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An apartment problem

Abstract In the 60 - ies of the last century, several optimization problems referring to the sequential methods were investigated. These tasks may include the Robbins' problem of optimal stopping, the secretary problem (see the discussion paper by Ferguson [18]), the parking problem or the job search problem. Subtle details of the wording in these issues cause that each of these terms include family of problems that differ significantly in detail. These issues focused attention of a large group of mathematicians. One of the related topic has been the subject of Professor Jerzy Zabczyk attention. Based on the discussions with Professor Richard Cowan[1] the model of choosing the best facility available from a random number of offers was established. In contemporary classification of the best choice problems it is the no-information, continuous time, secretary problem with the Poisson stream of options and the finite horizon.

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1. Preliminaries In the problem of apartments selection, mentioned in the title of this note, the objective is to choose the best apartment at a sequence, based only on the current rank and at the instant of its evaluation, when the prospects of apartments arrive in a random order and they are evaluated immediately. The successive arrival times form a Poisson process on $[0, T]$ with the known intensity λ (and T is also known). It makes the number of candidates random with the Poisson distribution, similarly as in the work by Presman and Sonin [29], but the problem assuming that the arrivals are at the jumps of continuous time Poisson process is different because, if k candidates have arrived by time t, the distribution of the number of additional candidates depends on the remaining time $T - t$ and this parameter was not included in [29]. These research considerations were performed at the beginning of seventies of the twentieth century. It was only ten years after the solution of the basic best candidate selection problem by Lindley [24]. For

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[1] In Warsaw during the 1982 International Congress of Mathematicians which was held in August 1983.

the better understanding the role of the Cowan and Zabczyk works [14, 15] the main results in the area with methods of their solution are presented first. At the end the research stimulated by and the models referring to an apartment problem of [14, 15] are discussed.

1.1. Five key models

1.1.1. No information best choice problem A known number, let us say n, of different items is presented one by one in a random order (see [19] for the review of early results in the topic). All possible orders are equally likely. The observer is able at any time to rank the items that have so far been presented according their attractiveness. As each item is presented he must either accept it, in which case the process stops, or reject it, when the next item in the sequence is presented and the observer faces the same choice as before. If the last item is presented it must be accepted. The observer's aim is to maximize the probability that the item he chooses is, in fact, the best of the n items available. The first published solution of this problem was given by Lindley [24]. He defined the state of the search process at any time by two numbers (r, s), where r is the number of items presented so far and s the current rank of the r-th, the last presented item. Letting $V(r, s)$ denote the maximum expected probability of choosing the best item when the state of the process is (r, s), the principle of dynamic programming yields the solvable equations.

A more rigorous formulation one can find in Dynkin & Yushkevich [16]. With a sequential observation of the applicants we connect some natural probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The elementary events are the permutation of all applicants and the probability measure $\mathbf{P}$ is the uniform distribution on Ω. The observable sequence of the relative ranks Y_k, $k = 1, 2, \ldots, N$, defines the sequence of the σ-fields $\mathcal{F}_k = \sigma\{Y_1, \ldots, Y_k\}$, $k = 1, 2, \ldots, N$ (further the convention is adopted for the sequence $X_1, \ldots, X_n = \overrightarrow{X_{1,n}}$. The random variables Y_k are independent and $P(Y_k = i) = 1/k$. Denote $\mathcal{S}^N$ the set of all Markov times τ with respect to the σ-fields $\{\mathcal{F}_k\}_{k=1}^N$ bounded by N. Now, the problem can be formulated as follows: we are searching $\tau^* \in \mathcal{S}^N$ that

$$\mathbf{P}\{Z_{\tau^*} = 1\} = \sup_{\tau \in \mathcal{S}^N} \mathbf{P}\{Z_\tau = 1\}.$$

The problem can be reduced to the optimal stopping problem for a homogeneous Markov chain with suitable payoff functions.

Let $W_1 = 1$. Define $W_t = \inf\{r > W_{t-1} : Y_r = 1\}$, $t > 1$, $(\inf \emptyset = \infty)$. $(W_t, \mathcal{F}_t, \mathbf{P}_{(1,1)})_{t=1}^N$ is the homogeneous Markov chain with the state space $\mathbb{E} = \{1, 2, ..., N\} \cup \{\infty\}$, $\mathcal{F}_t = \sigma(W_1, W_2, ..., W_t)$ and the following one-step transition probabilities: $p(r, s) = \mathbf{P}\{W_{t+1} = s \mid W_t = r\} = \frac{r}{s(s-1)}$ if $1 \le r < s \le N$, $p(r, \infty) = 1 - \sum_{s=r+1}^N p(r, s)$, $p(\infty, \infty) = 1$ and 0 otherwise. The payoff function for the problem defined on $\mathbb{E}$ has a form $f(r) = \frac{r}{N}$.

Let $\mathcal{T}^N = \{\tau \in \mathcal{S}^N : \tau = r \Rightarrow Y_r = 1\}$. It is the set of stopping times with respect to $\mathcal{F}_t, t = 1, 2, ...$ We have $\mathbf{P}\{Z_{\tau^*} = 1\} = \sup_{\sigma\in\mathcal{T}^N} \mathbf{E}_1 f(W_\sigma)$ and let σ^* be such that $\mathbf{P}\{Z_{\tau^*} = 1\} = \mathbf{E}_1 f(W_{\sigma^*})$. Denote $\tilde{c}(r) = \sup_{\tau>r} \mathbf{P}\{Z_\tau = 1\}$. We have

$$\tilde{c}(r) = \begin{cases} c_1(r) & \text{if } r_a \le r \le N \\ c_1(r_a) & \text{if } 1 \le r < r_a, \end{cases}$$

where $c_1(r) = \frac{r}{N}\sum_{s=r+1}^{N} \frac{1}{s-1}$, $r = 1, 2, \ldots, N$ and $r_a = \inf\{1 \le r \le N : \sum_{s=r+1}^{N} \frac{1}{s-1} \le 1\}$. When $N \to \infty$ such that $\frac{r}{N} \to x$ we obtain $\hat{c}_1(x) = \lim_{N\to\infty} c_1(r) = -x \ln x$, $a = \lim_{N\to\infty} \frac{r_a}{N} = e^{-1} \cong 0.3679$ (cf. Shiryaev [45], Freeman [19], Rose [34]).

A deeper analysis of the presented key problem in the field and its solution led to the basic question on the origin and nature of the compared values of the studied objects. The mathematical model should take this into account. The discussion in this direction one can find in Ferguson's paper [18]. In no information case it is assumed that the assigned labels are different and the decision maker has no idea about them before investigation. Just like the guest which has came for the first time to the city and is looking for a place to live.

1.1.2. The full information secretary problem In contrast to the model described in the section 1.1.1 let us consider the case where the statistician's knowledge about the evaluated properties is much wider. The origins of this secretary problem version lie with Cayley [10], where values are observed sequentially. Their values are a realization of an independent random variable with a continuous distribution function (*c.d.f.*) F. Following the description of the problem given by Bojdecki [3] let N be a fixed natural number. The results of items evaluation are the realization of the random variables $X_1, X_2, \ldots, X_N$. They are observed. Let us define $\mathcal{G}_n = \sigma\{X_1, X_2, \ldots, X_n\}$ and $\mathfrak{S}$ the set of all stopping times with respect to the family $\{\mathcal{G}_n\}_{n=1}^N$. The elements of $\mathfrak{S}$ can take ∞ which means that it does not take value less than N (we do not stop at all). The aim is to find a stopping time $\tau^* \in \mathfrak{S}$ such that

$$\mathbf{P}(\tau^* < +\infty, X_{\tau^*} = \max\{\overrightarrow{X_{1,N}}\}) = \sup_{\tau\in\mathfrak{S}} P(\tau < +\infty, X_\tau = \max\{\overrightarrow{X_{1,N}}\}). \quad (1)$$

Define $\eta_n = \mathbf{P}(X_n = \max(X_1, \ldots, X_N)|\mathcal{G}_n)$, $n = 1, 2, \ldots, N$. Clearly $\eta_n = \mathbb{I}_{\{X_n=\max(X_1,\ldots,X_N)\}} X_n^{N-n}$, where $\mathbb{I}_{\{\cdot\}}(\omega)$ is the indicator function of the event. If we assume that $\eta_\infty = 0$ then we have $\mathbf{E}\eta_\tau = \mathbf{P}(\tau < \infty, X_\tau = \max(X_1, \ldots, X_N))$, for any $\tau \in \mathfrak{S}$. It reduces the problem to the classical optimal stopping problem for the sequence $\{\eta_n\}_{n=1}^N$. In this case it is possible to reformulate it as the optimal stopping of a Markov chain. To this end, let $\mathfrak{S}_0$ denote the set of all $\sigma \in \mathfrak{S}$ such that $X_n = \max(X_1, \ldots, X_n)$ on$\{\sigma = n\}$ for $n = l, 2, \ldots, N$.

If $\tau \in \mathfrak{S}$, then τ' defined as τ if $X_\tau = \max(X_1, \ldots, X_\tau)$, and as $+\infty$ otherwise is in $\mathfrak{S}_0$. It is clear that $\mathbf{E}\eta_\tau \le \mathbf{E}\eta_{\tau'}$, so it suffices to consider stopping times belonging to $\mathfrak{S}_0$ only. Define the moments when the consecutive maxima appear, *i.e.* $\tau_1 = 1$, $\tau_{k+1} = \inf(n : N \geq n > \tau_k,\ X_n = \max(X_1, \ldots, X_n))$, for $k = 1, 2, \ldots, N-1$. This sequence of stopping times belongs to $\mathfrak{S}_0$. Define $\xi_k = (\tau_k, X_{\tau_k})$ on $\{\tau_k < +\infty\}$ and $\xi_k = \partial$ on $\{\tau_k = +\infty\}$ (∂ is a label for the final state and denotes that after $\tau_{k-1} < +\infty$ there were no "leaders"). The sequence $\{\xi_k\}_{k=1}^N$ is a Markov chain with respect to $\{\mathcal{G}_{\tau_k}\}_{k=1}^N$ with the state space $(\{1, 2, \ldots, N\} \times [0, 1]$ and the transition probabilities

$$\begin{aligned} \mathbf{P}(n, x; m, B) &= \mathbf{P}(\xi_{k+1} \in (m, B) | \xi_k = (n, x)) \\ &= \mathbb{I}_{\{n<m\}}(n, m) x^{n-m-1} \mu((x, 1] \cap B), \end{aligned}$$

for $n, m = 1, 2, \ldots, N$, $x \in [0, 1]$, B a Borel subset of $[0, 1]$, where $\mu(\cdot)$ is the Lebesque measure.

For any $\tau \in \mathfrak{S}_0$ define $\sigma = k$ on the set $\{\tau = \tau_k < +\infty\}$, $k = 1, 2, \ldots, N$ and $\sigma = +\infty$ on $\{\tau = +\infty\}$. Let $f(n, x) = x^{N-n}$ for $n = 1, 2, \ldots, N$, $x \in [0, 1]$ and 0 otherwise. If σ is a stopping time with respect of $\{\mathcal{G}_{\tau_k}\}_{n=1}^N$ then $\eta_\tau = \mathbb{I}_{\{\tau<+\infty\}} X_\tau^{N-\tau} = f(\xi_\sigma)$.

The solution of this reformulated optimal stopping problem for Markov chain is based on the observation that the problem is the monotone case (cf. [12] Sec. 3.5, p. 54). For such a case the construction of the optimal strategy is relatively simple. It is so often used in the investigation of the secretary problem extension that it is worth recalling the lemma formulating the solution of the optimal stopping problem for this class.

LEMMA 1.1 *Let ξ be a homogeneous Markov chain with state space $\mathbf{S}$, and $f : S \to R$, be a bounded function. Denote $\mathbf{Z} = \{s \in \mathbf{S} : \mathbf{T}f(s) \leq f(s)\}$. Assume that (i) $\forall_{s\in\mathbf{S}}\ \mathbf{P}(\exists_k \xi_k \in \mathbf{Z}) = 1$; (ii) $\forall_{s\in\mathbf{Z}}\ \mathbf{P}(\exists_k \xi_k \notin \mathbf{Z}) = 0$. Then $\sigma^\star = \inf\{k : \xi_k \in \mathbf{Z}\}$ is a solution of the problem of optimal stopping of ξ with the reward function f.*

In the full information secretary problem the set $\mathbf{Z} = \{\partial\} \cup \bigcup_{n=1}^N (\{n\} \times [x_n, 1])$ where $x_N = 0$ and x_n is the solution of the equation $\mathbf{T}f(n, x) = f(n, x)$ in $[0, 1]$ which for $n < N$ has the following explicit form

$$\sum_{k=n+1}^{N} \frac{x^{N-k+1} - 1}{N - k + 1} = 1. \tag{2}$$

The sequence $\{x_n\}$ is decreasing and the Markov chain ξ_n is going to the right and upwards. This implies that the assumptions of the lemma are satisfactory. Hence the solution comes in the form:

THEOREM 1.2 *The solution of the full information secretary problem has the form:*

$$\tau^\star = \inf\{n \leq N : X_n = \max\{X_1, \ldots, x_n\}, F(X_n) \geq x_n\}, \tag{3}$$

where $x_N = 0$ and x_n for $n < N$ are the unique roots of the equations (2).

Before the precise model of the full information secretary problem was formulated by Bojdecki [3], the solution has been presented by Gilbert & Mosteller [20] based on heuristic arguments. Sakaguchi [35] published his solution also. The references of early papers on the topics can be found in [19].

1.1.3. The partial information secretary problem Let $\mathcal{D}$ be a family of continuous distribution functions. The formulation of the problem is the same as in the full information case in the section 1.1.1. There are two extreme cases: (i) when $\mathcal{D}$ contains all continuous distribution functions (the secretary problem) and (ii) when $\mathcal{D}$ contains a single c.d.f. (the full information case). When $\mathcal{D}$ is neither the class of all continuous distribution functions nor a single c.d.f. it is the partial information case which was investigated *e.g.* by Petruccelli [28]. In his paper, for an arbitrary location, the scale and location-scale parameter family $\mathcal{D}$, he gave sufficient conditions for the existence of stopping rules τ_N having asymptotic probability of choosing the largest as in the full information case. On the other side Samuels [38] showed that for the family $\mathcal{D}$ of all uniform continuous distribution functions the minimax stopping rule is the same as in the no information case. Petruccelli [27] proved that for $\mathcal{D}$ containing all uniform $U[\theta-1/2,\theta+1/2]$ continuous distribution functions the minimax stopping rule τ_N gives $\lim_{n\to\infty}\mathbf{P}(X_{\tau_N}=\max\{\overrightarrow{X_{1,N}}\})\cong 0.4352$?

1.1.4. Minimizing the expected rank in the no information secretary problem If as in the case of the problem of the section 1.1.1 the aim of the decision maker is to minimize $\mathbf{E}(Z_\tau)$ by $\tau\in\mathfrak{S}^N$. This topic was the sole problem of the paper by Chow, Moriguti, Robbins, and Samuels [11]. First of all probabilistic properties of relative rank vs. absolute rank should be recalled. The relative ranks $\{Y_n\}_{n=1}^N$ are independent and they have uniform distribution on the set of their values: $\mathbf{P}(Y_k=j)=\frac{1}{k}$, for $j=1,2,\ldots,k$. The conditional distribution of the absolute rank Z_n given history is the following:

$$\mathbf{P}(Z_n=k|Y_1=j_1,\ldots,Y_n=j)=\mathbf{P}(Z_n=k|Y_n=j)=\frac{\binom{k-1}{j-1}\binom{N-k}{n-j}}{\binom{N}{n}},\tag{4}$$

for $j\le k\le n\le N$. $\mathbf{E}(Z_n|Y_n=j)=\frac{N+1}{n+1}j$. With backward induction the solution of the minimization problem of the expected rank was solved. Let $c_n=\inf_{\tau\ge n}\mathbf{E}(Z_\tau)=\inf_{\tau\ge n}\mathbf{E}(\frac{N+1}{\tau+1}Y_\tau)$. The sequence $\{c_n\}_{n=0}^{N-1}$ is non decreasing. Define $s_n=\left[\frac{n+1}{N+1}c_n\right]$.

Theorem 1.3 *The optimal stopping time in the expected rank minimization problem for the no information secretary problem*

$$v=\inf_{\tau\in\mathfrak{S}^N}\mathbf{E}(Z_\tau)\tag{5}$$

has the form $\tau^\star = \inf\{1 \leq n \leq N : Y_n \leq s_n\}$ *and the value* $v = c_0$.

$$\lim_{N\to\infty} c_0 = \prod_{j=1}^{\infty} \left(\frac{j+2}{j}\right)^{\frac{1}{j}+1} \cong 3.8695.$$

The form of the optimal stopping time for the problem is typical. It is so suggestive that it leads to errors in some heuristic investigations (cf. the section 1.2.2). The rank minimization problem for no information case returned as a stimulation for further research up to contemporary works.

1.1.5. Minimizing the expected rank in the full information secretary problem Let us observe sequentially the sequence $\{X_n\}_{n=1}^N$, like in the full information secretary problem presented in the section 1.1.2, and one must stop on exactly one of them at the instance when it appears. The aim is to choose the stopping rule τ with respect to the filtration $\mathcal{G}_n$ which minimizes the expected rank of the selected observation. Let us denote $\mathfrak{M}^N$ the set of stopping times with respect to the filtration $\{\mathcal{G}_n\}_{n=1}^N$. The formulation is as in (5) but with respect to the stopping times from $\mathfrak{M}^N$. This full-information expected-rank minimization problem is known as Robbins' problem (cf. [5]). The general solution is still unknown, and only some bounds are known for the limiting value as N tends to infinity. Recent work on the topic was published by Bruss [7] and Gnedin & Iksanov [21]. The problem was posed by Robbins at the Amherst conference in 1990. An amazing story about the history of it one can find in [5].

The problems formulated in the sections 1.1.1-1.1.5 show the role in the theory of optimal stopping and the respective difficulties. However, the importance of the Cowan-Zabczyk problem is mostly related to the models of the secretary problem with an unknown number of options. The aim of this note is to reopen awaken interest in this problem and by simply viewing it from what modification of the primary setting was formulated, to increase the probability that a reader may see the importance of the basic formulation and an unsolved question related to it.

1.2. An unknown number of options

1.2.1. Asymptotic behavior of the solution when the number of options tends to infinity In the formulated problems the number of available items is finite and known. An interesting question concerns the behaviour of solution when the number of options tends to infinity. In the no-information case both the threshold and the value of the problem, when the number of the option tends to ∞, get $a = \lim_{N\to\infty} \frac{r_a}{N} = e^{-1} \cong 0.3679$ and $\hat{c}_1(a) = \lim_{N\to\infty} c_1(r_a) = e^{-1}$. In the analysis of this question concerning extended problems an interesting technique was applied based on the approximation of difference equations by the differential equations (see paper by Mucci [25, 26]).

In the full information case the exact asymptotic value was the subject of research by many mathematicians. The rigorous explanation of the asymptotic behavior belongs to Samuels [37] (cf. also [2] p. 58-60). The proof is formal, without deeper probabilistic intuition. The asymptotic value is close to 0.58.

1.2.2. Discrete distributions of the number of options Let N denote the unknown, true number of items in the no–information secretary problem as in the section 1.1.1. The distribution $p_i = \mathbf{P}(N = i)$, $i = 1, 2, \ldots$ is known to the observer. Write

$$\pi_k = \mathbf{P}(N \geq k) = \sum_{j=k+1}^{\infty} p_j \tag{6}$$

Presman & Sonin [29] provided a treatment of the standard problem, using the Dynkin approach. The transition probabilities of the imbedded Markov chain are

$$q(r,s) = p(r,s)\frac{\pi_s}{\pi_r} = \frac{r\pi_s}{s(s-1)\pi_r} \text{ for } r < s < \infty, \tag{7}$$

$q(r,\infty) = \sum_{k=r}^{\infty} \frac{r}{k}\frac{p_k}{\pi_r}$. The ∞ as in the section 1.1.2 denote the absorbing state to cover the possibility that $r \leq N < s$. In this case, the k-th item is the actual best, and the probability of this, given that it is relatively best, is just $q(k,\infty)$. The problem is much harder than when the horizon is given. The one-step-look-ahead (OLA) policy is no longer optimal even for the bounded N. The set $\mathbf{Z}$ of states r at which it is optimal to stop may be thought of, trivially, as a succession of *islands* separated by a sea of continuation states. It was not obvious for Rasmussen [31] and Rasmussen & Robbins [32] where the analysis of the problem is not correct and results are wrong. Both papers are using the erroneous theorem 3.1 of [31]. Irle [23] gives a counter-example distribution of N for their result and using the method obtained by Rasch [30] based on the Howard's policy iteration method he repeated the results of Presman & Sonin [29].

The minimax problems against the set of distribution of the horizon N were the natural extensions and they were also investigated. The analysis for the bounded horizon can be found in [22]. Let $\mathcal{D}$ be the set of all such distributions. The problem of section 1.1.1 with the infimum over $\mathcal{D}$ and the supremum over all stopping times was solved there. The authors derive the explicit value, give the minimax-optimal stop rule and the distribution yielding the value. The minimax distribution of N (which puts, except for bound n, no probability mass on the unique stopping island for the deterministic $N = n$ case) is seemingly a new distribution on $\{1, 2, \ldots, n\}$.

1.2.3. The Poisson stream of options The problems posed in the previous section are provoking the extension for modeling the stream of options in the continuous time. The good model for that is the point process.

The simplest one is the Poisson process with the known intensity. The real search time should be bounded to a fixed or random interval. In the sixties of XX-th century, when the theory of Markov decision processes arose and the development of the theory of optimal stopping grew, the problem of selling houses was the perfect training ground for creating algorithms' applications. The examples of the contribution in the area are Elfving problem [17] (see also Breiman [4] for systematic presentation of the known models at that time). The Elfving problem is not the secretary problem. The full information secretary problem with Poisson stream of option was considered by Sakaguchi [36] (it is one of the four models investigated in the paper).

2. The Cowan-Zabczyk model

2.1. Lead-in Taking into account preliminaries, let us recall Tom Ferguson's claim that the Secretary Problem "has been extended and generalized in many different directions so that now one can say that it constitutes a *field within mathematics–probability–optimization*". Further, this paper will recall a list of extensions related to the model presented by Cowan and Zabczyk [14, 15] which motivated to very deep question and investigation. The story is not finished yet. The original formulation of the problem is following:

It is the continuous time, no information best choice problem. The Poisson stream of option with bounded horizon is observed by a decision maker. Each option is evaluated by comparison to the past option and the rank is assigned. All options have different value. The total number of options is random. At the moment of evaluation the decision maker can accept it or reject. Recall is not possible. The investigation stops when the item is chosen or the epoch T is reached.

The solution of the original problem is complicated. The natural questions are related to parameters of the model:

1. the horizon T (e.g. unknown, random, asymptotic solution when $T \to \infty$);
2. the parameter λ (e.g. unknown, the *a priory* distribution is known);
3. another information about the options.

2.2. Description of the model Cowan and Zabczyk [14, 15] proved, using the standard theory of optimal stopping of a Markov chain, that the optimal rule is to select the i–th candidate (the best so far), where i is the smallest k for which the k–th candidate is a leader and $\lambda(T-t) \leq y_k$, where the $\{y_k\}$ are the increasing sequence of numbers such that, if $\lambda(T-t) = y_k$, then the success probabilities for the rules *stop now* and *stop at the next leader* are equal. These probabilities were derived and the y_k are the solutions to the equations

$$\sum_{n=0}^{\infty} \frac{y_n}{n!(n+k)} = \sum_{n=1}^{\infty} \frac{y_n}{n!(n+k)} \sum_{j=0}^{n-1} \frac{1}{k+j}.$$

The presented issue became the basis for further research. Many of the questions that are typical for this matter, lived to see the answer. It is worth mentioning that some of them here, and others will be included in the paper. The work also inspired research on a similar model related to the full information secretary problem but the papers that treated the issue not always quoted the formulation of [14, 15].

The asymptotic behavior of the thresholds y_k was investigated by Ciesielski and Zabczyk [13]. It was shown that $\lim_{k\to\infty} \frac{y_k}{k}$ tends to $\mathrm{e}-1$. The Bayesian approach to the stream of option leads Bruss [9] to the observation that under some priors for λ, the non-informative priors, the optimal thresholds are stationary. This result is followed by Bruss [8] who gives a solution to the minimax version of the best choice problem with random number of options.

The full list of the papers related to the Cowan-Zabczyk problem is difficult to collate. Directly this model cites more than twenty works, and many more secondary results. It is obvious that [14, 15] stimulated Bruss & others' research on the version with the random horizon T (see Bruss [8], Bruss & Samuels [6]), the unknown intensity of the stream of options (see Bruss [9]) when the other aim than choosing the best considered [41]) and the bilateral problem on the Poisson stream of the options [42]. There are experimental research reported e.g in [44], [39] which are also stimulated by this seminal Cowan-Zabczyk problem.

3. Extenstions of the Cowan-Zabczyk models[2] It is the subjective selection of such problems. The main stream of questions recalls that the number of items under investigation is unknown and random. On the list there are results published before and after Cowan-Zabczyk's seminal papers (cf. the preliminaries in section 1.1). However, the Poisson stream of options as natural model for real time *apartment* search problem, considered as the optimal stopping problem, was appealing idea at the time. The secondary ideas were related to the unknown or random parameter of the Poisson process. Also the game version of the secretary problem with the Poisson stream of options is before hands.

A method of selecting the best element from a random sequence of an unknown length was investigated by Stewart [40]. By assuming that the arrival times of the elements are the independent and identically distributed exponential random variables, a procedure was established that maximizes

[2] Results directly stimulated by the papers [14, 15].

the probability of selecting the best element. Asymptotically, for large values of the actual length of the sequence, the optimal probability is e^{-1}. It is shown that the method behaves well even when the actual number of options is comparatively small, and that it is not particularly sensitive to errors in the specification of the arrival rate of the process.

Ano and Ando [1]considered a continuous–time generalization of the secretary problem. A decision maker finds an apartment during a fixed period $(0, T]$. Opportunities to inspect apartments occur at the epochs of a homogeneous Poisson process of an intensity λ. The decision maker can rank a given apartment among all those inspected to date. The objective is to maximize the probability of selecting the best apartment from those available in $(0, T]$. It extends Bruss' problem to the problem in which each owner of an apartment can accept the offer proposed by an apartment's searcher with a fixed known probability p, $(0 < p \leq 1)$ and the decision maker is allowed to make at most $m \geq 1$ offers. It is shown that the optimal stopping rule for the problem is to make an offer to the first relatively best option after a time $s^*_m = (T + a)\exp(-C^{(m)}(q)) - a$, where $q = 1 - p$. In particular, they give $C^{(1)}(q) = 1$, $C^{(2)}(q) = 1 + q/2$ and $C^{(3)}(q) = 1 + q/2 + q^2/3 + q^3/8$. Finally, they consider the case when the probability p depends on m.

In the paper [41] a continuous-time generalization of the secretary problem that was studied by Bruss [9] was extended. It generalizes the problem within two frameworks. In the first case the objective is to stop on the best or on the second best object whilst in the second case the goal is to stop on the second best object. The optimal strategies and the probabilities of success are derived.

The unique paper [42] with the game model deals with the continuous-time two-person game version of the secretary problem. Objects appear according to a compound Poisson process with unknown intensity λ having a prior exponential density. Each player can choose only one candidate object, with the aim being to choose the best object before cut-off date T. If both players would like to select the same object, priority is randomly assigned to Player 1 with probability α, or to Player 2 with probability $1-\alpha$. The player that has been rejected may select any object from those offered later. The resulting nonzero-sum stochastic game was studied and the Nash equilibria was constructed. Some explicit solutions are obtained and it is shown that the structure of the solution undergoes a phase transition based on the value of α. This extends previous models of Bruss [9] from one side and G. Ravindran & Szajowski [33] (or Szajowski [43]).

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Sekwncyjne poszukiwanie lokalu do wynajęcia.

Krzysztof Szajowski

Streszczenie W latach 60 -tych poprzedniego wieku analizowano wielu matematyków skupiało swoja uwagę na zadaniach optymalizacyjnych nawiązujących do sekwencyjnego przeszukiwania czy obserwacji. Do tych zadań można zaliczyć problem optymalnego zatrzymania Robbinsa, problem sekretarki, (dość obszerną analizę tego zagadnienia przeprowadził Ferguson [18]), zadanie optymalnego parkowania czy też problem poszukiwania pracy. Subtelne szczegóły tych zagadnień powodują, iż każde zagadnienie z wymienionych ma liczne wersje różniące się szczegółami, które powodują, iż mamy do czynienia całą rodziną modeli. Jedno z zagadnień zainteresowało professora Jerzy Zabczyk. W wyniku dyskusji z profesorem Richardem Cowanem (w Warszawie) stworzyli model poszukiwania najlepszego obiektu, gdy dostępnych obiektów jest losowa liczba. Wg współczesnej klasyfikacji problemów wyboru najlepszego obiektu jest to przypadek poszukiwania najlepszego obiektu przy braku informacji, z czasem ciągłym, gdy strumień zgłoszeń jest poissonowski a horyzont jest skończony, ustalony.

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Słowa kluczowe: czas zatrzymania, gry z zatrzymywaniem procesu, proces Markowa, złożony proces Poissona, gra o sumie niezerowej, losowy priorytet, randomizowane momenty zatrzymania.

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