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The power method for the generalized eigenvalue problem

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Abstract In this paper the Power Method for the generalized eigenvalue problem for matrix pencil $(A-\lambda B)x = 0$ is considered. At any step of this iterative process the system of linear algebraic equations $By = Ax$ has to be approximately solved with respect to y. We try to answer the question: how accurately we have to solve this system on each step of iteration,in order to guarantee resolution of the eigenproblem with given precision.

1.Introduction Let A, B denote $n \times n$ matrices. We have to find such pairs (λ, v) - where λ is scalar (eigenvalue) and v is a nonzero n-dimensional vector (eigenvector coresponding to λ) - which satisfies the equation:

$$(A - \lambda B)v = 0 \tag{1.1}$$

If matrix B is nonsingular this problem is equivalent to the standard eigenvalue problem for matrix $B^{-1}A$. Following iteration process is called the power method for matrix pencil (A, B):

$$\begin{aligned} &x_0 \\ &By_{k+1} = Ax_k \\ &x_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|} \end{aligned} \tag{1.2}$$

Where x_0 is an initial vector. As in the standard Power Method the vector sequence $\{x_k\}$ generated in this way in general converges to the eigenvector corresponding to the largest (in absolute value) eigenvalue of the pencil (A,B).

In this paper three conditions involving matrices A, B will be assumed:

C1) A, B are Hermitian $n \times n$ matrices.

C2) Matrix B is positively definite i.e:

$$\forall x \in C^n, \quad x \neq 0 \quad (Bx, x)_2 > 0$$

where $(\, , \,)_2$ denotes the standard scalar product in the space C^n.

C3) Matrix A is nonsingular.

The conditions (C1) and (C2) guarantee that in the spaceC^n there exists basis $\{v_i\}_{=1}^n$ consiststing of eigenvectors of the pencil (A, B). Moreover all eigenvalues λ_i (λ_i corresponds to v_i) are real and corresponding eigenvectors $\{v_i\}_{=1}^n$ may be chosen orthogonal in the following sense:

$$\forall i, j \in \{1 \ldots n\}(Bv_i, v_j)_2 = \delta_{i,j} \tag{1.3}$$

Condition C2 could be replaced by the following one :

C4) There exist a positively definite linear combination of matrices A,B i.e:

$\exists \mu_1, \mu_2 \in C$ such that matrix $\mu_1 A + \mu_2 B$ is positively definite.

It is easy to see that pencil $(\mu_1 A, \mu_1 A + \mu_2 B)$ has the same eigenvectors as (A, B). More details concesing these facts may be found in [1].

DEFINITION 1.1 Let B be $n \times n$ Hermitian, positively definite matrix,put:

$$(x, y)_B \stackrel{\text{def}}{=} (Bx, y), \quad (\|x\|_B)^2 \stackrel{\text{def}}{=} (x, x)_B \ \forall x, y \in C^n.$$

We call $\| \, \|_B$ the energetic norm related to the matrix B.

Relation (1.3) indicate that the vectors $\{v_i\}_{=1}^n$ form an orthonormal basis in C^n with respect to the scalar product $(\, , \,)_B$. This important property of eigenvectors of the matrix pencil (A, B) simplify investigation of the process (1.2) when the norm $\| \, \|_B$ is used.

The following theorem gives relation between norm $\| \, \|_B$ and the standard norm $\| \, \|_2$ in the space C^n.

THEOREM 1.1 *For each $x \in C^n$ and positively definite Hermitian $n \times n$ matrix B:*

$$\frac{\|x\|_2}{(\|B^{-1}\|_2)^{1/2}} \leq \|x\|_B \leq (\|B\|_2)^{1/2} \|x\|_2$$

Proof. The right inequality follows from the Schwarz inequality:

$$(\|x\|_B)^2 = (Bx, x)_2 \leq \|Bx\|_2 \|x\|_2 \leq \|B\|_2 (\|x\|_2)^2.$$

The left inequality follows from the above inequality applied to the norm $\| \, \|_{B^{-1}}$:

$$(\|x\|_2)^2 = (BB^{-1}x, x)_2 = (B^{-1}x, x)_B \leq \|B^{-1}x\|_B \|x\|_B =$$
$$= \|x\|_{B^{-1}} \|x\|_B \leq (\|B^{-1}\|_2)^{1/2} \|x\|_2 \|x\|_B$$

∎

In the continuation of this article the vectors $\{v_i\}_{=1}^n$ will denote eigenvectors of the pencil (A, B) which satisfy equations (1.3) and λ_i will denote the eigenvalue corresponding to v_i. It will be assumed that for every k $\|x_k\|_B = 1$

i.e: in every step of iteration (1.2):

$$x_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|_B} \quad \text{hold.}$$

2. Convergence of the sequence $\{x_k\}$ Existence of n linear independent eigenvectors of the matrix pencil (A, B) imply the linear convengence of iteration process (1.2).

Suppose that:

$$(2.1) \qquad \lambda_1 = \lambda_2 = \ldots = \lambda_r \qquad \text{and} \qquad |\lambda_1| > |\lambda_{r+1}| \geq \ldots \geq |\lambda_n|$$

The value: $\xi \stackrel{\text{def}}{=} \frac{|\lambda_{r+1}|}{|\lambda_1|}$ will be called coeficient of convergence for matrix pencil (A, B). We have $\xi \in [0, 1)$.

Let $\mathbf{V}_{\lambda 1}$be the subspace of C^n spanned by all eigenvectors corresponding to λ_1: $\mathbf{V}_{\lambda 1} = \text{span}(v_1 \ldots v_r)$.

THEOREM 2.1 *Let $x_0 = \sum_{i=1}^n \alpha_i v_i$ where $\alpha_i \in C$ for $i \in \{1 \ldots n\}$ and $\sum_{i=1}^r |\alpha_i| > 0$. Then for every $k \in N$ there exists vector $w \in \mathbf{V}_{\lambda 1}$, $\|w\|_B = 1$ such that*

$$\|x_k - w\|_B \leq \xi^k \frac{2\left[\sum_{i=r+1}^n |\alpha_i|^2\right]^{1/2}}{\left[\sum_{i=1}^r |\alpha_i^2|\right]^{1/2}}$$

Proof. From definition of the Power Method (1.2):

$$(1) \qquad x_k = \frac{(B^{-1}A)^k x_0}{\|(B^{-1}A)^k x_0\|_B}$$

$$(2)\, (B^{-1}A)^k x_0 = \sum_{i=1}^n \alpha_i (\lambda_i)^k v_i = |\lambda_1|^k \left[s \sum_{i=1}^r \alpha_i v_i + \sum_{i=r+1}^n \alpha_i (\lambda_i)^k |\lambda_1|^{-k} v_i\right]^2$$

Here s denotes the sign of $(\lambda_1)^k$. As $|\lambda_i(\lambda_1)| \leq \xi < 1$ (for $i > r$) from (1) and (2) follows easily that x_k converges to the vector: $w = \frac{s \sum_{i=1}^r \alpha_i v_i}{\|\sum_{i=1}^r \alpha_i v_i\|}$, hence $w \in \mathbf{V}_{\lambda 1}$ and $\|w\|_B = 1$.

To make it more clear put:

$$a = \sum_{i=1}^r \alpha_i v_i \qquad b = \sum_{i=r+1}^n \alpha_i (\lambda_i)^k |\lambda_1|^{-k} v_i.$$

Orthonormality of eigenvectors of the pencil (A, B) gives following inequalities:

$$(3) \qquad \|b\|_B = \left[\sum_{i=1}^n |\alpha_i|^2 |\lambda_i|^{2k} |\lambda_i|^{-2k}\right]^{1/2} \leq \xi^k \left[\sum_{i=r+1}^n |\alpha_i|^2\right]^{1/2}$$

$$(4) \qquad \|a+b\|_B = \left[\|a\|^2 + \|b\|^2\right]^{1/2} \geq \|a\|_B = \Big[\sum_{i=r+1}^{n} |\alpha_i|^2\Big]^{1/2}$$

From (1),(2) and definition of vectors a, b follows:

$$(5) \qquad \|x_k - w\|_B = \|(a+b)(\|a+b\|_B)^{-1} - a(\|a\|)^{-1}\|_B =$$
$$= (\|a+b\|_B\|a\|_B)^{-1}\|(\|a\|_B - \|a+b\|_B)a + \|a\|_B b\|_B \leq 2\|b\|_B/\|a+b\|_B$$

■

Thesis follows from (3),(4) and (5).

3. Residual vector and error of one step of the iteration At each step k of the Power Method (1.2) we have to solve the linear system with matrix B:

$$(3.1). \qquad By = Ad_k$$

Let's consider now one step of iteration starting from vector d_k ($\|d_k\|_B = 1$). Let $y_a \neq 0$ be an aproximation of the solution y ($y = B^{-1}Ad_k$) such that the residual vector $r = By_a - Ad_k$ of the linear system (3.1) associated with y_a satisfies the inequality:

$$(3.2) \qquad \|r\|_2 \leq \delta_r$$

Let:

$$(3.3) \qquad d_{k+1} = y/\|y\|_B$$

$$(3.4) \qquad z_{k+1} = y_a/\|y_a\|_B$$

Define the error e_k at this step as:

$$(3.5) \qquad e_k \stackrel{\text{def}}{=} \|z_{k+1} - d_{k+1}\|_B$$

Hence e_k is the error of single step of the power method,result of not exact solution of system (3.1) and calculed in the norm $\|\ \|_B$. Following theorem answers the question how the value e_k depends of δ_r.

THEOREM 3.1

$$e_k \leq 2\left[\|B^{-1}\|_2\right]^{1/2}\|B\|_2\|A^{-1}\|_2\delta_r$$

Proof. Assumptions imply:

$$By - Ad_k = 0$$
$$By_a - Ad_k = r \qquad \text{and} \qquad \|r\|_2 \leq \delta_r$$

Hence:

$$B(y_a - y) = r$$
$$y_a - y = B^{-1}r$$

$$\|y_a - y\|_B = \|B^{-1}r\|_B = \|r\|_{B^{-1}}$$

Applaying theorem (1.1) to $\| \|_{B^{-1}}$ we get:

$$\|y_a - y\|_B \le \left[\|B^{-1}\|_2\right]^{1/2}\|r\|_2 \le \left[\|B^{-1}\|_2\right]^{1/2}\delta_r \tag{1}$$

From (1) and from general properties of vector norm:

$$\begin{aligned} e_k &= \|y_a/\|y_a\| - y/\|y\|_B\|_B = \\ &= (\|y\|_B\|y_a\|_B)^{-1}\|(y_a\|y\|_B - y_a\|y_a\|_B + y_a\|y_a\|_B - y\|y_a\|_B)\| \le \\ &\le (\|y\|_B\|y_a\|_B)^{-1}\|y_a\|_B(|\|y_a\| - \|y\|_B| + \|y_a - y\|_B) \le \\ &\le 2(\|y\|_B)^{-1}\|y - y_a\|_B \le 2(\|y\|_B)^{-1}\left[\|B^{-1}\|_2\right]^{1/2}\delta_r \end{aligned}$$

The last inequality imply:

$$e_k \le \frac{2\left[\|B^{-1}\|_2\right]^{1/2}\delta_r}{\|\|B^{-1}Ad_k\|_B} \tag{2}$$

Applaying theorem 1.1 to norm $\| \|_{B^{-1}}$ we get:

$$\|B^{-1}Ad_k\|_B = \|Ad_k\|_{B-1} \ge \|Ad_k\|_2/((\|B\|)_2)^{1/2} \tag{3}$$

$\|d_k\|_2 = \|A^{-1}Ad_k\|_2 \le \|A^{-1}\|_2\|Ad_k\|_2$, hence

$$\frac{1}{\|Ad_k\|_2} \le \frac{\|A^{-1}\|_2}{\|d_k\|_2} \tag{4}$$

Appplaying again theorem 1.1 we get:

$$\frac{1}{\|d_k\|_2} \le \frac{(\|B\|_2)^{1/2}}{\|d_k\|_B} = (\|B\|_2)^{1/2} \tag{5}$$

The thesis follows by (2),(3),(4),(5). ■

4. Accuracy of aproximation of eigenvector For lemma 4.1,4.2,4.3 will be applied following notation. Consider again one step of power method for matrix pencil (A,B) starting from vector d_k, $\|d_k\|_B = 1$.

Let d_{k+1}, z_{k+1}, e_k be defined by equalities (3.3),(3.4),(3,5) respectlively. ie.: d_{k+1} is an exact vector calculated by Power Method after one step, z_{k+1} is a vector calculated after one step of power method when the linear system is solved not exactly and e_k is an error of this iteration.

Let $\alpha_{i,j}$ for $i \in 1\ldots n$, $j \in k, k+1$ be such complex numbers that

$$d_j = \sum_{i=1}^{n}\alpha_{i,j}v_i = w_j + \sum_{i=r+1}^{n}\alpha_{i,j}v_i \tag{4.1}$$

where $\{v_i\}_{i-1}^{n}$ are the eigenvectors of pencil (A, B) orthormal with respect to $(\, , \,)_B$. Since $\|d_j\|_B = 1$ for $j = k, k+1$ we have:

$$\sum_{i=1}^{n} |\alpha_{i,j}|^2 = 1 \tag{4.2}$$

Assume that cooresponding eigenvalues satisfy relations (2.1). Hence the vector $w_k \in \mathbf{V}_{\lambda 1}$ and w_k is the orthogonal projection (with respect to $(\, , \,)_B$) of d_k on subspace $\mathbf{V}_{\lambda 1}$ i.e: $w_k = \sum_{i=1}^{r} (d, v)_B v_i$. Hence:

$$\|w_k - d_k\|_B = \min_{w \in \mathbf{V}_{\lambda 1}} \|w - d_k\|_B \tag{4.3}$$

Now define for $j = k, k+1$:

$$\gamma_j \stackrel{\text{def}}{=} \|w_j\|_B \quad \text{and if } \gamma_j \neq 0\text{:} \quad u_j \stackrel{\text{def}}{=} \frac{w_j}{\|w_j\|_B}.$$

LEMMA 4.1 *If $\gamma_j \neq 0$ then $u_j \in \mathbf{V}_{\lambda 1}$, $\|u_j\|_B = 1$ and:*

$$\|u_j - d_j\|_B = \min_{\substack{w \in \mathbf{V}_{\lambda 1} \\ \|w\|_B = 1}} \|w - d_j\|_B$$

Proof. Relation (4.2),definition of γ_j and orthonormality of eige- nvectors v_i imply:

$$
\begin{aligned}
(1) \qquad (\|d_j - u_j\|_B)^2 &= (x_j - u_j, d_j - u_j)_B = \\
&= \sum_{i=1}^{r} |\alpha_{i,j} - \alpha_{i,j}(\gamma_j)^{-1}|^2 + \sum_{i=r+1}^{n} |\alpha_{i,j}|^2 = \\
&= [1 - (\gamma_j)^{-1}]^2 [\sum_{i=1}^{n} |\alpha_{i,j}|^2] + 1 - (\gamma_j)^2 = \\
&= [1 - (\gamma_j)^{-1}]^2 (\gamma_j)^2 + 1 - (\gamma_j)^2 = 2(1 - \gamma_j)
\end{aligned}
$$

Let $w \in \mathbf{V}_{\lambda 1}$ be an arbitrary vector such that $\|w\|_B = 1$. Put $w = \sum_{i=1}^{r} s_i v_i$ $(s_i \in C)$.Since v_i are orthonormal we have:

$$\sum_{i=1}^{r} |s_i|^2 = 1$$

Applaing Schwarz inequality we get:

$$
\begin{aligned}
(2) \quad (\|w - d_j\|_B)^2 &= \sum_{i=1}^{r} |\alpha_{i,j} - s_i|^2 + \sum_{i=r+1}^{n} |\alpha_{i,j}|^2 \geq \\
&\geq \sum_{i=1}^{r} (|\alpha_{i,j}| - |s_i|)^2 + 1 - (\gamma_j)^2 =
\end{aligned}
$$

$$= \sum_{i=1}^{r} |s_i|^2 + \sum_{i=1}^{r} |\alpha_{i,j}|^2 - 2(\sum_{i=1}^{r} |s_i||\alpha_{i,j}|) + 1 - (\gamma_j)^2 =$$

$$= 1 + (\gamma_j)^2 - 2(\sum_{i=1}^{r} |s_i||\alpha_{i,j}|) + 1 - (\gamma_j)^2 = 2(1 - (\sum_{i=1}^{r} |s_i||\alpha_{i,j}|)) \geq$$

$$\geq 2(1 - [\sum_{i=1}^{r} |s_i|^2]^{1/2} [\sum_{i=1}^{r} |\alpha_{i,j}|^2]^{1/2}) = 2(1 - \gamma_j)$$

Now thesis follows by (1) and (2). ∎

With the same notation we have for $j = k, k+1$: $\gamma_j \left[\sum_{i=1}^{r} |\alpha_{i,j}|^2\right]^{1/2}$, hence: $0 \leq \gamma_j \leq 1$ and $\gamma_j = 1$ if and only if $d_j \in \mathbf{V}_{\lambda 1}$.

From definition (1.2) of power method, γ_{k+1} depends on γ_k as follows:

$$\gamma_{k+1} = \frac{\gamma_k}{\left[(\gamma_k)^2 + \sum_{i=r+1}^{n} |\alpha_{i,k}|^2 |\lambda_i|^2 |\lambda_1|^{-2}\right]^{1/2}} \tag{4.4}$$

LEMMA 4.2 *If* $\gamma_k > 0$ *then:*

$$(\|d_{k+1} - u_{k+1}\|_B)^2 \leq (\|d_k - u_k\|_B)^2 - 2\gamma_k \left[\frac{1}{[(\gamma_k)^2 - (\xi\gamma_k)^2 + (\xi)^2]^{1/2}} - 1\right]$$

Proof. As in proof in lemma (4.1) we calculate:

$$(\|d_k - u_k\|_B)^2 = 2(1 - \gamma_k),$$

$$(\|d_{k+1} - u_{k+1}\|_B)^2 = 2(1 - \gamma_{k+1}).$$

Hence and from (4.4) follows that:

$$(\|d_k - u_k\|_B)^2 - (\|d_{k+1} - u_{k+1}\|_B)^2 = 2(\gamma_{k+1} - \gamma_k) =$$

$$2\gamma_k \left[\frac{1}{\left[(\gamma_k)^2 + \sum_{i=r+1}^{n} |\alpha_{i,k}|^2 |\lambda_i|^2 |\lambda_1|^{-2}\right]^{1/2}} - 1\right] \geq$$

$$2\gamma_k \left[\frac{1}{\left[(\gamma_k)^2 + (\sum_{i=r+1}^{n} |\alpha_{i,k}|^2)(\xi)^2\right]^{1/2}} - 1\right] \geq$$

$$2\gamma_k \left[\frac{1}{\left[(\gamma_k)^2 + (1 - (\gamma_k)^2)(\xi)^2\right]^{1/2}} - 1\right]$$

This inequality is equivalent to the inequality in thesis of lemma. ∎

Remark. The inequality in lemma 4.2 is equivalent to the following one:

$$\min_{\substack{w \in \mathbf{V}_{\lambda 1} \\ \|w\|_B = 1}} \|w - d_k\|_B \geq \min_{w \in \mathbf{V}_{\lambda 1}} \|w - d_{k+1}\|_B$$

Both terms are equal if and only if $\gamma_k = 1$ i.e $d_k \in \mathbf{V}_{\lambda 1}$.

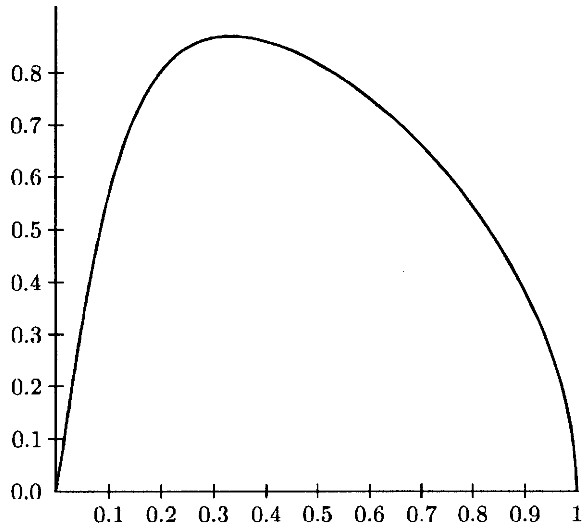
Let $f : [0,1] \to R$ be a function defined as follows:

$$f(\gamma) \stackrel{\mathrm{def}}{=} (2)^{1/2}\left[(1-\gamma)^{1/2} - \left[1 - \frac{\gamma}{\left[(\gamma)^2 - (\xi\gamma)^2 + (\xi)^2\right]^{1/2}}\right]\right]^{1/2}$$

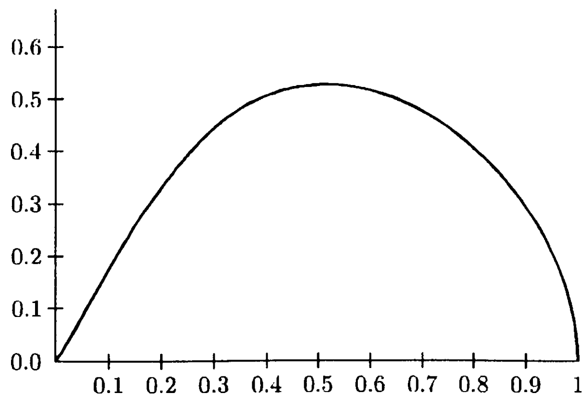
where $\xi \in [0,1)$ is the coeficient of convergence of matrix pencil (A, B).

This function for any $\xi \in [0,1)$ has the following properties: — f is nonnegative and takes the value 0 only for $\gamma = 0$ or $\gamma = 1$. — f is concave.

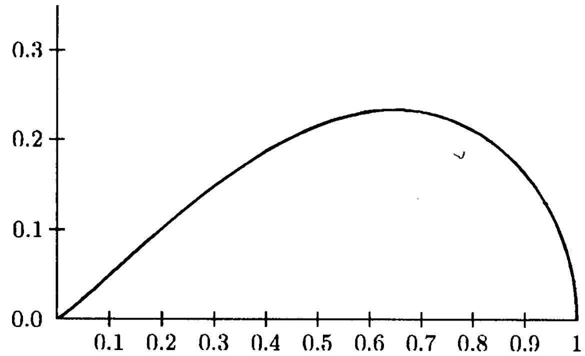
Graphs of $f(\gamma)$ for various values of parameters ξ are given below.



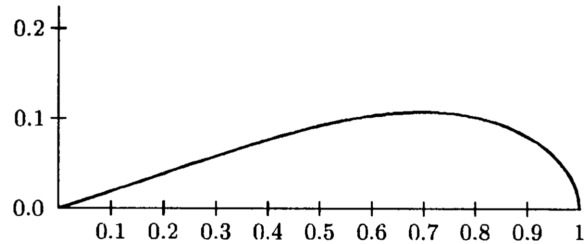
a) $\xi = 0.1$



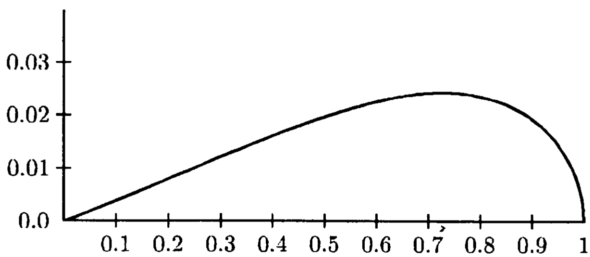
b) $\xi = 0.3$



c) $\xi = 0.6$



d) $\xi = 0.8$



e) $\xi = 0.95$

LEMMA 4.3

i) If $e_k \leq f(\gamma_k)$ *then:* $\|z_{k+1} - u_{k+1}\|_B \leq \|d_k - u_k\|_B$. *In other words if in this iteration we calculated not exact vector* d_{k+1} *but some vector* z_{k+1} *such that:* $\|z_{k+1}\|_B = 1$ *and* $\|z_{k+1} - d_{k+1}\|_B \leq f(\gamma_k)$ *then* z_{k+1} *is not worse aproximation of eigenvector of pencil* (A, B) *than* d_k, *in the following sense:*

$$\min_{\substack{w \in \mathbf{V}_{\lambda 1}, \\ \|w\|_B = 1}} \|w - d_k\|_B \geq \min_{\substack{w \in \mathbf{V}_{\lambda 1}, \\ \|w\|_B \cong 1}} \|w - z_{k+1}\|_B.$$

ii) Let σ be real number and $0 \le \sigma \le 1$. If $e_k \le \sigma f(\gamma_k)$ then: $\|z_{k+1} - u_{k+1}\|_B \le \|d_k - u_k\|_B - (1-\sigma) f(\gamma_k)$.

Proof.

(1) $\|z_{k+1} - u_{k+1}\|_B = \|z_{k+1} - d_{k+1} + d_{k+1} - u_{k+1}\|_B \le e_k + \|d_{k+1} - u_{k+1}\|_B$

From (1) and from lemma 4.2 follows that:

(2)

$$\|z_{k+1} - u_{k+1}\|_B \le e_k + \left[(\|d_k - u_k\|_B)^2 - 2\gamma_k \left[\frac{1}{[(\gamma)^2 - (\xi\gamma)^2 + (\xi)^2]^{1/2}} - 1\right]\right]^{1/2}$$

Consider the inequality:

(3)

$$e_k + \left[(\|d_k - u_k\|_B)^2 - 2\gamma_k \left[\frac{1}{[(\gamma)^2 - (\xi\gamma)^2 + (\xi)^2]^{1/2}} - 1\right]\right]^{1/2} \le \|d_k - u_k\|_B$$

Observe that $\|d_k - u_k\|_B = [2(1-\gamma_k)]^{1/2}$. After algebraic reduction, inequality (3) turns to be equvalent to following one:

(4) $$e_k \le f(\gamma_k)$$

So if (4) is true then (3) is true and because (3) implies that: $\|z_{k+1} - u_{k+1}\|_B \le \|d_k - u_k\|_B$ then (i) is proved. Implication (ii) follows simply from (3). and assumption about e_k in (ii). ∎

Now let $\{z_k\}$ be the sequence of vectors generated by Power Method (1.2) when in each step of iteration system $By = Ax$ is solved aproximately (i.e in every step z_k being normalized approximate solution of this system). For every k $\|z_k\|_B = 1$. e_k — is the error at single k-th step of iteration calculated in norm $\|\ \|_B$ i.e: if d_{k+1} is the exact vector calculated after one step of Power Method starting from z_k then: $e_k = \|z_{k+1} - d_{k+1}\|_B$. Let:

$$z_k = \sum_{i=1}^{n} \beta_{i,k} v_i, \qquad \forall i,k \quad \beta_{i,k} \in C,$$

$$\rho_k = \sum_{i=1}^{r} \left[|\beta_{i,k}|^2\right]^{1/2}.$$

THEOREM 4.1 *Let $\rho_0 > 0$; $0 < \varepsilon < (2)^{1/2}$*
Put:

$$\delta_1 \le \delta \stackrel{\text{def}}{=} 0.5 \min\left[f(\rho_0), f\left(1 - \frac{(\varepsilon)^2}{2}\right)\right]$$

$$l \in N : l \stackrel{\text{def}}{=} \mathrm{Ent}\left[(\|z_0 - u_0\|_B - \varepsilon)(\delta)^{-1}\right] + 1$$

If for every k $e_k \le \delta_1$ then:

$\exists m \in N, m \le l, \exists w \in \mathrm{V}_{\lambda 1}, \|w\|_B = 1$ *such that:* $\|z_m - w\|_B \le \varepsilon$.

If δ_1, ε *are so small that* $1 - 0.5(\varepsilon + \delta_1)^2 \geq \gamma_0$ *then for each* $k \geq m$:

$$\exists w \in \mathbf{V}_{\lambda 1}, \|w\|_B = 1 \quad \textit{such that:} \quad \|z_k - w\|_B \leq \varepsilon + \delta_1$$

Proof. Define the vectors u_k: $u_k \stackrel{\text{def}}{=} (\rho_k)^{-1}\left[\sum_{i=1}^n \beta_i v_i\right]$ i.e: u_k is normalized orthogonal projection (relatively $(\ ,\ )_B$) $of z_k$ on space $\mathbf{V}_{\lambda 1}$. Hence lemma 4.1 implies:

$$\forall k \quad u_k \in \mathbf{V}_{\lambda 1}, \quad \|u_k\|_B = 1 \text{ and}$$
$$\|u_k - z_k\|_B = \min_{\substack{w \in \mathbf{V}_{\lambda 1} \\ \|w\|_B = 1}} \|w - z_k\|_B$$

It follows from relation:

(1) $$\|z_k - u_k\|_B = [2(1 - \rho_k)]^{1/2}$$

that the inequalities:

(2) $$\|z_k - u_k\|_B \leq \varepsilon$$

(3). $$\rho_k \geq 1 - 0.5(\varepsilon)^2$$

are equivalent for each $\varepsilon : 0 \leq \varepsilon \leq 2^{1/2}$.

Hence for every $k \in N$:

(4) $$\text{if } \|z_k - u_k\|_B > \varepsilon \text{ then } \rho_k < 1 - 0.5(\varepsilon)^2$$

Now let's observe that:

If for $i \in \{0, 1 \ldots k - 1\}$: $\|z_i - u_i\|_B > \varepsilon$ then:

(5) $$\|z_k - u_k\|_B \leq \|z_0 - u_0\|_B - k\delta \text{ and } \rho_0 < \rho_1 < \ldots < \rho_{k-1} < \rho_k.$$

This implication follows from assumption about e_k and from lemma 4.3 (ii) with $\sigma = 0.5$.

From properties of function f and definition of δ we get:

(6) $$\delta \leq 0.5 f(\rho) \quad \text{for each} \quad \rho : \rho_0 \leq \rho \leq 1 - 0.5(\varepsilon)^2.$$

Now let's procede by induction to prove implication (5). for $k = 1$:

If $\|z_0 - u_0\|_B > \varepsilon$ then from (4) : $\rho_0 \leq 1 - 0.5(\varepsilon)^2$.

Hence according with (6): $e_1 \leq \delta \leq 0.5 f(\rho)$.

From (ii) in lemma (4.3):

$$\|z_1 - u_1\|_B \leq \|z_0 - u_0\|_B - 0.5 f(\rho_0) \leq \|z_0 - u_0\|_B - \delta$$

From (1) follows that: $\rho_0 < \rho_1$.

Assume that implication (5) is true for $i \leq k - 1$ and and let's proof it for $i = k$.

From assumption of induction we have:

$$\|z_{k-1} - u_{k-1}\|_B > \varepsilon; \qquad \|z_{k-1} - u_{k-1}\|_B \leq \|z_0 - u_0\|_B - (k-1)\delta;$$
$$\rho_0 < \rho_1 < \ldots < \rho_{k-1}$$

So: $\rho_0 \leq \rho_{k-1} \leq 1 - 0.5(\varepsilon)^2$ and hence $e_k \leq \delta \leq 0.5 f(\rho_{k-1}) \ldots$

Applying again lemma 4.3 (ii) we get:

$$\|z_k - u_k\|_B \le \|z_{k-1} - u_{k-1}\|_B - 0.5f(\rho_{k-1}) \le \|z_0 - u_0\|_B - k\delta.$$

Implication (5) is proved by induction.

The above says that until we do not calculate such good vector z_m that $\|z_m - u_m\|_B \le \varepsilon$ our procedure produces always better approximation at next step then it was beafore,the error being diminished at least by δ.

Relation (5) implies that exists a natural number m ($m \le l$) such that:

$$\|z_m - u_m\|_B \le \varepsilon.$$

For $k = m + 1$ since $\rho_k \ge 1 - 0.5(\varepsilon)^2$ relation (6) is not true and z_{m+1} may be worse approximation of eigenvector than z_m.

Let $s \ge m + 1$ be the first natural number such that:

$$\|z_{s-1} - u_{s-1}\|_B \le \varepsilon \ \text{ and } \ \|z_s - u_s\|_B > \varepsilon \tag{7}$$

Denote as d_s the exact vector calculated from z_{s-1} after one step of Power Method (1.2).

Let $w_s \in \mathbf{V}_{\lambda 1}$ be such vector that $\|w_s\|_B = 1$ and $\|w_s - d_s\|_B = \min_{\substack{w \in \mathbf{V}_{\lambda 1} \\ \|w\|_B = 1}} \|w - d_s\|_B$ i.e:

$$w_s = \frac{\sum_{i=1}^{r} (d_s, v_i)_B v_i}{\|\sum_{i=1}^{r} (d_s, v_i)_B v_i\|_B}$$

Because $\|d_s - w_s\|_B \le \|z_{s-1} - u_{s-1}\|_B \le \varepsilon$ (exact calculation gives always not worse aproximation of eigenvector after one step — see lemma 4.2) we have:

$$\|z_s - w_s\|_B \le \|z_s - d_s\|_B + \|d_s - w_s\|_B \le \delta_1 + \varepsilon$$

and:

$$\|z_s - u_s\|_B \le \|z_s - w_s\|_B \le \delta_1 + \varepsilon.$$

This inequality and (7) implies:

$$1 - 0.5(\varepsilon)^2 \ge \rho_s \ge 1 - 0.5(\varepsilon + \delta_1)^2 \ge \rho_0$$

Hence the lemma 4.3 may be applied again because according (6): $e_s \le \delta \le 0.5f(\rho_s)$. We have then:

$$\|z_{s+1} - u_{s+1}\|_B \le \|z_s - u_s\|_B - 0.5f(\rho_s) \le \|z_s - u_s\|_B - \delta \le \delta + \varepsilon - \delta = \varepsilon$$

Hence for every $k \ge m$:

$$\|z_k - u_k\|_B \le \varepsilon + \delta_1$$

(in fact in the worse case upper bound of $\|z_k - u_k\|_B$ may oscilate arternately between ε and $\varepsilon + \delta_1$). ■

References

[1] B. N. Parlett, The symetric eigenvalue problem.