $(g', g, g'')\colon \boldsymbol{X} \to \boldsymbol{Y}$ such that $f \sim g$ and $f'' = g''$.*

The proof is left to the reader.

(1.4) COROLLARY. *If $\boldsymbol{X}\colon 0 \to X' \xrightarrow{i} X \xrightarrow{p} X'' \to 0$ is a normal sequence, then the map i is a homotopy equivalence if and only if the complex X'' is contractible. Analogously p is a homotopy equivalence if and only if X' is contractible.*

Proof. Suppose $i\colon X' \to X$ is a homotopy equivalence. Then $l'i \sim 1_{X'}$, $il' \sim 1_X$ for some complex map $l'\colon X \to X'$. Using now Lemma 1.1 to $f' = l'i$ and $g' = 1$ we get a complex map $l\colon X \to X'$ such that $li = 1$ and $l \sim l'$. Hence we deduce that the sequence X splits. In particular there exists a complex map $k\colon X'' \to X$ such that $pk = 1_{X''}$. Since $il \sim il' \sim 1$, we have $il - 1 \sim 0$ and consequently $0 \sim p(il-1)k = -1_{X''}$. Thus we have shown X'' is contractible. Conversely, if X'' is contractible, then applying Lemma 1.2 to $f = 0_{X''}$, $f' = 0$ and $g' = 1$ we obtain a complex map $k\colon X'' \to X$ such that $pk = 1$ and $k \sim 0$. Hence $li = 1$ and $1_X = il + {} + kp$ for some complex map $l\colon X \to X'$. It follows $1 - il \sim 0$ i.e. the map i is a homotopy equivalence. The proof of the second part of the corollary is similar.

(1.5) Remark. Corollary 1.4 is a completion of [3], 2.18.

(1.6) LEMMA. *If $F\colon X \to Y$ is a complex map, then $F = gi$, where i is a normal monomorphism and g is a homotopy equivalence.*

Proof. $F = (X \xrightarrow{i} CX \oplus Y \xrightarrow{g} Y)$, where $i(x) = (x, 0, f(x))$ and g is the natural projection.

2. Brown's Theorem. A complex X is called *projective* if X_n is a projective module for all $n \in Z$ (in general it is not a projective object in K). Throughout this section $\mathscr{P}$ will denote one of the following full subcategories of the category K.

(A) The category of all left projective complexes.

(B) The category of all projective complexes X such that $X_n = 0$ for sufficiently small n.

(C) The category of all projective complexes. In this case we assume that R is the ring of the finite left global dimension.

Moreover, let $\mathscr{P}^{\square}$ denote the quotient category $\mathscr{P}/\sim$ and let $[P', P] = \mathrm{Hom}_{\mathscr{P}^{\square}}(P', P)$. If $f\colon X \to Y$ is a complex map, then we write $[f]$ for the homotopy class of f.

(2.1) Definition. A cohomology theory on $\mathscr{P}$ is a sequence of contravariant, homotopy preserving functors $H^n\colon \mathscr{P} \to Ab$(1), $n \in Z$, satisfying the following conditions:

(i) for any normal sequence $\boldsymbol{X}$: $0 \to X' \xrightarrow{i} X \xrightarrow{p} X'' \to 0$ there are homomorphisms $d_n(\boldsymbol{X})\colon H^n(X') \to H^{n+1}(X'')$ such that the sequence.

$$\ldots \to H^n(X'') \xrightarrow{p^*} H^n(X) \xrightarrow{i^*} H^n(X') \xrightarrow{d_n(\boldsymbol{X})} H^{n+1}(X'') \to \ldots$$

is exact,

(ii) if $(f', f, f'')\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of normal sequences in $\mathscr{P}$, then the diagram below is commutative:

$$\begin{array}{ccc} H^n(X') & \xrightarrow{d_n(\boldsymbol{X})} & H^{n+1}(X'') \\ \downarrow{\scriptstyle f'^*} & & \downarrow{\scriptstyle f''^*} \\ H^n(Y') & \xrightarrow{d_n(\boldsymbol{Y})} & H^{n+1}(Y) \end{array}$$

If $T\colon {}_RM \to Ab$ is a contravariant functor of finite type, then by [4], Theorem 6.10, the functors $\{D_s^q T\}_{q \in Z}$ restricted to the category $\mathscr{P}$ form a cohomology theory on P.

To give a typical example of cohomology theory on $\mathscr{P}$ we need the following definitions:

(2.2) Definition. A spectrum in the category K is a *sequence of complexes* E_n, $n \in Z$, together with complex maps $\varepsilon_n\colon SE_n \to E_{n+1}$.

(2.3) Definition. A spectrum $\{E_n, \varepsilon_n\}$ is called an *S-spectrum* if the group homomorphisms

$$[SP, SE_q] \xrightarrow{(\varepsilon_q)_*} [SP, E_{q+1}]$$

are isomorphisms for any $P \in \mathrm{ob}\mathscr{P}$ and all $q \in Z$.

(2.4) Example. Suppose $E = \{E_n, \varepsilon_n\}$ is an S-spectrum in $\mathscr{P}$ and define:

$$H^q(\cdot, E) = [\cdot, E_q]\colon \mathscr{P} \to Ab$$

for $q \in Z$. We shall show $\big(H^q(\cdot, E),\ q \in Z\big)$ is a cohomology theory on $\mathscr{P}$. For this purpose we define natural equivalences of functors

$$\sigma^q\colon H^{q+1}(\cdot, E)S \to H^q(\cdot, E)$$

(1) Ab denotes the category of Abelian groups.

as follows:

$$\sigma^q(P) = ([SP, E_{q+1}] \overset{(\varepsilon_q)_\bullet^{-1}}{\simeq} [SP, SE_q] \simeq [P, E_q]).$$

Furthermore, if $\boldsymbol{P} = (0 \to P' \xrightarrow{i} P \xrightarrow{p} P'' \to 0)$ is a normal sequence in $\mathscr{P}$ and $\theta(\boldsymbol{P})\colon P'' \to SP'$ is the complex map defined in Section 1, then the connecting homomorphism

$$d^q(\boldsymbol{P})\colon H^q(P', E) \to H^{q+1}(P'', E)$$

is a composition

$$H^q(P', E) \xrightarrow{\sigma^q(P')^{-1}} H^{q+1}(SP', E) \xrightarrow{H^{q+1}(\theta(\boldsymbol{P}))} H^{q+1}(P'', E).$$

In virtue of [4], Theorem 2.12, to prove that the functors $H^q(\cdot, E)$ together with d^q described above form a cohomology theory on $\mathscr{P}$ it is sufficient to show that the sequences of Abelian groups

$$[P'', E_q] \xrightarrow{p^*} [P, E_q] \xrightarrow{i^*} [P', E_q]$$

is exact for any normal sequence $0 \to P' \xrightarrow{i} P \xrightarrow{p} P'' \to 0$ in $\mathscr{P}$ and $q \in Z$. Clearly, $\operatorname{Im} p^* \subset \operatorname{Ker} i^*$. Let $[f] \in [P, E_q]$ and let $i^*([f]) = [fi] = 0$. Hence $fi \sim 0$ and we have the following commutative diagram with normal rows

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r, "i"] \arrow[d, "fi"] & P \arrow[r, "p"] \arrow[dl, "f"] & P'' \arrow[r] & 0 \\
& E_q & & &
\end{tikzcd}$$

Using the Homotopy Extension Property (Lemma 1.1) we get a complex map $g\colon P \to E_q$ such that $f \sim g$ and $gi = 0$. Then $g = g''p$ for some $g''\colon P'' \to E_q$ because $(P \xrightarrow{p} P'') = \operatorname{Coker} i$. Consequently, $[f] = [g] = p^*([g'']) \in \operatorname{Im} p^*$ and we see $\operatorname{Ker} i^* = \operatorname{Im} p^*$. Thus we have shown that $\{H^q(\cdot, E)\}_{q \in Z}$ is a cohomology theory.

In what follows it will be proved, by the use of the results of E. Brown's paper [2], that any cohomology theory $\{H^q\}_{q \in Z}$ on $\mathscr{P}$ satisfying condition $H^q(\bigoplus_i P_i) \approx \prod_i H^q(P_i)$ is isomorphic with the cohomology theory $\{H^q(\cdot, E)\}_{q \in Z}$ for some S-spectrum E in $\mathscr{P}$.

We start with

(2.5) Definition [2]. A pair (C, C_0), where C_0 is a subcategory of a category C, is called a *homotopy category*, if it satisfies the following conditions

1° C_0 is a small and full subcategory of C;

2° C_0 has finite sums, C has arbitrary sums;

3° if $f_i\colon A \to P_i$, $i = 1, 2$, are in C, then there are maps $g_i\colon P_i \to Z$ in C such that $g_1 f_1 = g_2 f_2$ and such that, if $g_i'\colon P_i \to Z'$ satisfy $g_1' f_1 = g_2' f_2$, then $g_i' = h g_i$ for some $h\colon Z \to Z'$ (h is not necessarily unique);

4° if $f^n\colon P^n \to P^{n+1}$, $n = 1, 2, \ldots$, are in C, then there are $P \in \operatorname{ob} C$ and maps $g^n\colon P^n \to P$ such that

(i) $\operatorname{Lim}_n (g^n)_*\colon \varinjlim_n [Z, P^n] \approx [Z, P]$ for all $Z \in \operatorname{ob} C_0$, where $[P, P']$ is a set of morphisms from P to P' and $(g^n)_*$ is given by $(g^n)_*(h) = g^n h$.

(ii) $\varprojlim_n (g^n)^*\colon [P, Z] \to \varprojlim_n [P^n, Z]$ is an epimorphism for all $Z \in \operatorname{ob} C$, where $(g^n)^*(h) = hg^n$.

(2.6) **Remark.** If C is an additive category, then it follows from Lemma 2.10 in [2] that condition 3° is equivalent to condition 3′:

3′ If $f\colon A \to P$ is in C, then there is $g\colon P \to Z$ in C such that $gf = 0$ and if $g'\colon P \to Z'$ satisfies $g'f = 0$, then $g' = hg$ for some $h\colon Z \to Z'$.

Any map g satisfying 3′ (for a given f) is called an *equalizer of the maps* $f, 0\colon A \to P$.

Let S be the category of sets.

(2.7) Definition [2]. If (C, C_0) is a homotopy category and $H\colon C \to S$ is a contravariant functor, then H is called a *homotopy functor*, if it satisfies the following conditions:

(a) The natural injections $P_i \to \bigoplus_i P_i$ induce an isomorphism

$$H(\bigoplus_i P_i) \approx \prod_i H(P_i).$$

(b) If $f_i\colon A \to P_i$ and $g_i\colon P_i \to Z$ are as in 3° of Definition 2.5 and $u_i \in H(P_i)$ satisfy $H(f_1)u_1 = H(f_2)u_2$, then there is $v \in H(Z)$ such that $H(g_i)v = u_i$ for $i = 1, 2$.

Let P_0 be the full subcategory of $\mathscr{P}$ whose objects are complexes P such that P_m is a finitely generated module for any n and but a finite number of P_n are zero.

(2.8) Proposition. *The pair* $(\mathscr{P}^\square, \mathscr{P}_0^\square)$ *is a homotopy category.*

Proof. Evidently conditions 1° and 2° of Definition 2.5 are satisfied. In view of Remark 2.6 we may prove 3′ instead of 3°. Let then $f\colon A \to P$ be any complex map in $\mathscr{P}$ and let g be the complex map from the pushout diagram

$$\begin{tikzcd} A \arrow[r, "j(A)"] \arrow[d, "f"] & CA \arrow[d] \\ P \arrow[r, "g"] & P \cup CA \end{tikzcd}$$

It is easy to see that $(P \cup CA)_n = (P_n \oplus CA_n)/\{(-f_n(a), a, 0)\colon a \in A_n\} \simeq P_n \oplus A_{n-1}$. Consequently $P \cup CA \in P$.

Furthermore CA is contractible so $gf \sim 0$, i.e. $[g][f] = 0$. If $g'f \sim 0$ for some complex map $g'\colon P \to Z'$ in P, then clearly $g'f$ can be factored through $j(A)$ i.e. $g'f = lj(A)$ for some $l\colon CA \to Z'$. By the pushout property of the above diagram there is a complex map $h: P\cup CA \to Z'$ such that $g' = hg$. Thus we have shown that $[g]$ is an equalizer of $[f]$ and 0. If $A, P \in \mathrm{ob}\mathscr{P}_0$, then the mentioned equality $(P\cup CA)_n = P_n \oplus A_{n-1}$ implies $P\cup CA \in \mathrm{ob}\mathscr{P}_0$.

It remains to prove condition 4°. Take for this purpose complex maps $f^n: P^n \to P^{n+1}$, $n = 1, 2, \ldots$, and observe that in view of Lemma 1.6 and the fact that $CX \oplus Y \in \mathrm{ob}\mathscr{P}$ provided $X, Y \in \mathrm{ob}\mathscr{P}$ we may assume that f^n are normal monomorphisms. Let $g^n: P^n \to P$ be a direct limit in $K({}_RM)$ of the direct system $\{P^n, f^{n,m}\}_{n,m\in Z}$, where $f^{n,m}: P^n \to P^m$ is $f^{m-1}\cdot \ldots \cdot f^n$ for $n < m$ and $f^{n,n} = 1$. At first we show that $P \in \mathrm{ob}\mathscr{P}$, i.e. $P_i = \varinjlim_n \{P_i^n, f_i^{n,m}\}$ is a projective module for any i. Since f^n are normal monomorphisms, the sequences

$$0 \to P_i^n \xrightarrow{f_i^{n,m}} P_i^m \to P_i^m/\mathrm{Im} f_i^{n,m} \to 0$$

split for $n \leqslant m$. Thus $P_i^m/\mathrm{Im} f_i^{n,m}$ are projective modules and the conclusion follows.

Now we prove that $[g^n]: P^n \to P$ satisfy (i) and (ii) of condition 4°. To prove (i), consider the homomorphism

$$g_* = \varinjlim_n (g^n)_*: \varinjlim_n [Z, P^n] \to [Z, P]$$

for $Z \in \mathrm{ob}\mathscr{P}_0$. If $[f] \in [Z, P]$, then $Z \in \mathrm{ob}\mathscr{P}_0$ and $P = \bigcup_{n=1}^{\infty} \mathrm{Im} g^n$ imply $\mathrm{Im} f \subset \mathrm{Im} g^n$ for some n. On the other hand, $g^n: P^n \to \mathrm{Im} g^n$ is an isomorphism since $\mathrm{Ker}\, g^n = \bigcup_m \mathrm{Ker} f^{n,m} = 0$.

Consequently, $f = g^n f'$ for some $f': Z \to P^n$ and hence g_* takes the class $[f']$ in $\varinjlim_n [Z, P^n]$ into $[f]$. Thus g_* is an epimorphism. Now suppose $g_*(a) = 0$ for $a \in \varinjlim_n [Z, P^n]$. If $[h] \in [Z, P^n]$ is a representative of a, then clearly $g^n h \sim 0$. Let $s: Z \to P$ be a chain homotopy joining $g^n h$ with 0. Then, similarly as above, $s = g^m s'$ for $m \geqslant n$ and $s': Z \to P^m$. It is easy to verify that s' is a chain homotopy joining $f^{n,m} h$ with 0. Consequently, $a = \overline{[h]} = \overline{[f^{n,m} h]} = 0$ and we see that g_* is a monomorphism. This completes the proof of (i).

Now we prove (ii). Let $\{[h^n]\} \in \varprojlim_n [P^n, Z]$. Then we have the homotopy commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & P^n \arrow[r, "f^n"] \arrow[d, "h^n"] & P^{n+1} \arrow[r] \arrow[dl, "h^{n+1}"] & P^{n+1}/\mathrm{Im} f^n \arrow[r] & 0 \\
0 \arrow[r] & Z & & &
\end{tikzcd}$$

with normal rows. If $n = 1$, then by the Homotopy Extension Property there is a complex map $l^2 \sim h^2: P^2 \to Z$ such that $l^2 f^1 = h^1$. Since $h^3 f^2$

$\sim h^2 \sim l^2$, then the same arguments show that $l^2 = l^3 f^2$ for some $l^3 \sim h^3$. Continuing this procedure we get a sequence of complex maps $l^n\colon P^n \to Z$ such that $l^{n+1} f^n = l^n$ and $l^n \sim h^n$ for all n. Obviously $\{l^n\}$ induces a complex map $h\colon P = \varinjlim_n P^n \to Z$ for which $\varprojlim_n (g^n)^*([h]) = \{[h^n]\}$. The proposition is proved.

(2.9) LEMMA. *If $f_*\colon [Y, P] \to [Y, P']$ is an isomorphism for all $Y \in \operatorname{ob}\mathscr{P}_0$, then $[f]\colon P \to P'$ in $\mathscr{P}^{\square}$ is an isomorphism, too.*

Proof. Suppose $f\colon P \to P'$ satisfy $f_*\colon [Y, P] \approx [Y, P']$ for all $Y \in \operatorname{ob}\mathscr{P}_0$. Then in particular $f_*\colon [S^n, P] \approx [S^n, P']$, where

$$(S^n)_i = \begin{cases} 0 & \text{for } i \neq n, \\ R & \text{for } i = n \end{cases}$$

(if $\mathscr{P}$ is a category of type (A) we take $n \geqslant 0$). Moreover, one can easily check that $[S^n, Z] \approx H_n(Z)$ (naturally) for any complex Z. Consequently, $H(f)\colon H(P) \to H(P')$ is an isomorphism and then applying [3], 3.3, we get that $[f]$ is an isomorphism.

Let $H\colon \mathscr{P}^{\square} \to Ab$ be a contravariant functor satisfying the following conditions:

(I). The natural injections $P_i \to \bigoplus_i P_i$ induce an isomorphism $H(\bigoplus_i P_i) \approx \prod_i H(P_i)$.

(II). If $0 \to P' \to P \to P'' \to 0$ is a normal sequence in $\mathscr{P}$, then $H(P'') \to H(P) \to H(P')$ is an exact sequence of Abelian groups.

Let $\zeta\colon Ab \to S$ be the covariant functor forgetting the group structure. We are to prove that the composition $\mathscr{P}^{\square} \xrightarrow{H} Ab \xrightarrow{\zeta} S$ is a homotopy functor in the sense of Definition 2.7 First observe that H is an additive functor by (I). Next we prove

(2.10) LEMMA. *If $H\colon \mathscr{P}^{\square} \to Ab$ is a contravariant functor satisfying condition* (II) *and $g'\colon P \to Z$ is an equalizer of $f, 0\colon A \to P$, then the sequence*

$$H(Z) \xrightarrow{H(g')} H(P) \xrightarrow{H(f)} H(A)$$

is exact.

Proof. We begin by proving that the lemma holds for the equalizer $g\colon P \to P \cup CA$ of $f, 0\colon A \to P$ constructed in the proof of Proposition 2.8. Since $(A \cup CA)_n = (P_n \oplus CA_n)/N_n$, where $N_n = \{(-f_n(a), (a, 0));\ a \in A_n\}$; then we have following homotopy commutative diagram:

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & P & \xrightarrow{g} & P \cup CA & & \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1} & & \\ 0 & \to & A & \xrightarrow{\bar{f}} & P \oplus CA & \xrightarrow{p} & P \cup CA & \to & 0 \end{array}$$

with $\bar{f}_n(a) = (-f_n(a), (a, 0))$, $i_n(x) = (x, 0)$ and p the natural projection.

Notice that the bottom row is a normal sequence of complexes and i is a homotopy equivalence. It results that the sequence

$$H(P\cup CA)\xrightarrow{H(g)}H(P)\xrightarrow{H(f)}H(A)$$

is exact.

If $g': P\to Z$ is an arbitrary equalizer of f and 0, then clearly there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] & P \arrow[r, "g"] \arrow[d, "g'"] & P\cup CA \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

Hence we get the commutative diagram

$$\begin{tikzcd} H(P\cup CA) \arrow[r, "H(g)"] & H(P) \arrow[r, "H(f)"] & H(A) \\ & H(Z) \arrow[ul, "H(h)"] \arrow[u, "H(g')"'] & \end{tikzcd}$$

with exact row, by the first part of the proof. This implies $\operatorname{Ker} H(f) \subset \operatorname{Im} H(g')$. Moreover, by the additivity of H, $H(f)H(g') = H(0) = 0$, i.e. $\operatorname{Im} H(g') \subset \operatorname{Ker} H(f)$. Thus $\operatorname{Ker} H(g') = \operatorname{Im} H(f)$ and the proof is completed.

(2.11) PROPOSITION. *If $H: \mathscr{P}^{\square}\to Ab$ is a contravariant functor satisfying conditions* (I) *and* (II), *then $\zeta H: \mathscr{P}^{\square}\to S$ is a homotopy functor.*

Proof. We need to prove only condition (b) of Definition 2.7. If $f_i: A\to P_i$ and $g_i: P_i\to Z$, $i = 1, 2$, are as in 3° of 2.5 and $u_i\in H(P_i)$ satisfy $H(f_1)u_1 = H(f_2)u_2$, then one can easily check that the map $g: P_1\oplus P_2\to Z$ given by $g(x, y) = g_1(x)+g_2(y)$ is an equalizer of $0, f: A\to P_1\oplus P_2$, where $f(a) = \big(f_1(a), -f_2(a)\big)$. Consider the following commutative diagram

$$\begin{tikzcd} H(Z) \arrow[r, "H(g)"] \arrow[dr, "\varphi"'] & H(P_1\oplus P_2) \arrow[r, "H(f)"] \arrow[d, "l"] & H(A) \\ & H(P_1)\oplus H(P_2) \arrow[ur, "\psi"'] & \end{tikzcd}$$

with $\varphi(z) = \big(H(g_1)z, H(g_2)z\big)$, $l(y) = \big(H(i_1)y, H(i_2)y\big)$, $\psi(x_1, x_2) = H(f_1)x_1 + H(-f_2)x_2$ and exact row by Lemma 2.10. Moreover, l is an isomorphism, then

$$H(Z)\xrightarrow{\varphi}H(P_1)\oplus H(P_2)\xrightarrow{\psi}H(A)$$

is an exact sequence. Consequently, the equality $H(f_1)u_1 = H(f_2)u_2$ implies $(u_1, u_2)\in \operatorname{Ker}\psi = \operatorname{Im}\varphi$. Hence $u_i = H(g_i)v$ for some $v\in H(Z)$ and the proposition follows.

Let $\hat{\mathscr{P}}_0$ denote the collection of all $P\in \operatorname{ob}\mathscr{P}$ such that a map $f: P\to P'$ is an isomorphism, if $f^*: [Y, P]\approx [P', Y]$ for all $Y\in \operatorname{ob}\mathscr{P}_0$.

Now we can prove the main result of this section.

(2.12) THEOREM. *If* $H: \mathscr{P}^{\square} \to Ab$ *is a contravariant functor satisfying conditions* (I) *and* (II), *then there is a complex* $E_H \in \mathrm{ob}\mathscr{P}$ *and a natural equivalence of functors*

$$\gamma: [\cdot, E_H] \to H.$$

Proof. By Proposition 2.11 $\zeta H: \mathscr{P}^{\square} \to S$ is a homotopy functor. Hence applying Theorem 2.8 in [2] we get a complex $E_H \in \mathrm{ob}\mathscr{P}$ and a natural transformation of functors $\gamma: [\cdot, E_H] \to \zeta H$ such that $E_H \in \hat{\mathscr{P}}_0$ implies E_H is unique up to an equivalence and γ is a natural equivalence. In virtue of Lemma 2.9 $\hat{\mathscr{P}}_0 = \mathrm{ob}\mathscr{P}$ and thus it is sufficient to show that $\gamma(P): [P, E_H] \to \zeta H(P) = H(P)$ is an isomorphism of Abelian groups (not only underlying sets). It follows from the proof of Theorem 2.8 in [2] that there is $u \in H(E_H)$ such that $\gamma(P)(f) = H(f)u$. Since H is an additive functor, we have $\gamma(P)(f-g) = H(f-g)(u) = H(f)u - H(g)u$, i.e. $\gamma(P)$ is a homomorphism of groups. The theorem is proved.

As a consequence we get an analogue of Theorem II in [1].

(2.13) THEOREM. *If* $\{H^q\}_{q\in Z}: \mathscr{P} \to Ab$ *is a cohomology theory on* $\mathscr{P}$ *with* H^q *satisfying condition* (I), *then there is an* S-*spectrum* $E = \{E_q, \varepsilon_q\}_{q\in Z}$ *in* $\mathscr{P}$ *such that* $\{H^q\}_{q\in Z}$ *is isomorphic with the cohomology theory* $\{H^q(\cdot, E)\}_{q\in Z}$ *described in example* 2.4.

For the proof we need the following

(2.14) LEMMA. *If* $\{H^q\}_{q\in Z}: \mathscr{P} \to Ab$ *is a cohomology theory on* $\mathscr{P}$ *and* $\boldsymbol{P} = (0 \to P' \to P \to P'' \to 0)$ *is a normal sequence of complexes in* $\mathscr{P}$, *then the connecting homomorphism* $d^q: H^q(P') \to H^{q+1}(P'')$ *is the composition*

$$H^q(P') \xrightarrow{\sigma^q(P')^{-1}} H^{q+1}(SP') \xrightarrow{H^{q+1}(\theta(\boldsymbol{P}))} H^{q+1}(P''),$$

where $\sigma^q: H^{q+1}S \to H^q$ *is a natural equivalence of functors and* $\theta(\boldsymbol{P}): P'' \to SP'$ *is the complex map from Section* 1.

Proof of the lemma. Since H^q are h-functors, then the normal sequence $0 \to P' \to CP' \to SP' \to 0$ with the contractible complex CP' induces an exact sequence

$$\ldots \to H^q(CP') \to H^q(P') \xrightarrow{d^q(P')} H^{q+1}(SP') \to H^{q+1}(CP') \to \ldots$$

with $H^q(CP') = H^{q+1}(CP') = 0$. Hence $d^q(P')$ are isomorphisms and it is easy to check using the connectedness of $\{H^q\}$ that $d^q(P')^{-1}$ determine a natural equivalences of functors $\sigma^q: H^{q+1}S \to H^q$. Furthermore, a simple computation shows that the diagram

$$\begin{array}{ccccccccc} 0 & \to & P' & \to & P & \longrightarrow & P'' & \to & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \theta(\boldsymbol{P})=\theta} & & \\ 0 & \to & P' & \to & CP' & \to & SP' & \to & 0 \end{array}$$

with normal rows and the complex map h given by $h_n(p', p'') = (p', \theta_n(p''))$, is commutative. It follows that the diagram

$$\begin{array}{ccc} H^q(P') & \xrightarrow{d^q} & H^{q+1}(P'') \\ \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle H^{q+1}(\theta(\boldsymbol{P}))} \\ H^q(P') & \xrightarrow{\sigma^q(P')^{-1}} & H^{q+1}(SP') \end{array}$$

is commutative and the lemma is proved.

Proof of the theorem. It is clear that the functor H^q (for a fixed q) satisfies the assumptions of Theorem 2.12. Hence $H^q \approx [\cdot, E_q]$ for some $E_q \in \mathrm{ob}\mathscr{P}$. Moreover, the natural equivalence $\sigma^q\colon H^{q+1}S \to H^q$ from Lemma 2.14 yields a natural equivalence $\bar{\sigma}^q\colon [S(\cdot), E_{q+1}] \to [\cdot, E_q]$. If we put $\varepsilon_q = \bar{\sigma}^q(E_q)^{-1}(1_{E_q})$, then $\bar{\sigma}^q(P)^{-1}$ is the composition

$$[P, E_q] \xrightarrow{S} [S(P), SE_q] \xrightarrow{(\varepsilon_q)_*} [S(P), E_{q+1}].$$

It follows that $(\varepsilon_q)_*$ are isomorphisms, i.e. $E = \{E_q, \varepsilon_q\}_{q \in Z}$ is an S-spectrum and that the diagram

$$\begin{array}{ccc} H^q(P') & \xrightarrow{d^q} & H^{q+1}(P'') \\ \wr\!\! & & \wr\!\! \\ [P', E_q] & \xrightarrow{\bar{d}^q} & [P'', E_{q+1}] \end{array}$$

with $\bar{d}^q = H^{q+1}(\theta(\boldsymbol{P}))(\sigma^q(P'))^{-1}$ is commutative. This completes the proof.

We stated before that if $\hat{T}\colon {}_R M \to {}_{R'} M$ is a contravariant functor of the finite type and $\mathscr{P}$ is the category of type (A), then the functors $D_s^q \hat{T}\colon K^- \to {}_{R'} M$ restricted to the category $\mathscr{P}$ form a cohomology theory. Moreover, if the functors $D_s^q \hat{T}$ satisfy condition (I), then by the above Theorem we have the following

(2.15) Corollary. *There exists a spectrum E in $\mathscr{P}$ such that the cohomology theory $\{D_s^q \hat{T}\}$ is isomorphic with the cohomology theory $\{H^q(\cdot, E)\}$ from Example* 2.4.

(2.16) Corollary ([3], 5.5). *If by definition $H^q(X, A) = H_q(\mathrm{Hom}(X, A))$ for $X \in \mathrm{ob}\mathscr{P}$, $A \in \mathrm{ob}_R M$ and $\mathscr{P}$ is the category of type* (B) *or* (C), *then*

$$H^q(P, A) = [P, K(A, q)]$$

for all $P \in \mathrm{ob}\mathscr{P}$ and some projective complex $K(A, q)$ with $H_q K(A, q) = A$, $H_i K(A, q) = 0$, $i \neq q$.

Proof. Since $H^q(\cdot, A)\colon \mathscr{P} \to Ab$ satisfies the assumptions of Theorem 2.12, $H^q(\cdot, A) = [\cdot, K(A, q)]$ for some $K(A, q) \in \mathrm{ob}\mathscr{P}$. Hence

$$H_i K(A, q) = [S^i, K(A, q)] = H^q(S^i, A) = \begin{cases} A & \text{if } i = q, \\ 0 & \text{if } i \neq q, \end{cases}$$

where S^i $(i \in Z)$ are the complexes defined on p. 292.

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