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On generalized modular spaces I

Abstract. The present paper proceeds in 10 sections. Section 1 introduces a concept of a modular space; central cores of it are modular bases and a comparision relation of these bases. Section 2 shows that the theory of linear-topological spaces is a special case of the theory of modular spaces. In Section 3 an upper linear-topological space and in Section 4 a lower linear-topological space is defined for a given modular space. Section 5 deals with the convergence of Moore–Smith sequences in modular spaces. Section 6 introduces separations axioms for modular spaces. Section 7 concerns bounded sets in modular spaces. In Section 8 locally bounded modular spaces and in Section 9 locally convex modular spaces are investigated. Finally, in Section 10 the concept of a locally convex modular space generated by a given modular space is investigated.

Introduction. The central core of the contemporary functional analysis is a concept called a *linear-topological space* ([2], [3], [16], [17]). This concept cannot, however, deal with such spaces investigated in the functional analysis as modular spaces ([11], [12], [13], [14]) and in particular as Orlicz spaces ([15], [4], [9], [10], [6], [7], [8], [1], [5]). The purpose of this paper is a construction of such a model which could contain both theories of modular and linear-topological spaces.

I. GENERALIZED MODULAR SPACES

1.1. Let X be a real or complex linear space. A non-void family B of subsets of X will be called a *modular base* in X if the two following conditions are satisfied:

(M1) *for every two sets* $U_1, U_2 \in B$ *there exists a set* $U_3 \in B$ *such that* $\Gamma(U_3) \subset U_1 \cap U_2$,

(M2) *every set* $U \in B$ *is absorbent in* X.

In the above conditions for a given set $U \subset X$ by $\Gamma(U)$ we denoted a set of all vectors $z \in X$ representable in the form $z = ax+\beta y$, where $x, y \in U$ and a, β are numbers such that $|a|+|\beta| \leqslant 1$. A set $U \subset X$ is *absorbent* in X if for every vector $x \in X$ there is a number $a \neq 0$ such that $ax \in U$.

1.2. Let B_1 and B_2 be two modular bases in X. We shall say that a base B_2 is *non-weaker* than a base B_1, written $B_1 \dashv B_2$, if there exists a number $a \neq 0$ such that for every set $U_1 \in B_1$ there is a set $U_2 \in B_2$ satisfying $aU_2 \subset U_1$.

Here, for a given number a and a given set $U \subset X$ by aU we denote as usual, a set of all these vectors $y \in X$ representable in the form $y = ax$, where $x \in U$.

It is easy to see that the relation $\dashv$ is reflexive and transitive in the class of all modular bases in X. Therefore the equivalence of modular bases means the following:

We say that modular bases B_1 and B_2 in X are *equivalent*, written henceforth as $B_1 \sim B_2$, if simultaneously $B_1 \dashv B_2$ and $B_2 \dashv B_1$.

In the sequel $B^{\sim}$ will denote a class of all modular bases equivalent to a modular base B. In accordance with this we shall write $B_1^{\sim} = B_2^{\sim}$ if $B_1 \sim B_2$ and $B_1^{\sim} \leqslant B_2^{\sim}$ whenever $B_1 \dashv B_2$.

1.3. Let U be a subset of X. By bal. U we denote the set of all vectors $y \in X$ which could be written in the form $y = ax$, where $x \in U$ and a is such a number that $|a| \leqslant 1$. Whenever it occurs that $U = \text{bal.}\,U$ we shall call U, in accordance with already widely used terminology, a *balanced set*. Again, let B be a family of subsets of X. By bal.B we denote the family of all sets $U' \subset X$ such that $U' = \text{bal.}\,U$, where $U \in B$.

We shall show that

For every modular base B in X the family bal.B *is also a modular base in X and* bal.$B \sim B$.

Proof. Let B be a modular base in X. Clearly, the family bal.B satisfies condition (M2). Let U_1' and U_2' be any two sets in bal.B. Then there exists sets $U_1, U_2 \in B$ such that $U_1' = \text{bal.}\,U_1$ and $U_2' = \text{bal.}\,U_2$. By virtue of (M1) for B it follows that there is a set $U_3 \in B$ such that $\Gamma(U_3) \subset U_1 \cap \cap U_2$. Observe that

$$U_1' \cap U_2' \supset U_1 \cap U_2 \supset \Gamma(U_3) = \Gamma(\text{bal.}\,U_3).$$

The set bal.U_3 is an element of bal.B. This means that the family bal.B satisfies (M1), hence it is a modular base in X. It remains to prove that bal.$B \sim B$. From the inclusion $U \subset \text{bal.}\,U$ it clearly follows that bal.$B \dashv B$. Let now U be an arbitrary set in B. By (M1) for B we get that there exists a set $U_1 \in B$ such that $U \supset \Gamma(U_1) \supset \text{bal.}\,U_1$. Hence the required relation $B \dashv \text{bal.}B$ follows.

1.4. A generalized modular space is said to be the pair $[X, B^{\sim}]$, where X denotes a linear space and B a modular base in X.

Occurring in this definition sign $\sim$ over B is essential; $B^{\sim}$ denotes the whole class of modular bases in X equivalent to B. Instead of *gen-*

eralized modular space $[X, B^{\sim}]$ we shall also say *modular space* $[X, B^{\sim}]$ or *modular space* X *with respect to the class of modular bases* $B^{\sim}$.

Let $[X, B_1^{\sim}]$ and $[X, B_2^{\sim}]$ be two modular spaces over the same linear space X. We shall write $[X, B_1^{\sim}] \leqslant [X, B_2^{\sim}]$ whenever $B_1^{\sim} \leqslant B_2^{\sim}$.

The pair $[X, B]$, where X is a linear space and B a modular base in X, we shall call the *representative of a modular space* or a *free modular space*. Let $[X, B_1]$ and $[X, B_2]$ be two free modular spaces. The space $[X, B_2]$ is said to be *non-weaker than the space* $[X, B_1]$, denoted $[X, B_1] \dashv [X, B_2]$, if $B_1 \dashv B_2$. Whenever $B_1 \sim B_2$ then we call the spaces $[X, B_1]$ and $[X, B_2]$ *equivalent* and we denote $[X, B_1] \sim [X, B_2]$. According to this we have $[X, B^{\sim}] = [X, B]^{\sim}$.

2.1. A linear-topological space is the pair $[X, T]$, where X is a linear space and T a topology in X such that the algebraic operations in X are continuous in this topology. This topology T is called a *linear topology in* X ([2], [3], [16], [17]).

The two following well-known theorems from the theory of linear-topological spaces are crucial for our considerations:

I. *In every linear-topological space* $[X, T]$ *the family* B *of all neighbourhoods of the origin satisfies the following conditions*:

(LT1) *for every set* $U_1 \in B$ *there exists* $U_2 \in B$ *with* $U_2 + U_2 \subset U_1$,

(LT2) *for every set* $U_1 \in B$ *there exists* $U_2 \in B$ *with* $\mathrm{bal}.U_2 \subset U_1$,

(LT3) *for each two sets* $U_1, U_2 \in B$ *there exists* $U_3 \in B$ *such that* $U_3 \subset U_1 \cap \cap U_2$,

(LT4) *every set* $U \in B$ *is absorbent in* X.

For any two sets $U_1, U_2 \subset X$ by $U_1 + U_2$ we denote, as usual, the set of all $x \in X$ representable in the form $x = x_1 + x_2$, where $x_1 \in U_1$ and $x_2 \in U_2$.

II. *For every non-void family* B *of sets in the linear space* X, *satisfying* (LT1) *through* (LT4), *there is a unique topology* T *in* X *satisfying the following conditions*:

1° $[X, T]$ *is a linear-topological space*,

2° *for every neighbourhood of the origin* V *in* T *there is a set* $U \in B$ *such that* $U \subset V$,

3° *for every* $U \in B$ *there is a neighbourhood of the origin* V *in* T *such that* $V \subset U$.

In this topology T the closure of any set $Z \subset X$ is

$$\bar{Z} = \bigcap_{U \in B} (Z + U).$$

2.2. Hereafter any non-void family B of sets in X satisfying (LT1)–(LT4) shall be called a *linear-topological base* or shortly an *LT-base* in X.

Let U be an arbitrary set in X. By $\Delta(U)$ we denote the set of all vectors $z \in X$ representable in the form $z = \alpha x + \beta y$, where $x, y \in U$ and α, β are numbers such that $\sup\{|\alpha|, |\beta|\} \leqslant 1$. Note that using this notation, instead of four conditions for an LT-base (LT1)–(LT4) only two suffice:

(LT1*) *for any two sets* $U_1, U_2 \in B$ *there is* $U_3 \in B$ *such that* $\Delta(U_3) \subset U_1 \cap \cap U_2$,

(LT*2) $=$ (M2).

Moreover, note that for any set $U \subset X$ always holds $\Gamma(U) \subset \Delta(U)$. Hence

Every LT-base B *in* X *is a modular base in* X.

2.3. *For any modular base* B *in* X *the three following conditions are equivalent*:

(Δ^0) *for every* $a \neq 0$ *and for every* $U_1 \in B$ *there is a set* $U_2 \in B$ *such that* $aU_2 \subset U_1$,

(Δ_2) *for every* $U_1 \in B$ *there is* $U_2 \in B$ *such that* $2U_2 \subset U_1$,

(Δ^*) *there exists* a *with* $|a| > 1$ *such that for every* $U_1 \in B$ *there is a set* $U_2 \in B$ *such that* $aU_2 \subset U_1$.

Proof. Clearly, (Δ^0) implies (Δ_2), and (Δ_2) implies (Δ^*). Thus let a modular base B fulfils (Δ^*) and let β be any number different from 0 and U_1 any set in B. Then we find a positive integer n such that $|a|^n \geqslant |\beta|$. By (Δ^*) there is $U_2 \in B$ such that $aU_2 \subset U_1$. Repeating this n times we see that there exists a set $U_{n+1} \in B$ such that $aU_{n+1} \subset U_n$. In view of condition (M1) for B there is $U_{n+2} \in B$ such that $\mathrm{bal.}U_{n+2} \subset U_{n+1}$. Hence

$$U_1 \supset aU_2 \supset a^2 U_3 \supset \ldots \supset a^n U_{n+1} \supset a^n \mathrm{bal.}U_{n+2} \supset \beta U_{n+2}.$$

This shows that (Δ^*) implies (Δ^0).

2.4. *A modular base* B *in* X *is an LT-base if and only if it satisfies one of the three equivalent conditions of* 2.3, *e.g.* (Δ_2).

Proof. Let B be an LT-base in X. For every $U \subset X$ always holds $2U \subset \Delta(U)$. This and (LT1*) imply that B satisfies (Δ_2). Conversely, let B be a modular base in X satisfying (Δ_2). For every set $U \subset X$ always holds $\Delta(U) \subset 2\Gamma(U)$. Hence conditions (Δ_2) and (M1) ensure that B satisfies (LT1*). This means that B is an LT-base in X.

2.5. *Let* B_1 *and* B_2 *be two modular bases in* X. *If* $B_1 \sim B_2$ *and* B_1 *is an LT-base, then* B_2 *is also an LT-base.*

Proof. $B_1 \sim B_2$ means that there are numbers $a_1, a_2 \neq 0$ such that for every $U_1 \in B_1$ there is $U_2 \in B_2$ such that $a_1 U_2 \subset U_1$, and for every $U_2' \in B_2$ there exists $U_1' \in B_1$ such that $a_2 U_1' \subset U_2'$. Let U_2' be an arbitrary member of the base B_2. Then there is $U_1' \in B_1$ such that $a_2 U_1' \subset U_2'$. B_1 is an LT-base, hence it satisfies (Δ^0). It means that there exists a set

$U_1 \in B_1$ with $2a_1^{-1}a_2^{-1}U_1 \subset U_1'$. Then we can find $U_2 \in B_2$ such that $a_1 U_2 \subset U_1$. Assembling these inclusion we get

$$U_2' \supset a_2 U_1' \supset a_2(2a_1^{-1}a_2^{-1}U_1) = 2a_1^{-1}U_1 \supset 2U_2.$$

This means that B_2 satisfies (Δ_2), hence it is an LT-base.

2.6. *Let B_1 and B_2 be two LT-bases in X. The relation $B_1 \dashv B_2$ holds if and only if for every $U_1 \in B_1$ there is $U_2 \in B_2$ such that $U_2 \subset U_1$.*

Proof. If for every $U_1 \in B_1$ there is $U_2 \in B_2$ such that $U_2 \subset U_1$, then clearly $B_1 \dashv B_2$. Conversely, let us assume that $B_1 \dashv B_2$. Then there is $a \neq 0$ such that for any $U_1 \in B_1$ there is $U_2' \in B_2$ with $aU_2' \subset U_1$. B_2 is an LT-base and hence it satisfies (Δ^0). It means that there is $U_2 \in B_2$ such that $a^{-1}U_2 \subset U_2'$. It follows that $U_2 \subset aU_2' \subset U_1$.

2.7. Let X be a fixed linear space. By Ω_1 we denote the set of all linear topologies T on X and by Ω_2 the set of all classes $B^{\sim}$ in X including as a member an LT-base. Taking into account 2.1. I and 2.2 we define a mapping $F: \Omega_1 \to \Omega_2$, as follows: to a linear topology $T \in \Omega_1$ there corresponds a class $F(T) = B^{\sim}$, where B is a family of all neighbourhoods of the origin in T. In view of 2.1. II, 2.5 and 2.6 we can verify that this is a one-to-one mapping of Ω_1 onto Ω_2. Hence

There exists a one-to-one correspondence between the set Ω_1 of all linear topologies T on X and the set Ω_2 of all classes $B^{\sim}$ containing as a member an LT-base. This correspondence establishes the above defined mapping F.

This together with 2.1.I imply

If T_1 and T_2 are two linear topologies on X, and $B_1^{\sim}$, $B_2^{\sim}$ their corresponding by F classes of modular bases, then $T_1 \subset T_2$ holds if and only if $B_1^{\sim} \leqslant B_2^{\sim}$ holds.

In the sequel this correspondence we shall simply regard as an identity, i.e. the topological-linear space $[X, T]$ will be identified with the modular space $[X, B^{\sim}]$, where $B^{\sim} = F(T)$, and conversely the modular space $[X, B^{\sim}]$ such that $B^{\sim}$ contains as a member an LT-base will be identified with the linear-topological space $[X, T]$ such that $F(T) = B^{\sim}$. This identification is in fact nothing else but a formal equivalence of the two different methods of defining a linear-topological space: one in which the linear topology T for a linear space is explicitly given and the other in which the class $B^{\sim}$ of equivalent LT-bases is given. In this way the theory of linear-topological spaces becomes a part of the theory of modular spaces.

The theory of modular spaces is more general than the theory of linear-topological spaces. This can be showed on a simple example. The pair $[C, B^{\sim}]$, where C is a complex number system and B the family consisting of only one set $U = \{a \in C: |a| < 1\}$, is a modular space but not a linear topological space. In this case the modular base B does not satisfy condition (Δ_2); for the set U in B there is no set $U' \in B$ such that $2U' \subset U$.

3.1. Let B be a modular base in X. By $B^{\vee}$ we denote a family of all subsets $V \subset X$ representable in the form $V = aU$, where $U \in B$ and a is a number different from 0.

We shall show that

For any modular base B in X the family $B^{\vee}$ is an LT-base in X.

Proof. Let V_1 and V_2 be any two sets in $B^{\vee}$. Then there exist $U_1, U_2 \in B$ and $a_1, a_2 \neq 0$ such that $V_1 = a_1 U_1$ and $V_2 = a_2 U_2$. By virtue of (M1) for B there is a set $U_3 \in B$ such that $\Gamma(U_3) \subset U_1 \cap U_2$. Then for $a = \inf\{|a_1|, |a_2|\}$ we have

$$V_1 = a_1 U_1 \supset a_1 \Gamma(U_3) = |a_1| \Gamma(U_3) \supset a\Gamma(U_3) = 2\Gamma(\tfrac{1}{2}aU_3) \supset \Delta(\tfrac{1}{2}aU_3)$$

and similarly $V_2 \supset \Delta(\frac{1}{2}aU_3)$. Hence $V_1 \cap V_2 \supset \Delta(\frac{1}{2}aU_3)$. The set $\frac{1}{2}aU_3$ belongs to $B^{\vee}$. It follows that $B^{\vee}$ satisfies (LT1*). Obviously, $B^{\vee}$ also satisfies (LT2*). Therefore $B^{\vee}$ is an LT-base in X.

3.2. *Let B_1 and B_2 be two modular bases in X. The relation $B_1^{\vee} \dashv B_2^{\vee}$ holds if and only if for every set $U_1 \in B_1$ there are a set $U_2 \in B_2$ and a number $a \neq 0$ such that $aU_2 \subset U_1$.*

Proof. Let $B_1^{\vee} \dashv B_2^{\vee}$ and let U_1 be a set in B_1. U_1 is in $B_1^{\vee}$ since $B_1 \subset B_1^{\vee}$. In view of 2.6 from $B_1^{\vee} \dashv B_2^{\vee}$ we deduce that there is a set $V_2 \in B_2^{\vee}$ such that $V_2 \subset U_1$. Hence for every set $U_1 \in B_1$ there are a set $U_2 \in B_2$ and a number $a \neq 0$ such that $aU_2 \subset U_1$. On the other hand let for every $U_1 \in B_1$ there exist $U_2 \in B_2$ and $a \neq 0$ such that $aU_2 \subset U_1$, and let V_1 belong to B_1. Then V_1 is of the form $V_1 = a_1 U_1$, where $U_1 \in B_1$ and $a_1 \neq 0$. According to the hypotheses for a set U_1 we can find $U_2 \in B_2$ and $a \neq 0$ satisfying $aU_2 \subset U_1$. This means that $V_1 \supset a\, a_1 U_2$. The set $aa_1 U_2$ is clearly in $B_2^{\vee}$. Hence $B_1^{\vee} \dashv B_2^{\vee}$.

3.3. *Let B_1 and B_2 be two modular bases in X.* If $B_1 \dashv B_2$, *then also $B_1^{\vee} \dashv B_2^{\vee}$. Hence if $B_1 \sim B_2$, then also $B_1^{\vee} \sim B_2^{\vee}$.*

This is immediate from 3.2.

3.4. *For any modular base B in X the following hold:*

1° $B \dashv B^{\vee}$,

2° *if B_1 is an LT-base in X such that $B \dashv B_1$, then $B^{\vee} \dashv B_1$.*

Proof. $B \dashv B^{\vee}$ is a simple consequence of $B \subset B^{\vee}$. It remains to prove 2°. Let then B_1 be an LT-base in X such that $B \dashv B_1$. Let V be a set in $B^{\vee}$. This V can be written as $V = aU$, where $U \in B$ and $a \neq 0$. $B \dashv B_1$ implies that there are $a_1 \neq 0$ and $U_1' \in B_1$ such that $a_1 U_1' \dashv U$.

As an LT-base B_1 satisfies condition (Δ^0) and so there is a set $U_1 \in B_1$ with $a^{-1}a_1^{-1}U_1 \subset U_1'$. Hence we get $U_1 \subset aa_1U_1' \subset V$. Thus the proof that $B^{\check{}} \dashv B_1$ is accomplished.

3.5. *For any modular base B in X the relation $B \sim B^{\check{}}$ holds if and only if B is an LT-base.*

Proof. If $B \sim B^{\check{}}$, then by 3.1 and 2.5 we get that B is an LT-base. Conversely, let B an LT-base. Then from 3.4.2° and the obvious relation $B \dashv B$ it follows that $B^{\check{}} \dashv B$. Then 3.4.1° implies that $B \dashv B^{\check{}}$ and so $B \sim B^{\check{}}$.

3.6. From 3.1 and 3:3 follows that for a modular space $[X, B^{\tilde{}}]$ the pair $[X, B^{\check{}\tilde{}}]$ uniquely determines the linear-topological space. The result of 3.4 says that $[X, B^{\check{}\tilde{}}]$ is the *linear-topological space best* for a given modular space $[X, B^{\tilde{}}]$ with respect on the right-hand side of the relation $\leqslant$. In the sequel the space $[X, B^{\check{}\tilde{}}]$ will be called the *upper linear-topological space* generated by the modular space $[X, B^{\tilde{}}]$.

4.1. Let B be a modular base in X. By $B^{\hat{}}$ we denote the family of all sets $W \subset X$ which can be represented as

$$W = \bigcup_{n=1}^{\infty} (U_1 + U_2 + \ldots + U_n) = \lim_{n\to\infty} (U_1 + U_2 + \ldots + U_n),$$

where $\{U_n\}$ is an (infinite) sequence of sets in B.

We shall demonstrate that

For any modular base B in X the family $B^{\hat{}}$ is an LT-base in X.

Proof. Let W_1 and W_2 be arbitrary sets in $B^{\hat{}}$. Then there exist sequences $\{U_n'\}$ and $\{U_n''\}$ of sets in B such that

$$W_1 = \bigcup_{n=1}^{\infty} (U_1' + \ldots + U_n') \quad \text{and} \quad W_2 + \bigcup_{n=1}^{\infty} (U_1'' + \ldots + U_n'').$$

By virtue of (M1) for B we see that for $m = 1, 2, \ldots$ there exist sets $U_m \in B$ such that

$$\mathrm{bal.}\, U_m \subset \Gamma(U_m) \subset U_{2m-1}' \cap U_{2m-1}'' \cap U_{2m}' \cap U_{2m}''.$$

The set

$$W = \bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n)$$

clearly belongs to $B^{\hat{}}$. We shall show that $\Delta(W) \subset W_1 \cap W_2$. Let z be an arbitrary member in $\Delta(W)$. Then, according to the definition, $z = ax + \beta y$, where $x, y \in W$ and a, β are numbers such that $\sup\{|a|, |\beta|\} \leqslant 1$. We observe that there is a positive integer n such that $x, y \in (U_1 + \ldots +$

$+U_n)$. Hence

$$\begin{aligned} z &= ax+\beta y \in aU_1+\ldots+aU_n+\beta U_1+\ldots+\beta U_n \\ &\subset \text{bal.}\, U_1+\text{bal.}\, U_1+\text{bal.}\, U_2+\text{bal.}\, U_2+\ldots+\text{bal.}\, U_n+\text{bal.}\, U_n \\ &\subset (U_1'\cap U_1'')+(U_2'\cap U_2'')+\ldots+(U_{2n-1}'\cap U_{2n-1}'')+(U_{2n}'\cap U_{2n}''). \end{aligned}$$

Thus we get

$$z \in U_1'+U_2'+\ldots+U_{2n-1}'+U_{2n}' \subset W_1$$

$$\text{and} \qquad z \in U_1''+U_2''+\ldots+U_{2n-1}'+U_{2n}'' \subset W_2.$$

It means that $z \in W_1\cap W_2$. Therefore $B^\wedge$ satisfies (LT1*).

Now, let W be an arbitrary set in $B^\wedge$. Then there exists a sequence $\{U_n\}$ of sets in B such that W is the above given form. We see that $U_1 \subset W$. Now, by virtue of (M2) for B it follows that the set W is absorbent in X. Thus, $B^\wedge$ satisfies also (LT2*), and so it is an LT-base in X.

4.2. *For any modular base B in X the following hold:*

1° $B^\wedge \dashv B$,

2° *if B_1 is an LT-base in X such that $B_1 \dashv B$, then $B_1 \dashv B^\wedge$.*

Proof. A quick look at that part of the proof of 4.1 in which we have proven that for $B^\wedge$ holds (LT2*) reveals that for every set $W \in B^\wedge$ there is a set $U_1 \in B$ with $U_1 \subset W$. Hence $B^\wedge \dashv B$.

Now let B_1 be an LT-base in X such that $B_1 \dashv B$ and let V be any set in B_1. By (LT1) (also (LT1*)) for B_1 we get the existence of a set $V_1 \in B_1$ with $V \supset V_1+V_1$. Again, by the same condition shows that there is $V_2 \in B_1$ such that $V_1 \supset V_2+V_2$. Continuing this procedure we obtain a sequence $\{V_n\}$ of sets in B_1 such that $V_n \supset V_{n+1}+V_{n+1}$ for $n=1,2,\ldots$ and $V \supset V_1+V_1$. Since $B_1 \dashv B$ it follows that there exists a number $a \neq 0$ with the property that for every sequence $\{V_n\}$ of sets of B_1 there is a sequence $\{U_n\}$ of sets of B such that $aU_n \subset V_n$ for $n=1,2,\ldots$ Hence, for every n we have

$$\begin{aligned} V &\supset V_1+V_1 \supset V_1+V_2+V_2 \supset \ldots \supset V_1+V_2+\ldots+V_n+V_n \\ &\supset V_1+V_2+\ldots+V_n \supset a(U_1+U_2+\ldots+U_n) \end{aligned}$$

and

$$V \supset a\bigcup_{n=1}^{\infty}(U_1+U_2+\ldots+U_n).$$

The set on the right-hand side of the above inclusion clearly belongs to $B^\wedge$ and a is independent the of choice of V in B_1. Therefore the following holds $B_1 \dashv B^\wedge$.

4.3. *Let B_1 and B_2 be two modular bases in X. If $B_1 \dashv B_2$, then also $B_1^{\wedge} \dashv B_2^{\wedge}$. Hence if $B_1 \sim B_2$, then also $B_1^{\wedge} \sim B_2^{\wedge}$.*

Proof. Let $B_1 \dashv B_2$. By 4.2.1° we have $B_1^{\wedge} \dashv B_1$. This together with our assumption yields $B_1^{\wedge} \dashv B_2$. This, in view of 4.2.2° implies that $B_1^{\wedge} \dashv B_2^{\wedge}$.

4.4. *For any modular base B in X the relation $B \sim B^{\wedge}$ holds if and only if B is an LT-base.*

Proof. If $B \sim B^{\wedge}$, then by 4.1 and 2.5 we get that B is an LT-base. Conversely, let B be an LT-base. Then by 4.2.2° and obvious relation $B \dashv B$ we get $B \dashv B^{\wedge}$. In view of 4.2.1° it follows $B^{\wedge} \dashv B$ and so $B \sim B^{\wedge}$.

4.5. *Let B_1 and B_2 be two modular bases in X. The relation $B_1^{\vee} \dashv B_2^{\wedge}$ holds if and only if for every $a \neq 0$ and every $U_1 \in B_1$ there exists $U_2 \in B_2$ such that $aU_2 \subset U_1$.*

Proof. Let $B_1^{\vee} \dashv B_2^{\wedge}$. Then by 3.4.1° we get $B_1 \dashv B_2^{\wedge}$. This means that there is a number $a_1 \neq 0$ such that for every $U_1 \in B_1$ there exists $W_2 \in B_2$ with $a_1 W_2 \subset U_1$. Since B_2 is an LT-base it satisfies (Δ^0), and so, for every $a \neq 0$ there is $W_2' \in B_2$ with $aa_1^{-1} W_2' \subset W_2$. Referring again to the proof of 4.1 we see that then there exists U_2 in B_2 such that $U_2 \subset W_2'$. It is an easy matter to verify that then $aU_2 \subset a_1 W_2 \subset U_1$.

Conversely, let B_1 and B_2 be modular bases in X satisfying the condition of the theorem. Let V_1 be an arbitrary set in B_1. Then $V_1 = \beta U_1$, where $U_1 \in B_1$ and $\beta \neq 0$. From our hypothesis it follows that there exists $U_2 \in B_2$ with $U_2 \subset \beta U_1 = V_1$. This implies that $B_1^{\vee} \dashv B_2$ and hence by 3.1 and 4.2.2° we get $B_1^{\vee} \dashv B_2^{\wedge}$.

4.6. It is a consequence of 4.1 and 4.3 that for a modular space $[X, B^{\sim}]$ the pair $[X, B^{\wedge\sim}]$ determines the unique linear-topological space. The result of 4.2 says that $[X, B^{\wedge\sim}]$ is the linear-topological space best for a given modular space $[X, B^{\sim}]$ with respect on the left-hand side of the relation $\leqslant$. In the sequel the space $[X, B^{\wedge\sim}]$ will be called the *lower linear-topological space* generated by the modular space $[X, B^{\sim}]$.

4.7. For any modular base B in X the four following conditions are equivalent:

1° *B is an LT-base,* 2° $B \sim B^{\vee}$,

3° $B \sim B^{\wedge}$, 4° $B^{\vee} \sim B^{\wedge}$.

Therefore for a modular space $[X, B^{\sim}]$ the following conditions are equivalent:

1° $[X, B^{\sim}]$ *is a linear-topological space,*

2° $[X, B^{\sim}] = [X, B^{\vee\sim}]$,

3° $[X, B^{\sim}] = [X, B^{\wedge\sim}]$,

4° $[X, B^{\vee\sim}] = [X, B^{\wedge\sim}]$.

This is an immediate consequence of 3.5, 4.4, 3.4.1° and 4.2.1°.

5.1. Let B be a modular base, $S = \{x_\sigma\}$ an *Moore–Smith sequence* (MS-sequence) over a directed set $[\Sigma, \dashv]$ and x_0 a point in X. MS-sequence S is said to *converge to x_0 with respect to the base B*, written $S \xrightarrow{B} x_0$ or $x_\sigma \xrightarrow{B} x_0$, if there is a number $a \neq 0$ such that for every $U \in B$ there exist $\sigma_0 \in \Sigma$ such that for every $\sigma \in \Sigma$ with $\sigma_0 \dashv \sigma$ the following holds $a(x_\sigma - x_0) \in U$.

Notice that an MS-sequence $S = \{x_\sigma\}$ converges to x_0 with respect to B if and only if the MS-sequence $\{x_\sigma - x_0\}$ converges to 0 with respect to B.

5.2. *Let B be a modular base, $S_1 = \{x'_\sigma\}$, $S_2 = \{x''_\sigma\}$ two MS-sequences over the same directed set $[\Sigma, \dashv]$ and x_1, x_2 two points in X. If S_1 converges to x_1 and S_2 converges to x_2 both with respect to B, then for any numbers α, β the MS-sequence $\alpha S_1 + \beta S_2 = \{\alpha x'_\sigma + \beta x''_\sigma\}$ converges to $\alpha x_1 + \beta x_2$ with respect to B.*

Proof. Let $S_1 \xrightarrow{B} x_1$ and $S_2 \xrightarrow{B} x_2$. It means that there are two numbers $a_1, a_2 \neq 0$ such that for every $U' \in B$ we can find $\sigma_1, \sigma_2 \in \Sigma$ such that $a_1(x'_\sigma - x_1) \in U'$ for $\sigma_1 \dashv \sigma \in \Sigma$ and $a_2(x''_\sigma - x_2) \in U'$ for $\sigma_2 \dashv \sigma \in \Sigma$. Let α, β be any numbers. Take $a_3 \neq 0$ such that $|\alpha a_3| \leqslant \frac{1}{2}|a_1|$ and $|\beta a_3| \leqslant \frac{1}{2}|a_2|$. In view of (M1) for B it follows that for every $U \in B$ there is $U' \in B$ with $\Gamma(U') \subset U$. Take then $\sigma_3 \in \Sigma$ such that $\sigma_1 \dashv \sigma_3$ and $\sigma_2 \dashv \sigma_3$. It is easy to verify that for $\sigma_3 \dashv \sigma \in \Sigma$

$$a_3\big(\alpha x'_\sigma + \beta x''_\sigma - (\alpha x_1 + \beta x_2)\big) \in \Gamma(U') \subset U.$$

Hence $\alpha S_1 + \beta S_2 \xrightarrow{B} \alpha x_1 + \beta x_2$.

5.3. *Let B_1 and B_2 be two modular bases in X. Relation $B_1 \dashv B_2$ holds if and only if for every MS-sequence S in X: $S \xrightarrow{B_2} 0$ implies $S \xrightarrow{B_1} 0$. Whence $B_1 \sim B_2$ occurs if and only if for every MS-sequence S in X: $S \xrightarrow{B_1} 0$ if and only if $S \xrightarrow{B_2} 0$.*

Proof is omitted.

5.4. *Let B be a modular base in X. For any MS-sequence $S = \{x_\sigma\}$ the condition $S \xrightarrow{B^\vee} 0$ is equivalent to: for every number $a \neq 0$ and every $U \in B$ there is a $\sigma_0 \in \Sigma$ such that $ax_\sigma \in U$ for $\sigma_0 \dashv \sigma \in \Sigma$.*

Proof. Assume that $S \xrightarrow{B^\vee} 0$. Then there is $a_1 \neq 0$ such that for every set $V \in B^\vee$ there exists a $\sigma_0 \in \Sigma$ satisfying $a_1 x_\sigma \in V$ for $\sigma_0 \dashv \sigma \in \Sigma$. For every $a \neq 0$ and every $U \in B$ we have $a_1 a^{-1} U \in B^\vee$. Hence holds the condition of the theorem: for every number $a \neq 0$ and every set $U \in B$ there is a $\sigma_0 \in \Sigma$ such that $ax_\sigma \in U$ for $\sigma_0 \dashv \sigma \in \Sigma$. Conversely, let S satisfies this condition. Every set V in $B^\vee$ is of the form $V = \beta U$, where $U \in B$ and $\beta \neq 0$.

Then for every $V \in B^{\check{}}$ there is a $\sigma_0 \in \Sigma$ with $x_\sigma \in V$ for $\sigma_0 \dashv \sigma \in \Sigma$. This means that $S \xrightarrow{B^{\check{}}} 0$.

5.5. *For any modular base B in X the following conditions are equivalent:*

1° *B is an LT-base,*

2° *for every MS-sequence S in X: $S \xrightarrow{B} 0$ if and only if $S \xrightarrow{B^{\check{}}} 0$,*

3° *for every MS-sequence S in X: $S \xrightarrow{B} 0$ if and only if $S \xrightarrow{B^{\hat{}}} 0$,*

4° *for every MS-sequence S in X: $S \xrightarrow{B^{\check{}}} 0$ if and only if $S \xrightarrow{B^{\hat{}}} 0$,*

5° *for every MS-sequence $\{x_\sigma\}$ in X the condition: for every $U \in B$ there is $\sigma_1 \in \Sigma$ such that $x_\sigma \in U$ for $\sigma_1 \dashv \sigma \in \Sigma$, implies: for every $a \neq 0$ and every $U \in B$ there is a $\sigma_2 \in \Sigma$ such that $ax_\sigma \in U$ for $\sigma_2 \dashv \sigma \in \Sigma$.*

6° *for every MS-sequence $\{x_\sigma\}$ in X the condition: for every $U \in B$ there is $\sigma_1 \in \Sigma$ such that $x_\sigma \in U$ for $\sigma_1 \dashv \sigma \in \Sigma$, implies: for every $U \in B$ there is a $\sigma_2 \in \Sigma$ such that $2x_\sigma \in U$ for $\sigma_2 \dashv \sigma \in \Sigma$,*

7° *there exists a number $|a| > 1$ such that for every MS-sequence $\{x_\sigma\}$ in X the condition: for every $U \in B$ there is a $\sigma_1 \in \Sigma$ such that $x_\sigma \in U$ for $\sigma_1 \dashv \sigma \in \Sigma$, implies: for every $U \in B$ there is a $\sigma_2 \in \Sigma$ such that $ax_\sigma \in U$ for $\sigma_2 \dashv \sigma \in \Sigma$,*

8° *for every MS-sequence $\{x_\sigma\}$ in X satisfying the condition: for every $U \in B$ there is a $\sigma_1 \in \Sigma$ such that $x_\sigma \in U$ for $\sigma_1 \dashv \sigma \in \Sigma$, there exists a number $|a| > 1$ such that for every $U \in B$ there is a $\sigma_2 \in \Sigma$ such that $ax_\sigma \in U$ for $\sigma_2 \dashv \sigma \in \Sigma$.*

Proof is omitted.

5.6. *A modular base B in X is an LT-base if and only if for every MS-sequence $S = \{x_\sigma\}$ in X any two out of the three following condition are equivalent:*

1° *for every number $a \neq 0$ and every $U \in B$ there exists a $\sigma_0 \in \Sigma$ such that $ax_\sigma \in U$ for $\sigma_0 \dashv \sigma \in \Sigma$,*

2° *for every $U \in B$ there is a $\sigma_0 \in \Sigma$ such that $x_\sigma \in U$ for $\sigma_0 \dashv \sigma \in \Sigma$,*

3° *there is a number $a \neq 0$ such that for every $U \in B$ there exists a $\sigma_0 \in \Sigma$ such that $ax_\sigma \in U$ for $\sigma_0 \dashv \sigma \in \Sigma$, i.e. $S \xrightarrow{B} 0$.*

This follows from 5.5.

5.7. Let B be a modular base and $S = \{x_\sigma\}$ an MS-sequence in X. S is said to *satisfy the Cauchy condition with respect to B* if there exists a number $a \neq 0$ such that for every $U \in B$ there is a $\sigma_0 \in \Sigma$ satisfying $a(x_{\sigma_1} - x_{\sigma_2}) \in U$ for $\sigma_0 \dashv \sigma_1, \sigma_2 \in \Sigma$.

Let B be a modular base and $S = \{x_\sigma, \sigma \in \Sigma, \dashv\}$ and MS-sequence in X. We equip the cartesian product $\Sigma \times \Sigma$ with the relation $\dashv_1$ as follows: $(\sigma_1', \sigma_2') \dashv_1 (\sigma_1'', \sigma_2'')$ whenever $\sigma_1' \dashv \sigma_1''$ and $\sigma_2' \dashv \sigma_2''$. The pair $[\Sigma\times\Sigma, \dashv_1]$ is clearly a directed set. Denote $x_{(\sigma_1,\sigma_2)} = x_{\sigma_1} - x_{\sigma_2}$ for $\sigma_1, a_2 \in \Sigma$. Notice that S satisfies the Cauchy condition with respect to B if and only if the MS-sequence $S_1 = \{x_{(\sigma_1,\sigma_2)}, (\sigma_1, \sigma_2) \in \Sigma\times\Sigma, \dashv_1\}$ converges to 0 with respect to B. This, on account of 5.3 implies that

If B_1 and B_2 are two modular bases in X such that $B_1 \dashv B_2$, then every MS-sequence S in X satisfying the Cauchy condition with respect to B_2 also satisfies this condition with respect to B_1. Hence, if B_1 and B_2 are such modular bases in X that $B_1 \sim B_2$, then any MS-sequence S in X satisfies the Cauchy condition with respect to B_1 if and only if does with respect to B_2.

5.8. *Let B be a modular base in X. Each MS-sequence S in X covergent with respect to B satisfies the Cauchy condition with respect to B.*

Proof. Assume that $S = \{x_\sigma\}$ converges with respect to B to some $x_0 \in X$. Then, according to the definition of convergence, there is a number $a \neq 0$ such that for every set $U' \in B$ there exists a $\sigma_0 \in \Sigma$ such that $a(x_\sigma - x_0) \in U'$ for $\sigma_0 \dashv \sigma \in \Sigma$. Let U be any set in B. By (M1) for B there is a set $U' \in B$ with $\Gamma(U') \subset U$. Then

$$\tfrac{1}{2}a(x_{\sigma_1} - x_{\sigma_2}) = \tfrac{1}{2}a(x_{\sigma_1} - x_0) - \tfrac{1}{2}a(x_{\sigma_2} - x_0) \in \Gamma(U') \subset U$$

for $\sigma_0 \dashv \sigma_1, \sigma_2 \in \Sigma$. Hence satisfies the Cauchy condition with respect to B.

5.9. Let $[X, B^{\sim}]$ be a modular space, S an MS-sequence and $x_0 \in X$. S is said to *converge to x_0* in the space $[X, B^{\sim}]$ if $S \xrightarrow{B} x_0$.

It is clear in view of 5.3 that in the above definition the phrase "$S \xrightarrow{B} x_0$" can be replaced by "$S \xrightarrow{B_1} x_0$", where B_1 is any base in the class $B^{\sim}$. Considering conditions 2° and 3° in 5.6 we get that when $[X, B^{\sim}]$ is a linear-topological space this definition of convergence of S to x_0 coincides with that commonly used in the theory of linear-topological spaces.

Let $[X, B^{\sim}]$ be a modular space and S an MS-sequence in X. Similarly as above S is said to *satisfy the Cauchy condition* in $[X, B^{\sim}]$ whenever it does with respect to B. The remark of 5.7 says now that in this definition B can be replaced by any base B_1 in the class $B^{\sim}$. The result of 5.8 means that every MS-sequence S in X convergent in $[X, B^{\sim}]$ satisfies the Cauchy condition in this space.

A modular base B in X is said to be *complete* if every MS-sequence S in X satisfying the Cauchy condition with respect to B is convergent with respect to it. Whenever a modular base possesses only the property that every (countable) sequence $\{x_n\}$ in X satisfying the Cauchy condition with respect to B is convergent with respect to it, is called *sequentially complete*.

The space $[X, B^{\sim}]$ is called *complete* if the base B is complete and sequentially complete if B is sequentially complete. Obviously B can be replaced by any base B_1 in $B^{\sim}$.

6.1. Let B be a modular base in X. Consider the following three conditions imposed on B:

$(T_1^{\vee})$ *for every $x \in X$, $x \neq 0$, there is an $a \neq 0$ and $U \in B$ such that $x \notin aU$,*

(T_1) *for every $x \in X$, $x \neq 0$, there is a set $U \in B$ with $x \notin U$,*

$(T_1^{\wedge})$ *for every $x \in X$, $x \neq 0$, there is a sequence $\{U_n\}$ of sets in B such that*

$$x \notin \bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n).$$

Notice that if B satisfies $(T_1^{\wedge})$, then it also satisfies (T_1) and this in turn implies that $(T_1^{\vee})$ holds for B. Moreover, $(T_1^{\vee})$ holds for B if and only if (T_1) holds for $B^{\vee}$ and $(T_1^{\wedge})$ holds for B if and only if (T_1) holds for $B^{\wedge}$.

6.2. *A modular base B in X satisfies (T_1) if and only if the following condition holds: for each $x \in X$, $x \neq 0$, and for every $a \neq 0$ there is a set $U \in B$ with $ax \notin U$.*

Proof. If B satisfies this above condition, then, obviously, B satisfies (T_1). Now let x be any member in X different than 0 and a any number different than 0. Then $ax \in X$ and $ax \neq 0$. By (T_1) for B it follows that there is a set $U \in B$ such that $ax \notin U$. Therefore the required condition holds for B.

6.3. *Let B_1 and B_2 be two modular bases in X. If $B_1 \dashv B_2$ and (T_1) (resp. $(T_1^{\vee})$, $(T_1^{\wedge})$) holds for B_1, then (T_1) (resp. $(T_1^{\vee})$, $(T_1^{\wedge})$) holds for B_2. Hence if $B_1 \sim B_2$, then B_1 satisfies (T_1) (resp. $(T_1^{\vee})$, $(T_1^{\wedge})$) if and only if B_2 does.*

Proof. In view of the remark in 6.1 as well as of 3.3 and 4.3 it is clear that it suffices to prove only for (T_1).

Let B_1 and B_2 be modular bases in X such that $B_1 \dashv B_2$ and let (T_1) holds for B_1. Recall that $B_1 \dashv B_2$ means that there is $a \neq 0$ such that for every set $U_1 \in B_1$ one can find a set $U_2 \in B_2$ with $aU_2 \subset U_1$. By virtue of 6.2, it follows from the fact that B_1 satisfies (T_1) that for each $x \in X$, $x \neq 0$, there is $U_1 \in B_1$ with $ax \notin U_1$. Thus for each $x \in X$, $x \neq 0$, there exists a set $U_2 \in B_2$ such that $x \notin U_2$. This clearly means that B_2 satisfies (T_1).

6.4. *For an LT-base B in X condition $(T_1^{\vee})$, (T_1) and $(T_1^{\wedge})$ are equivalent.*

This is immediate from 6.3 and $\bar{4.7}$ and the remark of 6.1.

6.5. *Let B be a modular base in X. Every MS-sequence (countable sequence) S in X convergent with respect to B has exactly one limit if and only if B satisfies (T_1).*

Proof. Let B be a modular base in X satisfying (T_1) and $S = \{x_\sigma\}$ be any MS-sequence (sequence) in X convergent with respect to B. Assume that $S \xrightarrow{B} x_1$ and $S \xrightarrow{B} x_2$, where $x_1, x_2 \in X$. Then there exists $a_1 \neq 0$ such that for every $U' \in B$ there is a $\sigma_1 \in \Sigma$ with the property that $a_1(x_\sigma - x_1) \in U'$ for $\sigma_1 \dashv \sigma \in \Sigma$, and there exists $a_2 \neq 0$ such that for every $U' \in B$ there is a $\sigma_2 \in \Sigma$ with the property that $a_2(x_\sigma - x_2) \in U'$ for $\sigma_2 \dashv \sigma \in \Sigma$. Now let U be any set in B. By (M1) for B it follows that there is a set $U' \in B$ with $\Gamma(U') \subset U$. Take $a = \frac{1}{2}\inf\{|a_1|, |a_2|\}$ and $\sigma_0 \in \Sigma$ such that $\sigma_1 \dashv \sigma_0$ and $\sigma_2 \dashv \sigma_0$. We verify that

$$a(x_2 - x_1) = (aa_1^{-1})a_1(x_{\sigma_0} - x_1) - (aa_2^{-1})a_2(x_{\sigma_0} - x_2) \in \Gamma(U') \subset U.$$

Hence as B satisfies (T_1) we get $a(x_2 - x_1) = 0$ and so $x_2 = x_1$.

Conversely, let for a modular base B in X condition (T_1) fails to hold. Then there is $x \in X$, $x \neq 0$, such that $x \in U$ for every $U \in B$. Consider the sequence $S = \{x_n\}$, where $x_n = x$ for $n = 1, 2, \ldots$ Observe that $x_n \in U$ and $x_n - x = 0 \in U$ for every $U \in B$ and $n = 1, 2, \ldots$ Whence $S \xrightarrow{B} 0$ and $S \xrightarrow{B} x$, where $x \neq 0$.

6.6. A modular space $[X, B^{\sim}]$ is said to *satisfy condition* (T_1) (resp. $(\mathrm{T}_1^{\vee})$, $(\mathrm{T}_1^{\wedge})$) whenever B satisfies (T_1) (resp. $(\mathrm{T}_1^{\vee})$, $(\mathrm{T}_1^{\wedge})$).

Obviously in this definition B can be replaced by any B_1 in $B^{\sim}$. Instead of "the modular space $[X, B^{\sim}]$ satisfies (T_1)-condition" we simply say "$[X, B^{\sim}]$ is T_1-space", and similarly for $(\mathrm{T}_1^{\vee})$ and $(\mathrm{T}_1^{\wedge})$.

7.1. Let B be a modular base and Z a set in X. Z is said to be *bounded* with respect to B if for every $U \in B$ there is $a \neq 0$ such that $aZ \subset U$.

This definition is identical to that of the theory of linear-topological spaces.

7.2. *For any modular base B and any set Z in X the following conditions are equivalent:*

1° *Z is bounded with respect to B,*

2° *for every sequence $\{x_n\}$ in Z and for every sequence of numbers $\{a_n\}$ converging to 0, the sequence $\{a_n x_n\}$ converges to 0 with respect to B,*

3° *every sequence $\{n^{-1}x_n\}$, where $x_n \in Z$ for $n = 1, 2, \ldots$, converges to 0 with respect to B.*

Proof. Suppose that 2° does not hold. Then there are sequences $\{x_n\}$ in Z and $\{a_n\}$ of numbers converging to 0 such that $\{a_n x_n\}$ does not converge to 0 with respect to B. Furthermore there exist a subsequence $\{a_{n_k} x_{n_k}\}$ of $\{a_n x_n\}$ and $U \in B$ such that $a_{n_k} x_{n_k} \notin U$ for $k = 1, 2, \ldots$ By (M1) for B we find a set $U' \in B$ with bal.$U' \subset U$. For any number $a \neq 0$ let k be such that $|a_{n_k}| \leqslant |a|$ and note that then $a x_{n_k} \notin U'$. Thus $aZ \not\subset U$ for every $a \neq 0$. This proves that 1° does not hold. Therefore 1° implies 2°.

That 2° implies 3° is clear.

Now suppose that 1° fails to hold. Then we have $U \in B$ such that for any positive integer n the difference $Z \setminus n^2 U$ is non-void. Take the sequence $\{n^{-1}x_n\}$, where $x_n \in Z \setminus n^2 U$ for $n = 1, 2, \ldots$ and let a be any number different than 0. By (M1) for B there is $U' \in B$ with $\mathrm{bal.}U' \subset U$. Note that for $n \geqslant |a|^{-1}$ we have $an^{-1}x_n \notin U'$. This implies that $\{n^{-1}x_n\}$ does not converge to 0 with respect to B. Therefore 3° does not hold. Thus 3° implies 1°.

7.3. *Let B_1 and B_2 be two modular bases in X. If $B_1 \dashv B_2$, then every set $Z \subset X$ bounded with respect to B_2 is also bounded with respect to B_1. Hence, if $B_1 \sim B_2$, then $Z \subset X$ is bounded with respect to B_1 if and only if it is bounded with respect to B_2.*

Proof. Recall that $B_1 \dashv B_2$ means that there is a number $a_1 \neq 0$ such that for every $U_1 \in B_1$ there exists $U_2 \in B_2$ with $a_1 U_2 \subset U_1$. Let $Z \subset X$ be bounded with respect to B_2. Then for every $U_2 \in B_2$ there is $a_2 \neq 0$ such that $a_2 Z \subset U_2$. Thus we get $a_1 a_2 Z \subset U_1$ for $U_1 \in B_1$. This means that Z is bounded with respect to B_1.

7.4. *Let B be a modular base and Z a set in X. Z is bounded with respect to B if and only if it is bounded with respect to $B^{\vee}$.*

Proof. If Z is bounded with respect to $B^{\vee}$, then, by 3.4.1° and 7.3, it is bounded with respect to B. Conversely, let Z be bounded with respect to B. Then for every $U \in B$ there is $a_1 \neq 0$ such that $a_1 Z \subset U$. Let V be any set in $B^{\vee}$. It can be represented as $V = a_2 U$, where $U \in B$ and $a_2 \neq 0$. Hence $a_1 a_2 Z \subset V$. It follows that Z is bounded with respect to $B^{\vee}$.

7.5. *Every sequence $\{x_n\}$ in X satisfying the Cauchy condition with respect to a modular base B is bounded with respect to B*

Proof. Let $\{x_n\}$ be a sequence in X satisfying the Cauchy condition with respect to B. Then there is $a_0 \neq 0$ such that for every $U' \in B$ there is a positive integer n_0 such that $a_0(x_n - x_{n_0}) \in U'$ for $n \geqslant n_0$. According to (M2) for B every $U' \in B$ is absorbent in X and so there exist numbers $a_n \neq 0$, $n = 1, 2, \ldots, n_0$, such that $a_n x_n \in U'$ for $n = 1, 2 \ldots, n_0$. Let U be any set in B. By (M1) for B we find a set $U' \in B$ such that $\mathrm{bal.}U' \subset \Gamma(U') \subset U$. Take $a = \frac{1}{2}\inf\{|a_0|, |a_1|, \ldots, |a_{0n}|\}$. We obtain

$$ax_n \in aa_0^{-1} U' \subset \mathrm{bal.}U' \subset U \qquad \text{for } n = 1, 2, \ldots, n_0$$

and

$$ax_n = (aa_0^{-1})a_0(x_n - x_{n_0}) + (aa_{n_0}^{-1})a_{n_0}x_{n_0} \in \Gamma(U') \subset U$$

for $n = n_0+1, n_0+2, \ldots$ Hence $\{x_n\}$ is bounded with respect to B.

7.6. Let $[X, B^{\sim}]$ be a modular space and Z a set in X. Z is said to be *bounded* in $[X, B^{\sim}]$ if it is bounded with respect to B. It follows from 7.3 that B in this definition can be replaced by any base B_1 in the class $B^{\sim}$. The result of 7.4 says that Z is *bounded in a modular space* $[X, B^{\sim}]$ if and only if it is bounded in the upper linear-topological space $[X, B^{\vee\sim}]$ generated by $[X, B^{\sim}]$.

8.1. A modular base B in X is said to be *almost locally bounded*, whenever there exists a set $U_0 \in B$ bounded with respect to B. A modular base B such that every $U \in B$ is bounded with respect to B is called *locally bounded*.

8.2. *Let B_1 and B_2 be two modular bases in X. If $B_1 \sim B_2$ and B_1 is almost locally bounded, then so is B_2.*

Proof. Let $U_1^0 \in B_1$ be bounded with respect to B_1. Since $B_1 \dashv B_2$ then there are $a \neq 0$ and $U_2^0 \in B_2$ such that $aU_2^0 \subset U_1^0$. Thus U_2^0 is bounded with respect to B_1. Since also $B_2 \dashv B_1$, by 7.3, we get that U_2^0 is bounded with respect to B_2.

8.3. *A modular base B in X is almost locally bounded if and only if the base $B^{\vee}$ is almost locally bounded and B is locally bounded if and only if $B^{\vee}$ is locally bounded.*

Proof. Let B be almost locally bounded modular base in X. Then there is a $U_0 \in B$ bounded with respect to B. By definition of $B^{\vee}$, $U_0 \in B^{\vee}$, and by 7.4 it is bounded with respect to $B^{\vee}$. Hence $B^{\vee}$ is almost locally bounded. Conversely, let $B^{\vee}$ be almost locally bounded. Then there is V_0 in $B^{\vee}$ bounded with respect to $B^{\vee}$. V_0 is of the form $V_0 = aU_0$, where $U_0 \in B$ and $a \neq 0$. We conclude therefore that U_0 is bounded with respect to $B^{\vee}$ and, by 7.4 it is also bounded with respect to B. This means that B is almost locally bounded.

Now let B be locally bounded. Note that every $V \subset X$ of the form $V = aU$, where $U \in B$ and $a \neq 0$, is bounded with respect to B. By 7.4 we get that every $V \in B^{\vee}$ is bounded with respect to $B^{\vee}$. Hence $B^{\vee}$ is also locally bounded. Conversely, if $B^{\vee}$ is locally bounded, then since $B \subset B^{\vee}$, it follows immediately that B is locally bounded.

8.4. *For every almost locally bounded modular base B in X there is an equivalent locally bounded modular base B_1 in X. It can be assumed that $B_1 \subset B$.*

Proof. Let B be modular base almost locally bounded in X. Then there is a set $U_0 \in B$ bounded with respect to B. Let B_1 denote the family of all these $U \in B$ such that $U \subset U_0$. Clearly B_1 is the required modular base in X.

8.5. *If a modular base B in X is locally bounded, then any $Z \subset X$ is bounded with respect to B if and only if there exist $U \in B$ and $a \neq 0$ such that $aZ \subset U$.*

The simple proof is omitted.

8.6. A modular space $[X, B^{\sim}]$ is said to be *locally bounded* if B is almost locally bounded. It follows from 8.2 that in this definition B can be replaced by any base B_1 in the class $B^{\sim}$. The result 8.4 says that $[X, B^{\sim}]$ is locally bounded if and only if in the class $B^{\sim}$ exists a locally bounded base.

The result of 8.3 states that the modular space $[X, B^{\sim}]$ is locally bounded if and only if the upper linear-topological space $[X, B^{\vee\sim}]$ is locally bounded.

9.1. A modular base B in X is said to be *locally convex* if every set $U \in B$ is absolutely convex, i.e. $\Gamma(U) \subset U$.

Notice that a non-void family B of sets in X is a *locally convex modular base* if and only if it satisfies the three following conditions:

1° *every $U \in B$ is absolutely convex,*

2° *for every $U_1, U_2 \in B$ there is $U_3 \in B$ such that $U_3 \subset U_1 \cap U_2$,*

3° *every $U \in B$ is absorbent in X.*

9.2. *If B is a non-void family of absolutely convex and absorbent sets in X, then the family B^* of all sets $U \subset X$ representable in the form*

$$U = U_1 \cap U_2 \cap \ldots \cap U_n,$$

where $U_1, U_2, \ldots, U_n \in B$, is a locally convex modular base in X.

This is a consequence of the remark in 9.1.

9.3. *If a modular base B in X is locally convex, then $B^{\vee}$ and $B^{\wedge}$ are also locally convex.*

Proof. Let B be a locally convex modular base in X. Then for every set $U \in B$ we have $\Gamma(U) \subset U$. Hence for every $U \in B$ and $a \neq 0$ we get that

$$\Gamma(aU) = a\Gamma(U) \subset aU.$$

Thus every set $V \in B^{\vee}$ is absolutely convex. This means that $B^{\vee}$ is locally convex. Let now $\{U_n\}$ be any sequence of sets in B. We observe that

$$\Gamma\Big(\bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n)\Big) \subset \bigcup_{n=1}^{\infty} \big(\Gamma(U_1) + \ldots + \Gamma(U_n)\big)$$
$$\subset \bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n).$$

Thus every set $W \in B^{\wedge}$ is absolutely convex. This means that $B^{\wedge}$ is locally convex.

9.4. A modular space $[X, B^{\sim}]$ is said to be *locally convex* if in the class $B^{\sim}$ there is a locally convex base.

The result of 9.3 says that if the modular space $[X, B^{\sim}]$ is locally convex, then the linear-topological spaces $[X, B^{\vee\sim}]$ and $[X, A^{\wedge\sim}]$ are also locally convex.

10.1. Let B be a modular base in X. By abs.conv.B we denote the family of all sets $V \subset X$ of the form $V = \text{abs.conv.}U$, where $U \in B$.

For any set $U \subset X$ by abs.conv.U we denote, as is widely accepted, the smallest absolutely convex set containing U, i.e. the set of all these $x \in X$ which can be represented in the form

$$x = a_1 x_1 + \ldots + a_n x_n,$$

where $x_1, \ldots, x_n \in U$ and $a_1, \ldots, a_n$ are numbers such that

$$|a_1| + \ldots + |a_n| \leqslant 1 \quad (\text{or} = 1).$$

We shall show that

For any modular base B in X the family abs.conv.B *is a locally convex modular base in X.*

Proof. Obviously, the family abs.conv.B satisfies 1° and 3° of the remark in 9.1. Let now V_1 and V_2 be any sets in abs.conv.B. Then there exist sets $U_1, U_2 \in B$ such that $V_1 = \text{abs.conv.}U_1$ and $V_2 = \text{abs.conv.}U_2$. By (M1) for B we get that there exists a set $U_3 \in B$ such that $U_3 \subset \Gamma(U_3) \subset U_1 \cap U_2$. Note that $V_3 = \text{abs.conv.}U_3$ belongs to abs.conv.B and has the property that

$$V_3 \subset \text{abs.conv.}(U_1 \cap U_2) \subset \text{abs.conv.}U_1 \cap \text{abs.conv.}U_2 = V_1 \cap V_2.$$

This proves that abs.conv.B satisfies 2° of the remark in 9.1. Hence this family is a locally convex modular base in X.

10.2. *Let B_1 and B_2 be two modular bases in X. If $B_1 \dashv B_2$, then also* abs.conv.$B_1 \dashv$ abs.conv.B_2. *Hence, if $B_1 \sim B_2$, then also* abs.conv.$B_1 \sim$ abs.conv.B_2.

Proof. $B_1 \dashv B_2$ means that there is $a \neq 0$ such that for every $U_1 \in B_1$ there exists $U_2 \in B_2$ with $aU_2 \subset U_1$. Hence we get that then also

$$a\,\text{abs.conv.}U_2 = \text{abs.conv.}(aU_2) \subset \text{abs. conv } U_1.$$

Thus abs.conv.$B_1 \dashv$ abs.conv.B_2.

10.3. *For any modular base B in X the following hold:*

1° abs.conv.$B \dashv B$,

2° *if B_1 is a locally convex modular base in X such that $B_1 \dashv B$, then* $B_1 \dashv$ abs.conv.B.

Proof. For every $U \subset X$ we have $U \subset \text{abs.conv.}U$. Hence abs.conv. $B \dashv B$. Now, let B_1 be a locally convex modular base in X such that $B_1 \dashv B$. Then $B_1 \dashv B$ indicates that there is $a \neq 0$ such that for every $U_1 \in B_1$ there exists $U \in B$ with $aU \subset U_1$. This implies that

$$a\,\text{abs.conv.}U = \text{abs.conv.}(aU) \subset \text{abs.conv.}U_1 = U_1.$$

Hence $B_1 \dashv \text{abs.conv.}B$.

10.4. *If B is an LT-base in X, then* abs.conv.B *is also an LT-base in X.*

Proof. Let B be an LT-base in X. Then B satisfies (Δ_2), i.e. for every $U \in B$ there is $U' \in B$ such that $2U' \subset U$. Hence then we have also $2\text{abs.conv.}U' \subset \text{abs.conv.}U$. This proves that abs.conv.B satisfies (Δ_2) and so it is an LT-base in X.

10.5. *For any modular base B in X the following relations hold:*

$$\text{abs.conv.}B^{\check{}} = (\text{abs.conv.}B)^{\check{}} \quad \textit{and} \quad \text{abs.conv.}B^{\hat{}} \sim (\text{abs.conv.}B)^{\hat{}}.$$

Proof. For any $U \in B$ and $a \neq 0$ we have

$$\text{abs.conv.}(aU) = a\,\text{abs.conv.}U.$$

Thus abs.conv $B^{\check{}} = (\text{abs.conv.}B)^{\check{}}$.

For any sequence $\{U_n\}$ in B we see that

$$\bigcup_{n=1}^{\infty} (\text{abs.conv.}U_1 + \ldots + \text{abs.conv.}U_n) \supset \text{abs.conv.}\Big(\bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n)\Big).$$

Hence $(\text{abs.conv.}B)^{\hat{}} \dashv \text{abs.conv.}B^{\hat{}}$. On the other hand from 4.2.1° and 10.2 we get $\text{abs.conv.}B^{\hat{}} \dashv \text{abs.conv.}B$. By 10.4 and 4.2.2° it follows that $\text{abs.conv.}B^{\hat{}} \dashv (\text{abs.conv.}B)^{\hat{}}$. Thus $\text{abs.conv.}B^{\hat{}} \sim (\text{abs.conv.}B)^{\hat{}}$.

10.6. *If a modular base B in X is almost locally bounded (resp. locally bounded), then so is* abs.conv.B.

Proof. Recall that B is almost locally bounded means that there is a set $U_0 \in B$ such that for every $U \in B$ there is $a \neq 0$ with $aU_0 \subset U$. Hence then we have also $a\,\text{abs.conv.}U_0 \subset \text{abs.conv.}U$. This means that abs.conv.B is almost locally bounded.

Similarly, if B is locally bounded, then for any $U_1, U_2 \in B$ there is $a \neq 0$ such that $aU_1 \subset U_2$. Thus also $a\,\text{abs.conv.}U_1 \subset \text{abs.conv.}U_2$. This proves that abs.conv.B is locally bounded.

10.7. It follows from 10.1 and 10.2 that for a modular space $[X, B^{\tilde{}}]$ the pair $[X, (\text{abs.conv.}B)^{\tilde{}}]$ determines the unique locally convex modular space. The space $[X, (\text{abs.conv.}B)^{\tilde{}}]$ will be called the *locally convex modular space generated by* $[X, B^{\tilde{}}]$. The result of 10.5 states that for any modular space $[X, B^{\sim}]$ the following equalities hold:

$$[X, (\text{abs.conv}.B^{\vee})^{\sim}] = [X, (\text{abs.conv}.B)^{\vee\sim}],$$
$$[X, (\text{abs.conv}.B^{\wedge})^{\sim}] = [X, (\text{abs.conv}.B)^{\wedge\sim}].$$

10.6 implies that the locally bounded modular space $[X, B^{\sim}]$ generates the locally convex modular space $[X, (\text{abs.conv}.B)^{\sim}]$ also locally bounded.

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