ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria III: MATEMATYKA STOSOWANA XXXX (1997)

HENRYK BRZESKWINIEWICZ AND KRZYSZTOF KŁACZYŃSKI

Poznań

Consistency examination of linear inequality system

(*Praca wpłynęła do Redakcji 24.07.1997*)

Summary. An investigation of the consistency of linear inequality system is considered. It has been proven that the system of linear inequalities $\boldsymbol{Ax} \geq \boldsymbol{b}$ is consistent if and only if for any generalized inverse $\boldsymbol{A}^{-}$ of matrix $\boldsymbol{A}$ the system of equations $(\boldsymbol{I} - \boldsymbol{AA}^{-})\boldsymbol{v} = -(\boldsymbol{I} - \boldsymbol{AA}^{-})\boldsymbol{b}$ has a nonnegative solution due to vector $\boldsymbol{v}$. Consistency of the above system does not depend on the choice of matrix $\boldsymbol{A}^{-}$.

The paper presents also methods of the examination in the existence of nonnegative solutions of linear equation system.

1. Introduction and preliminaries. Let us consider a system of inequalities

$$\boldsymbol{Ax} \geq \boldsymbol{b}, \tag{1.1}$$

where $\boldsymbol{A}$ is a $m \times p$ matrix, $\boldsymbol{b}$ is a $m \times 1$ vector, and the notation $\boldsymbol{Ax} \geq \boldsymbol{b}$ denotes the fact that vector $\boldsymbol{Ax} - \boldsymbol{b}$ has all nonnegative elements. System of (1.1) form is encountered in many practical problems. One of them is the estimation of parameters in the fixed linear statistical model, in which system $\boldsymbol{Ax} \geq \boldsymbol{b}$ presents an additional nonstochastic information about vector of unknown parameters. Such model finds application both in econometric problems (see. e.g. Ito, 1980; Schmidt, 1981) and in biometric problems (see e.g. Armstrong and Frome, 1976). An analysis of this model is presented by Judge and Takayama (1966), Lowell and Prescott (1970), Levis and Odell (1971), Liew (1976a and 1976b), Escobar and Skarpenees (1984), Werner (1990) and Kłaczyński (1994). The system of linear inequalities in the form of (1.1) is sometimes added to a mixed statistical linear model (see Sealy,

---

Key words and phrases: consistency of linear inequality system, nonnegative (semipositive) solutions of linear equation systems, linear models, generalized inverse of matrix.

1971), in which system (1.1) can indicate, as in Brown (1978) an additional nonstochastic information about the vector of variance components.

System of inequality (1.1) occurs also in linear programming problems as a system of linear constraints (see Bazaraa and Shetty, 1979; Chap. I and II).

In this paper for the research of inconsistency system of linear inequalities and their solution the notion of generalized inverse of matrix is used. It will give an idea of linear inequalities from the matrix point of view. Furthermore, in case where the generalized inverse of matrix is known (it frequently occurs in linear models) it may have a practical importance.

The consistency of the linear inequality system (1.1) will be solved on the basis of the existence or nonexistence of nonnegative solutions of the corresponding linear equation system. Hence, it contains some information about the linear equations, mainly in the context of the existence of their nonnegative solutions.

2. Results. Let $\boldsymbol{A}^-$ denote a generalized inverse of matrix $\boldsymbol{A}$, i.e. any matrix satisfying condition $\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A}$.

First, we consider a system of equations

$$\boldsymbol{R}\boldsymbol{x} = \boldsymbol{s}, \tag{2.1}$$

where $\boldsymbol{R}$ is $q \times p$ matrix and $\boldsymbol{s}$ is $q \times 1$ vector. In linear - equation theory well known is

LEMMA 1. (a) *The equation system* (2.1) *is consistent if and only if there occurs one of the following conditions*:

(i) $r(\boldsymbol{R}.\boldsymbol{s}) = r(\boldsymbol{R})$ (*see known Kronecker-Capelly Theorem*),
(ii) $\boldsymbol{s} \in \mathcal{R}(\boldsymbol{R})$ *or, equivalently*,
(iii) $\boldsymbol{R}\boldsymbol{R}^-\boldsymbol{s} = \boldsymbol{s}$, *for some* $\boldsymbol{R}^-$ (*see Rao and Mitra*, 1971, *p.* 23),

where $\mathcal{R}(\cdot)$ *and* $r(\cdot)$ *denotes the column space and rank of a matrix argument, respectively.*

(b) *Consistent system* (2.1) *has a general solution in the form*

$$\boldsymbol{x} = \boldsymbol{R}^-\boldsymbol{s} + (\boldsymbol{I} - \boldsymbol{R}^-\boldsymbol{R})\boldsymbol{v}, \tag{2.2}$$

where $\boldsymbol{v}$ *is any vector from space* $\mathcal{R}^p$ (*see Rao and Mitra*, 1971, *p.* 23).

Considering the consistency of system $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$, defined in (1.1), we start with a simple statement, which is a direct consequence of linear-equation theory.

LEMMA 2. *The following conditions are equivalent*:

(i) $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$ *for some vector* $\boldsymbol{x}$,
(ii) $\exists \boldsymbol{v} \geq \boldsymbol{0} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \boldsymbol{v}$ *for some vector* $\boldsymbol{x}$,

(iii) $\exists \boldsymbol{v} \geq \boldsymbol{0} : \boldsymbol{b} + \boldsymbol{v} \in \mathcal{R}(\boldsymbol{A})$ (*see* Lemma 1(a) (ii)),
(iv) $\exists \boldsymbol{v} \geq \boldsymbol{0} : r(\boldsymbol{A}, \boldsymbol{b} + \boldsymbol{v}) = r(\boldsymbol{A})$ (*see* Lemma 1(a) (i)),
(v) $\exists \boldsymbol{v} \geq \boldsymbol{0} : \boldsymbol{A}\boldsymbol{A}^{-}(\boldsymbol{b} + \boldsymbol{v}) = \boldsymbol{b} + \boldsymbol{v}$ *for some* $\boldsymbol{A}^{-}$ (*see* Lemma 1(a) (iii)).

As conclusions to the Lemma, let us formulate the sufficient conditions for consistency of system (1.1).

Lemma 2 (iii) with $\boldsymbol{v} = \boldsymbol{0}$ implies that if

$$\boldsymbol{b} \in \mathcal{R}(\mathbf{A}), \tag{2.2}$$

then system (1.1) is consistent.

Note relation (2.2) is not a necessary condition of the consistency of system (1.1).

For example the system of inequalities

$$\begin{aligned} -x_1 - x_2 &\geq -4 \\ 6x_1 + 2x_2 &\geq 8 \\ x_1 + 5x_2 &\geq 8 \end{aligned}$$

is consistent (e.g. vector $\boldsymbol{x} = (1, 2)'$ is its solution), but condition (2.2) is not satisfied.

It is easily visible that relation (2.2) is particularly satisfied if matrix $\boldsymbol{A}$ is of a full row rank (i.e. if $r(\boldsymbol{A}) = m$).

Now we are going to show that about the consistency of system (1.1) decides the existence of nonnegative solution of some linear equation system.

THEOREM 2.1. *The system of inequalities* (1.1) *is consistent if and only if the system of equations*

$$(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{-})\boldsymbol{v} = -(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{-})\boldsymbol{b} \tag{2.3}$$

has a nonnegative solution due to vector $\boldsymbol{v}$ *for any generalized inverse* $\boldsymbol{A}^{-}$ *of matrix* $\boldsymbol{A}$. *Consistency of system* (2.3) *does not depend on the choice of matrix* $\boldsymbol{A}^{-}$.

Proof. From Lemma 2(v) we have $(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{-})(\boldsymbol{b} + \boldsymbol{v}) = \boldsymbol{0}$, which leads to (2.3).

To complete the proof consider two following systems of equations

$$(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}_1^{-})\boldsymbol{v} = -(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}_1^{-})\boldsymbol{b} \tag{2.4}$$

$$(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}_2^{-})\boldsymbol{v} = -(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}_2^{-})\boldsymbol{b}, \tag{2.5}$$

where $\boldsymbol{A}_1^{-}$ and $\boldsymbol{A}_2^{-}$ are two different generalized inverses of matrix $\boldsymbol{A}$. Since $(\boldsymbol{I} - \boldsymbol{A}_i\boldsymbol{A}_i^{-})(\boldsymbol{I} - \boldsymbol{A}_i\boldsymbol{A}_i^{-}) = \boldsymbol{I} - \boldsymbol{A}_i\boldsymbol{A}_i^{-}$, $i = 1, 2$, it follows from Lemma 1(b) that for any vector $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ from space $\mathcal{R}^p$

$$\boldsymbol{v}_1 = -(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}_1^{-})\boldsymbol{b} + \boldsymbol{A}\boldsymbol{A}_1^{-}\boldsymbol{z}_1 \quad \text{and}$$

$$\boldsymbol{v}_2 = -(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}_2)\boldsymbol{b} + \boldsymbol{A}\boldsymbol{A}_2^{-}\boldsymbol{z}_2$$

are general solutions of (2.4) and (2.5), respectively. If we assume that $\boldsymbol{v}_1 \geq \mathbf{0}$ then substituting $\boldsymbol{z}_2 = \boldsymbol{A}\boldsymbol{A}_1^- \boldsymbol{z}_1 - \boldsymbol{A}\boldsymbol{A}_2^- \boldsymbol{b} + \boldsymbol{A}\boldsymbol{A}_1^- \boldsymbol{b}$ we get $\boldsymbol{v}_2 = \boldsymbol{v}_1 \geq \mathbf{0}$, which completes the proof.

The Theorem 2.1 and Lemma 1(b) yields

COROLLARY 2.1. *Consistent system* (1.1) *has a general solution in the form*

$$\boldsymbol{x}_0 = \boldsymbol{A}^-(\boldsymbol{b} + \boldsymbol{v}_0) + (\boldsymbol{I} - \boldsymbol{A}^-\boldsymbol{A})\boldsymbol{z}, \tag{2.6}$$

where $\boldsymbol{v}_0$ *is nonnegative solution of* (2.3) *and* $\boldsymbol{z}$ *is any vector from space* $\mathcal{R}^p$.

Proof. Substituting (2.6) into (1.1) we obtain

$$\boldsymbol{A}\boldsymbol{x}_0 = \boldsymbol{A}\boldsymbol{A}^-(\boldsymbol{b} + \boldsymbol{v}_0) + \boldsymbol{A}(\boldsymbol{I} - \boldsymbol{A}^-\boldsymbol{A})\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{v}_0 \geq \boldsymbol{b},$$

what completes the proof.

Note, that in practical situations nonnegative solution of $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{s}$ (or of (2.3)) can be obtained basing on the solution of the following linear programming problem:

$$\text{minimization } \mathbf{1}'\boldsymbol{u} \tag{2.7.}$$

subject to the constraints

$$\boldsymbol{R}\boldsymbol{x} + \boldsymbol{u} = \boldsymbol{s} \quad \boldsymbol{x} \geq \mathbf{0} \quad \text{and} \quad \boldsymbol{u} \geq \mathbf{0}. \tag{2.8}$$

It is known (see Gale, 1960, Chap. IV.5 and Bazaraa and Shetty, 1979, p. 70) that nonnegative solution of $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{s}$ exists if and only if the optimal solution of the problem (2.7)–(2.8) is attained at the point, in which the minimized function reaches the value zero.

References

Armstrong, R. D. and Frome, E. L. (1976): *A branch-and-bound solution of a restricted least squares problem*, Technometrics 18, 447–450.

Brown, K. G. (1978): *Estimation of variance components using residuals*, JASA 73, 141–146.

Bazaraa, M. S. and Shetty, C. M. (1979): *Nonlinear Programming. Theory and Algorithms*, John Wiley & Sons., New York.

Escobar, L. A. and Skarpenees, B. (1984): *A closed form solution for the least squares regression problem with linear inequality constraints*, Commun. Statist. - Theor. Meth. 13, 1127–1134.

Gale, D. (1960): *The theory of linear economic models*, McGraw-Hill Book Company, Inc. New York.

Ito, T. (1980): *Methods of estimation for multi-market disequilrium models*, Econometrica 48, 97–125.

Judge, G. G. and Takayama, T. (1966): *Inequality restrictions in regression analysis*, JASA 61, 166–181.

Kłaczyński, K. (1994): *On estimation of parameter functions in a weakly singular linear model with linear inequality restrictions* (in Polish), Matematyka Stosowana 37, 49–66.

Lewis, T. O. and Odell, P.L. (1971): *Estimation in Linear Models*, Prentice Hall, Inc. Englewood Cliffs, New Jersey.

Liew, Ch. K. (1976 a): *Inequality constrained least-squares estimation*, JASA 71, 746–751.

Liew, Ch. K. (1976 b): *A two-stage least-squares estimation with inequality restrictions on parameters*, The Review of Economics and Statistics LVIII No. 2, 234–238.

Lowell, M. C. and Prescott, E. (1970): *Multiple regression with inequality constraints: Pretesting bias, hypothesis testing and efficiency*, JASA 65, 913–925.

Rao, C. R. and Mitra, K. M. (1971): *Generalized Inverse of Matrix and its Applications*, Wiley, New York.

Schmidt, P. (1981): *Constraints on the parameters in simultaneous tobit and probit models*, in Structural Analysis of Discrete Data with Econometric Applications (C.F. Manski et al., Eds.), MIT Press, Mas., Cambridge 1981.

Sealy, J. (1971): *Linear spaces and unbiased estimation application to the mixed linear model*, The Annals of Mathematical Methods 41, 1735–1748.

Theil, H. (1971): *Principles of Econometrics*, Wiley, New York.

Waterman, M.S. (1974): *A restricted least squares problem*, Technometrics 16, 135–136.

Werner, H. J. (1990): *On Inequality Constrained Generalized Least Squares Estimation*, Linear Algebra Appl. 127, 379–392.

DEPARTMENT OF MATHEMATICAL AND STATISTICAL METHODS
AGRICULTURAL UNIVERSITY OF POZNAŃ
UL. WOJSKA POLSKIEGO 28
POLAND