ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XVII (1974)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XVII (1974)

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Some boundary value problems with transformed argument

In this paper we shall present a method of solving of a class of boundary value problems with deviating argument. This method is showed just on few simple examples, but its applications for more general equations are obvious and the proofs are going on the same line.

1. Let S_a be a rotation through the angle a of the plane $\boldsymbol{R}^2$, i. e. a linear operator defined by means of the matrix

$$S_a = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}.$$

In sequel we shall consider only rotation through an angle commensurable with 2π. Without loss of generality we can assume that

$$a = 2\pi/N. \tag{1.1}$$

Under this hypothesis

$$S_{2\pi/N}^N = I, \quad \text{where I denotes the unit matrix.} \tag{1.2}$$

Indeed, by an easy induction we have $S_a^k = S_{ka}$ for $k = 1, 2, \ldots$ The last formula justifies assumption (1.1). So that $S_{2\pi/N}^N = S_{N\cdot 2\pi/N} = S_{2\pi} = I$.

A domain $E \subset \boldsymbol{R}^2$ is said to be *invariant with respect to the rotation* S_a if $S_a E = E$.

Consider the Laplace equation

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x, y) \in E, \tag{1.3}$$

where E is a domain invariant with respect to the rotation $S_{2\pi/N}$, with a boundary condition of the Dirichlet type:

$$\lim_{p\to p_0} \sum_{k=0}^{N-1} a_k u(S_{2\pi/N}^k p) = b(p_0), \quad p = (x, y) \in E, \tag{1.4}$$

where p_0 belongs to the boundary ∂E of the domain E.

We are looking for complex-valued functions, continuous on $\bar{E} = E \cup \partial E$, satisfying equation (1.3) inside E and condition (1.4) on the boundary ∂E.

Denote by X the space of all twice continuously differentiable functions in E. In the space X we introduce a linear operator $\tilde{S}_a$ induced by the operator S_a in the following manner:

$$(1.5) \qquad (\tilde{S}_a u)(p) = u(S_a p), \quad \text{where } u \in X \text{ and } p = (x, y) \in E.$$

PROPOSITION 1.1. *For all a the operators S_a commute with the Laplacian on the space X.*

Indeed, let $p = (x, y) \in E$ and $p' = S_a p = (x', y')$. Then

$$(\Delta \tilde{S}_a u)(p) = \frac{\partial^2 u(S_a p)}{\partial x'^2} + \frac{\partial^2 u(S_a p)}{\partial y'^2} = \frac{\partial^2 u(S_a p)}{\partial x^2} + \frac{\partial^2 u(S_a p)}{\partial y^2} = (\tilde{S}_a \Delta u)(p)$$

for all $u \in X$ and $p \in E$.

The operator $\tilde{S}_{2\pi/N}$ is an involution of order N on the space X. Indeed, since (1.2) holds, then $(\tilde{S}^N_{2\pi/N} u)(p) = u(S^N_{2\pi/N} p) = u(p)$ for all $u \in X$ and $p \in E$. Thus

$$(1.5) \qquad \tilde{S}^N_{2\pi/N} = I \quad \text{on } X.$$

Observe that the operator $S_{2\pi/N}$ induces also an involution of order N defined in the same manner as the previous one on the space $\tilde{X}$ of all continuous functions on the boundary ∂E. We can denote this involution by the same letters $\tilde{S}_{2\pi/N}$, because it does not lead to any misunderstanding. We thus have $(\tilde{S}_{2\pi/N} u)(q) = u(S_{2\pi/N} q)$ for all $u \in \tilde{X}$ and $q \in \partial E$. Since $\tilde{S}_{2\pi/N}$ is an involution of order N on both spaces X and $\tilde{X}$, we have N disjoint projectors giving a partition of unity (see [1]):

$$\tilde{P}_j = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon^{-kj} \tilde{S}^k_{2\pi/N} = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon^{-kj} \tilde{S}_{2\pi k/N}, \quad \text{where } \varepsilon = e^{2\pi i/N} \ (j = 1, \ldots, N).$$

Thus, if $p = (x, y)$ belongs either to E or to ∂E, we have

$$(1.6) \qquad (\tilde{P}_j u)(p) = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon^{-kj} (\tilde{S}_{2\pi k/N} u)(p) = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon^{-kj} u(S_{2\pi k/N} p)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon^{-kj} u\left(x \cos\frac{2\pi k}{N} - y \sin\frac{2\pi k}{N},\ x \sin\frac{2\pi k}{N} + y \cos\frac{2\pi k}{N}\right)$$

$$(j = 1, 2, \ldots, N).$$

Both spaces X and $\tilde{X}$ are direct sums

$X = X_1 \oplus \ldots \oplus X_N$, $\tilde{X} = \tilde{X}_1 \oplus \ldots \oplus \tilde{X}_N$, where $X_j = \tilde{P}_j X$; $\tilde{X}_j = \tilde{P}_j \tilde{X}$, such that $\tilde{S}_{2\pi/N} u = \varepsilon^j u$ for $u \in X_j$ (resp. for $u \in \tilde{X}_j$) $(j = 1, 2, \ldots, N)$. Therefore Proposition 1.1 implies that equation (1.3) with the boundary condition (1.4), where the function $b(q)$ is continuous on ∂E, is equivalent to the system of N independent Laplace equations

$$(1.7) \qquad \Delta u_{(j)} = 0 \quad \text{in } E, \quad \text{where } u_{(j)} = \tilde{P}_j u \ (j = 1, 2, \ldots, N)$$

with the boundary condition of the Dirichlet type for the j-th equation:

$$(1.8) \qquad \lim_{p \to x_0} u_{(j)}(p) = \tilde{b}_j(p_0), \quad p_0 \in \partial E \ (j = 1, 2, \ldots, N),$$

where

$$(1.9) \ \tilde{b}_j = \frac{1}{b_j} \tilde{P}_j b, \quad \text{provided that} \quad b_j = \sum_{k=0}^{N-1} a_k \varepsilon^{kj} \neq 0 \quad (j = 1, 2, \ldots, N).$$

In the last conditions deviations of argument do not appear.

We obtain the following

THEOREM 1.1. *Let $E \subset \boldsymbol{R}^2$ be a domain invariant with respect to the rotation $S_{2\pi/N}$ with the boundary satisfying the Liapounov condition. If $b_j = \sum_{k=0}^{N-1} a_k \varepsilon^{kj} \neq 0$ for $j = 1, 2, \ldots, N$, where $\varepsilon = e^{2\pi/N}$ and if the function $b(p)$ is continuous on ∂E, then there exists a twice continuously differentiable in E and continuous on $\bar{E}$ solution of equation* (1.3) *which satisfies the boundary condition* (1.4). *This solution is of the form*

$$u(t) = \sum_{j=1}^{N} u_j,$$

where u_j is a solution of the j-th equation (1.7) *with the j-th condition* (1.8).

The proof follows immediately from the fact that the j-th equation (1.7) with the j-th boundary condition (1.8) is the classical Dirichlet boundary problem, which has a unique solution having the required properties.

Without any change of the proof we can consider the boundary condition (1.4) with variable coefficients a_k such that

$$(1.10) \qquad a_k(S_{2\pi/N} p) = a_k(p) \quad \text{for } p \in \bar{E} \ (k = 0, 1, \ldots, N-1),$$

because the operator of multiplication by such a function commutes with the involution $\tilde{S}_{2\pi/N}$ on X and $\tilde{X}$. Indeed, if $a(S_{2\pi/N} p) = a(p)$, then

$$[(\tilde{S}_{2\pi/N} a - a \tilde{S}_{2\pi/N}) u](p) = a(S_{2\pi/N} p) u(S_{2\pi/N} p) - a(p) u(S_{2\pi/N} p)$$
$$= [a(S_{2\pi/N} p) - a(p)] u(S_{2\pi/N} p) = 0.$$

It is enough to assume that $b_j(p) = \sum_{k=0}^{N-1} a_k(p) \varepsilon^{kj} \neq 0$ on ∂E.

Now consider the boundary value problem of Neumann type for equation (1.3). It means that we are looking for complex-valued functions continuous on E twice continuously differentiable in E, satisfying equation (1.3) in E and the condition

$$\lim_{p\to p_0}\sum_{k=0}^{N-1}\frac{\partial u(q)}{\partial n}\Big|_{q=S_{2\pi k/N}p} = b(p_0), \qquad p_0\in\partial E, p\in E, \tag{1.11}$$

where by $\partial/\partial n$ we mean the derivative in the direction of the interior normal to the domain E. We assume that $b(p)$ is a continuous function on ∂E. In the same manner, as for the Dirichlet problem, we conclude that equation (1.3) with the boundary condition (1.11) is equivalent to N independent equations (1.7) with the boundary condition for the j-th equation:

$$\lim_{p\to p_0}\sum_{k=0}^{N-1}\frac{\partial u_{(j)}(q)}{\partial n}\Big|_{q=S_{2\pi k/N}p} = \tilde{b}_j(p_0), \quad p_0\in\partial E,\ p\in E\ (j=1,2,\ldots,N), \tag{1.12}$$

where

$$\tilde{b}_j = \frac{1}{b_j}\tilde{P}_j b \quad\text{and}\quad b_j(p_0) = \sum_{k=0}^{N-1} a_k(p_0)\varepsilon^{kj}\neq 0 \quad \text{for } p_0\in\partial E\ (j=1,2,\ldots,N).$$

Observe that

$$\int_{\partial E}\tilde{b}_j(p)\,dS = 0 \qquad \text{for } j=1,2,\ldots,N-1. \tag{1.13}$$

Indeed, let us denote by E_k and ∂E_k these parts of the domain E and of its boundary respectively, which are contained in the angle $2\pi(k-1)/N<\varphi\leqslant 2\pi k/N$. It is obvious that

$$\int_{\partial E}\tilde{b}_j(p)\,dS = \sum_{k=1}^{N}\int_{\partial E_k}\tilde{b}_j(p)\,dS.$$

But $E_k = S^k_{2\pi/N}E_1$, because of the property $S_{2\pi/N}E=E$. Hence $E_1 = S^{-k}_{2\pi/N}E_k$ and

$$\begin{aligned}\int_{\partial E}\tilde{b}_j(p)\,dS &= \sum_{k=1}^{N}\int_{\partial E_k}\tilde{b}_j(p)\,dS = \int_{\partial E_1}\frac{1}{b_j}\sum_{k=1}^{N} b_{(j)}(S^k_{2\pi/N}p)\,dS\\
&= \int_{\partial E_1}\frac{1}{b_j}\sum_{k=1}^{N}(\tilde{S}^k_{2\pi/N}\tilde{P}_j b)(p)\,dS = \int_{\partial E_1}\frac{1}{b_j}\Big[\sum_{k=1}^{N}\varepsilon^{kj}\Big](\tilde{P}_j b)(p)\,dS\\
&= \int_{\partial E_1}\frac{1}{b_j}\,\delta_{N,j}\tilde{b}_{(j)}(p)\,dS = \delta_{N,j}\int_{\partial E_1}\tilde{b}_j(p)\,dS\\
&= \begin{cases} 0 & \text{for } j=1,2,\ldots,N-1,\\ \int_{\partial E}\tilde{b}_j(p)\,dS & \text{for } j=N\end{cases}\end{aligned}$$

($\delta_{N,j}$ being the Kronecker symbol) in virtue of Formula 1.6 of Chapter II. Therefore for an arbitrary $b \in \tilde{X}$ we have equalities (1.13) and writing $\tilde{b}(p) = \sum_{j=1}^{N} \tilde{b}(p)$ we obtain

$$\int_{\partial E} \tilde{b}(p)\,dS = \int_{\partial E} \tilde{b}_N(p)\,dS. \tag{1.14}$$

Since all previous considerations are valid for functions $a_k(p)$ invariant under the rotation $S_{2\pi/N}$, we obtain finally

THEOREM 1.2. *Let $E \subset \mathbf{R}^2$ be a domain invariant with respect to the rotation $S_{2\pi/N}$ with the boundary ∂E satisfying the Liapounov condition. Let the functions $a_0(p), \ldots, a_{N-1}(p)$ continuous on E be invariant under the rotation $S_{2\pi/N}$, i. e. $a_k(S_{2\pi/N}p) = a_k(p)$ $(k = 0, 1, \ldots, N-1)$ and let*

(i) $b_j(p) = \sum_{k=0}^{N-1} a_k(p)\varepsilon^{kj} \neq 0$ *on ∂E, where $\varepsilon = e^{2\pi i/N}$,*

(ii) $\int_{\partial E} b(p)\,dS = 0$, *where $b(p)$ is a continuous function on ∂E.*

Then there exists a function $u(t)$ continuous on $\bar{E}$, twice continuously differentiable in E and satisfying equation (1.3) *in E and condition* (1.11) *on ∂E. This function is of the form:*

$$u(p) = \sum_{j=1}^{N} u_j(p),$$

where $u_j(p)$ is a solution of the j-th equation (1.7) *satisfying the j-th boundary condition* (1.12).

Indeed, the j-th equation (1.7) has a solution satisfying the j-th condition (1.12) only if it is satisfied the condition $\int_{\partial E} \tilde{b}_j(p)\,dS = 0$ $(j = 1, 2, \ldots, N)$. The assumptions and formulae (1.13), (1.14) assert this condition to be satisfied.

In a similar manner we can examine the equation $\Delta u + au = v$ in E with the boundary conditions either (1.4) or (1.11), where $a(p)$ and $v(p)$ are continuous for $p \in \bar{E}$ and $a(S_{2\pi/N}p) = a(p)$.

2. Now we shall consider the following boundary problem with rotation of the Fourier type:

To determine a complex valued function $u(x, y, t)$ which satisfies

(1) the heat equation

(2.1) $\Delta u = \dfrac{\partial u}{\partial t} = 0$ for $p = (x, y) \in E \subset \mathbf{R}^2$ and $t > 0$, where E is a domain invariant under rotation $S_{2\pi/N}$ with the boundary satisfying Liapounov condition,

(2) an initial condition:

$$\lim_{t\to+0} u(p,t) = v(p) \quad \text{for all } p\in E; \tag{2.2}$$

(3) a boundary condition with rotation:

$$\lim_{p\to p_0} \sum_{k=0}^{N-1} a_k(p,t)\,u(S^k_{2\pi/N}p,t) = b(p_0,t) \quad \text{for } t>0,\ p_0\in\partial E,\ p\in E. \tag{2.3}$$

We assume that the functions $a_k(p,t)$ are invariant with respect to the rotation $S_{2\pi/N}$:

$$a_k(S_{2\pi/N}p,t) = a_k(p,t) \quad \text{for} \quad p\in\bar{E} \text{ and } t>0\ (k=0,1,\ldots,N) \tag{2.4}$$

We introduce, as before, the operator $\tilde{S}_{2\pi/N}$ defined by means of the equality

$$(\tilde{S}_{2\pi/N}u)(p,t) = u(S_{2\pi/N}p,t) \quad \text{for all } p\in\bar{E} \text{ and } t>0. \tag{2.4}$$

Since this operator acts only on the variable p, we can treat the time t in our problem as a parameter and solve it exactly in the same manner as in Section 1. We obtain the following

THEOREM 2.1. *Let $E\subset\boldsymbol{R}^2$ be a domain invariant with respect to the rotation $S_{2\pi/N}$ with the boundary ∂E satisfying the Liapounov condition. Let the functions $v(p)$, $b(p,t)$, $a_k(p,t)$ $(k=0,1,\ldots,N-1)$ be determined and continuous in the regions: $p\in E$; $p\in\partial E$, $t>0$; $p\in\bar{E}$, $t>0$ respectively. Let $b_j(p,t) = \sum_{k=0}^{N-1} a_k(p,t)\,\varepsilon^{kj} \neq 0$ for $p\in\partial E$ and $t>0$ and let condition* (2.4) *be satisfied. Then equation* (2.1) *with the initial condition* (2.2) *and with the boundary condition* (2.3) *has a unique solution which is of the form*

$$u(p,t) = \sum_{j=1}^{N} u_j(p,t),$$

where $u_j(p,t)$ is the solution of the j-th equation

$$\Delta u_{(j)} - \frac{\partial u_{(j)}}{\partial t} \quad \textit{for } p\in E \textit{ and } t>0\ (j=1,2,\ldots,N), \tag{2.6}$$

satisfying the j-th initial condition

$$\lim_{t\to+0} u_{(j)}(p,t) = v_{(j)}(p) \quad \textit{for all } p\in E\ (j=1,2,\ldots,N) \tag{2.7}$$

and the j-th boundary condition

$$\lim_{p\to p_0} u_{(j)}(p,t) = \tilde{b}_j(p_0,t) \quad \textit{for } t>0,\ p_0\in\partial E\ (j=1,2,\ldots,N), \tag{2.8}$$

where

$$\tilde{b}_j(p_0, t) = \frac{1_i}{b_j(p_0, t)} b_{(j)}(p_0, t) \tag{2.9}$$

and for an arbitrary function $w(p)$ we denote

$$W_{(j)}(p, t) = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon^{-kj} W(S^k_{2\pi/N} p, t) \qquad (j = 1, 2, \ldots, N).$$

For the proof we introduce the space X of all functions twice continuously differentiable with respect to variables x, y and continuously differentiable with respect to t for $(x, y) \in E$ and $t > 0$ and the space $\tilde{X}$ of functions continuous for $(x, y) \in \partial E$ and $t \geqslant 0$.

When the distribution $b(p_0, t)$ on the boundary ∂E changes periodically in the time with a period ω, i. e. when $b(p_0, t) = b(p_0, t+\omega)$ for all $p_0 \in \partial E$ and $t > 0$, we can solve the corresponding boundary value problem using multiinvolutions (see [2]). In this case no initial condition will be admitted because the considered functions will be periodic with respect to the variable t with periods commensurable with ω. Condition (2.3) can be admitted in a more general form:

$$\lim_{p \to p_0} \sum_{k=0}^{N-1} \sum_{j=0}^{M-1} a_{kj}(p, t) u\left(S^k_{2\pi/N} p, t - j\frac{\omega}{M}\right) = b(p_0, t), \qquad p_0 \in \partial E, t > 0,$$

M is a positive integer, $a_{kj}(p, t)$, $b(p_0, t)$ are invariant under the rotation $S_{2\pi/N}$ with respect to the variables $p \in E$, $p_0 \in \partial E$ respectively and ω/M-periodic with respect to the variable t. Here the second involution appears, namely the shift operator $(Su)(p, t) = u(p, t+\omega/M)$ for $p \in E$, $t \geqslant 0$, which is an involution of order M in the space of ω-periodic functions.

References

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