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Adaptive Control of Discrete Time-Varying LQG

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Abstract. The adaptive version of the discrete time-varying linear quadratic control is considered under the assumption that the coefficients have limits as time tends to infinity sufficiently fast in certain sense and the limiting system is observable and stabilizable. It is proved that time invariant LS estimator can be used to estimate the limits of the coefficients and that it is strongly consistent under some conditions well known from the time invariant case. The estimator of the parameters is used to define an adaptive control law and it is shown that the control law is optimal.

1. Introduction. Consider the stochastic system

$$x_{k+1} = A_k x_k + B_k u_k + w_{k+1}, k \geq 0, \tag{1}$$

where x_k is n-dimensional state vector, u_k is m-dimensional control vector, w_k, $k \geq 1$, is n-dimensional white noise sequence with zero means and covariance $\sum_k$. Moreover we assume that

$$\sup_k E \|w_k\|^\beta < \infty \text{ a.s. for some } \beta > 2. \tag{2}$$

The initial state x_0 is a given random vector, independent of w_k, $k \geq 1$. Together with (1) we consider the cost functional

$$J(x_0, u) = \lim_{N\to\infty} \frac{1}{N} E \sum_{k=0}^{N} (\langle Q_k x_k, x_k\rangle + \langle H_k u_k, u_k\rangle), \tag{3}$$

which is minimized, where the weight matrices $Q_k \geq 0, H_k > 0$. The objective of this paper is to find optimal control law for system (1) with cost functional (3) under the following hypotheses:

(A1) $\lim_{k\to\infty} A_k = A$, $\lim_{k\to\infty} B_k = B$, $\lim_{k\to\infty} \sum_k = \sum$, $\lim_{k\to\infty} Q_k = Q$, $\lim_{k\to\infty} H_k = H$.

(A2) $H > 0$.
(A3) A is a stable matrix.
(A4) (A, B) is controllable and $(A, \sqrt{Q})$ is observable.
(A5) $\sum > 0$.
(A6) The sequences A_k, B_k are unknown as well as the their limits A, B.

Under the assumptions (A1)–(A5) the solution of the problem (1) with cost functional (3) is given by the following theorem [3].

THEOREM 1. *Assume that assumptions* (A1)–(A5) *hold. Then the feedback control*

$$u_k = Lx_k \; , \tag{4}$$

where

$$L = -(H + B'PB)^{-1}B'PA \tag{5}$$

is optimal for system (1) *with cost functional* (3), *where P is the unique solution of the algebraic Riccati equation*

$$P = A'PA - A'PB(H + B'PB)^{-1}B'PA + Q. \tag{6}$$

Moreover the minimal value of the cost functional is given by $tr(\sum P)$.

II. Parameter Estimator. Theorem 1 implies that for the purpose of optimal control we should know only the limits A and B of the sequences of system coefficients. Now we are going to define their estimator. Set

$$\theta = \begin{bmatrix} A' \\ B' \end{bmatrix}, \qquad \vartheta_k = \begin{bmatrix} A'_k \\ B'_k \end{bmatrix}, \tag{7}$$

$$\varphi_k = \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \tag{8}$$

$$R_k = \Big(\sum_{i=0}^{k-1} \varphi_i\varphi'_i + \frac{1}{d}I\Big)^{-1}, \qquad d = n + m, \tag{9}$$

$$a_k = (1 + \varphi'_k R_k \varphi_k)^{-1} \tag{10}$$

$$r_k = 1 + \sum_{i=0}^{k-1} \|\varphi_i\|^2 \tag{10}$$

$$\lambda_k = \max\{\lambda : \lambda \in \sigma(R_k)\}$$

Let the estimator θ_n of θ at the moment n be given by the following recursive formula

$$\theta_{k+1} = \theta_k + a_k R_k \varphi_k (x'_{k+1} - \varphi'_k \theta_k), \tag{12}$$

$$R_{k+1} = R_k - a_k R_k \varphi_k \varphi'_k R_k \tag{13}$$

with arbitrary θ_0 and $R_0 = dI$. It is worth noting that in the time invariant case formulas (7)–(13) give the standard least-squares estimator. We shall use the following theorem from mathematical analysis ([4], vol I, pp. 53).

LEMMA 2 (Stolz's Theorem). *If $x_k \in R$, $y_k \in R$, $y_k \to \infty$, $y_{k+1} > y_k$ and the limit*

$$\lim_{k\to\infty} \frac{x_k - x_{k-1}}{y_k - y_{k-1}}$$

exists, then

$$\lim_{k\to\infty} \frac{x_k}{y_k} = \lim_{k\to\infty} \frac{x_k - x_{k-1}}{y_k - y_{k-1}}.$$

The following inequalities will be used in further considerations:

LEMMA 3 [5]. *If $x \geq 0, y \geq 0$ and $0 < \delta \leq 1$, than*

$$\delta x^{\delta-1}(x-y) \leq x^\delta - y^\delta. \tag{14}$$

Holder's Inequality. Let 1< $p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If for random variables ξ and η, $E|\xi|^p < \infty$ and $E|\eta|^q < \infty$, then

$$E|\xi\eta| \leq (E|\xi|^p)^{\frac{1}{p}} (E|\eta|^q)^{\frac{1}{q}}. \tag{15}$$

C_r-inequality. Let x_i, $i = 1, \ldots, n$ be nonnegative real number, then

$$\left(\sum_{i=1}^n x_i\right)^r \leq C_r \sum_{i=1}^n x_i^r, \tag{16}$$

for all real numbers r, where

$$C_r = \begin{cases} 1 & \text{for } r \leq 1, \\ n^{r-1} & \text{for } r > 1. \end{cases} \tag{16}$$

THEOREM 4. *Assume that*

$$\lim_{k\to\infty} r_k = \infty, \tag{18}$$

there exists a finite random variable c and constant $\alpha \in [0, \frac{1}{2})$ such that

$$\lambda_k \leq \frac{c}{r_k^{1-\alpha}}, \tag{19}$$

and

$$\|\theta - \vartheta_k\| = o\left(r_k^{-\frac{1}{2}}\right). \tag{20}$$

Then

$$\lim_{k\to\infty} \theta_k = \theta \ a.s. \tag{21}$$

with convergence rate

$$\|\theta_k - \theta\| = o\left(r_k^{\delta-\frac{1}{2}}\right), \tag{22}$$

for any $\delta \in (\alpha, \frac{1}{2}]$.

Proof. It is not difficult to check that

$$\theta_k = R_k \sum_{i=0}^{k-1} \varphi_i x'_{i+1} + R_k R_0^{-1} \theta_0. \tag{23}$$

Using notation (7) and (8), we can rewrite (1) as follows

$$x_k = \vartheta'_{k-1} \varphi_{k-1} + w_k. \tag{24}$$

Substituting (24) into (23) we see

$$\begin{aligned}\theta_k &= R_k \sum_{i=0}^{k-1} \varphi_i \varphi'_i \vartheta_i + R_k \sum_{i=0}^{k-1} \varphi_i w'_{i+1} + R_k R_0^{-1} \theta_0 \\ &= \theta - \frac{1}{d} R_k \theta - R_k \sum_{i=0}^{k-1} \varphi_i \varphi'_i (\theta - \vartheta_i) + R_k \sum_{i=0}^{k-1} \varphi_i w'_{i+1} + R_k R_0^{-1} \theta_0 \quad (25)\end{aligned}$$

and

$$\begin{aligned}\|\theta_k - \theta\| \le \frac{1}{d} \|R_k\| \, \|\theta\| &+ \left\| R_k \sum_{i=0}^{k-1} \varphi_i \varphi'_i (\theta - \vartheta_i) \right\| \\ &+ \left\| R_k \sum_{i=0}^{k-1} \varphi_i w'_{i+1} \right\| + \left\| R_k R_0^{-1} \theta_0 \right\|. \quad (26)\end{aligned}$$

Analogously as in the proof of Theorem 3.1 in [1] we can verify

$$\lim_{k\to\infty} r_k^{\frac{1}{2}-\delta} \left(\frac{1}{d} \|R_k\| \, \|\theta\| + \left\| R_k \sum_{i=0}^{k-1} \varphi_i w'_{i+1} \right\| + \left\| R_k R_0^{-1} \theta_0 \right\| \right) = 0. \tag{27}$$

Using (19) the second term on the right-hand side of (26) is estimated by

$$\left\| R_k \sum_{i=0}^{k-1} \varphi_i \varphi'_i (\theta - \vartheta_i) \right\| \le \frac{c \sum_{i=0}^{k-1} \|\varphi_i\|^2 \, \|\theta - \vartheta_i\|}{r_k^{1-\alpha}}$$

and thus

$$r_k^{\frac{1}{2}-\delta} \left\| R_k \sum_{i=0}^{k-1} \varphi_i \varphi'_i (\theta - \vartheta_i) \right\| \le \frac{c \sum_{i=0}^{k-1} \|\varphi_i\|^2 \, \|\theta - \vartheta_i\|}{r_k^{\frac{1}{2}+\delta-\alpha}}. \tag{28}$$

To end the proof is enough to show that

$$\lim_{k\to\infty} \frac{\sum_{i=0}^{k-1} \|\varphi_i\|^2 \, \|\theta - \vartheta_i\|}{r_k^{\frac{1}{2}+\delta-\alpha}} = 0. \tag{29}$$

It can be done using the Stolz's Theorem. To examine the limit

$$\lim_{k\to\infty} \frac{\|\varphi_{k-1}\|^2 \, \|\theta - \vartheta_{k-1}\|}{r_k^{\frac{1}{2}+\delta-\alpha} - r_{k-1}^{\frac{1}{2}+\delta-\alpha}}. \tag{30}$$

we use (14) and it leeds to the following inequality

$$\frac{\|\varphi_{k-1}\|^2\,\|\theta-\vartheta_{k-1}\|}{r_k^{\frac{1}{2}+\delta-\alpha}-r_{k-1}^{\frac{1}{2}+\delta-\alpha}} \le \frac{\|\theta-\vartheta_{k-1}\|}{(1+\delta-\alpha)r_k^{\delta-\alpha-\frac{1}{2}}}.$$

By assumption (20) the term on the right-hand side tends to zero when $k\to\infty$. ■

III. Adaptive control law. In this section we first define the adaptive control and then show that the estimator of θ is strongly consistent. For this purpose we must make an additional assumption about rate of convrrgence of ϑ_k.

(A7) There exists a constant $\nu>0$ such that

$$\lim_{k\to\infty}\|\theta-\vartheta_k\| = o((k^{1+\nu}\ln^{\nu}k)^{-1/2}). \tag{31}$$

We define the adaptive control in the same way as in [2] for time invariant case. Define P_k, L_k as follows

$$P_k = \widehat{A}_k'P_k\widehat{A}_k - \widehat{A}_k'P_k\widehat{B}_k(\overline{R}+\widehat{B}_k'P\widehat{B}_k)^{-1}\widehat{B}_k'P_k\widehat{A}_k + Q \tag{32}$$

$$L_k = -(H+\widehat{B}_k'P_k\widehat{B}_k)^{-1}\widehat{B}_k'P_k\widehat{A}_k, \tag{33}$$

where

$$\theta_k = \begin{bmatrix}\widehat{A}_k'\\ \widehat{B}_k'\end{bmatrix},\qquad \widehat{A}_k'\in R^{n\times n},\qquad \widehat{B}_k'\in R^{n\times m}.$$

Fix $\varepsilon\in(0,\min(\frac{1}{5},\nu))$ and take any matrix $K_0\in R^{n\times m}$ as the initial value for the adaptive feedback gain K_k,which is defined by

$$K_k = \begin{cases} L_k \text{ if } \|L_k\|<\ln^{\varepsilon/2}k,\ (\widehat{A}_k,\widehat{B}_k) \text{ is controllable} \\ \qquad\quad \text{and } \widehat{A}_k,\sqrt{Q}) \text{ is observable} \\ K_{k-1} \text{ otherwise.}\end{cases} \tag{34}$$

Introduce a sequence of i.i.d. random vectors ε_k independent of w_k such that

$$E\varepsilon_1=0,\qquad E\varepsilon_1\varepsilon_1'=I,\qquad E\|\varepsilon_1\|^4<\infty$$

and define

$$v_n = n^{-\delta/2}\varepsilon_n,\qquad \delta\in\left(0,\frac{1-5\varepsilon}{2}\right),\qquad v_0=0. \tag{35}$$

The time axis is partitioned by stopping times τ_k, t_k, $\tau_k<t_k<\tau_{k+1}$, $k=1,\dots,$ and the adaptive control u_k is defined to be

$$u_k = K_k^0x_k+v_k \tag{36}$$

with

$$K_k^0 = \begin{cases} K_k & \text{if } k \text{ belongs to some } [\tau_r,t_r) \\ 0 & \text{if } k \text{ belongs to some } [t_r,\tau_{r+1}).\end{cases} \tag{37}$$

The stopping times are given as follows. We take $\tau_1 = 3$ and define

$$t_k = \sup\Big\{s > \tau_k : \sum_{i=\tau_k}^{j-1} \|x_i\|^2 \le \|x_{\tau_k}\|^2 \ln^\varepsilon \tau_k + (j-1)^{1+\frac{\varepsilon}{2}},$$

$$\bigwedge j \in (\tau_k, s]\Big\} \qquad (38)$$

$$\tau_{k+1} = \sup\Big\{s > t_k : \sum_{i=\tau_k+1}^{s} \|x_i\|^2 \le \frac{s^{1+\varepsilon}}{2^k},\ \frac{\|x_s\|^2 \ln^\varepsilon s}{s^{1+\frac{\varepsilon}{2}}} \le 1\Big\} \qquad (39)$$

IV. Strong consistency of parameter estimates. In our future considerations we will need the following facts.

LEMMA 5. *If matrices Ω_k converge to a stable matrix, then there exists constants $0 < \mu < 1$ and $c > 0$ such that*

$$\Big\| \prod_{i=k}^{j} \Omega_i \Big\| < c\mu^{k-j+1}, \quad \bigwedge k > j, \quad \bigwedge j \ge 0, \qquad (40)$$

where

$$\prod_{i=p}^{q} X_i = \begin{cases} X_q X_{q+1} \ldots X_p, & \text{for } q \ge p \\ I, & \text{for } q < p. \end{cases}$$

LEMMA 6. *Let F_k be a family of non-decreasing sub σ-algebras, f_k be a random F_k-measurable vector and let (w_k, F_k) be a martingale difference sequence satisfying* (2). *Then*

$$\sum_{i=1}^{k} f_i w'_{i+1} = O\Big(\Big(\sum_{i=1}^{k} \|f_i\|^2 \ln \sum_{i=1}^{k} \|f_i\|^2\Big)^{1/2}\Big) \quad \text{a.s.} \qquad (41)$$

For the proof of Lemma 5 we refer the reader to [1], p.191 and for the proof of Lemma 6 to [6].

LEMMA 7. *Under the control defined by* (36)–(39) *the following estimate takes place*

$$\frac{1}{k^{1+\varepsilon}} \sum_{i=1}^{k} \|x_i\|^2 = O(1) \quad a.s. \qquad (42)$$

$$r_k = O(k^{1+\varepsilon} \ln^\varepsilon k). \qquad (43)$$

Proof. Let $\beta_1 = \min(\beta, 4)$ and fix $s \in [2, \beta_1)$. As in the proof of Lemma 3.1 in [2] we can show that

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \|w_i\|^s < \infty, \qquad (44)$$

$$\limsup_{k\to\infty} \frac{1}{k^{1-\frac{\delta s}{2}}} \sum_{i=1}^{k} \|v_i\|^s < \infty. \tag{45}$$

Hence by the boundes of B_k we have

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \|B_{i-1}v_{i-1} + w_i\|^s < \infty. \tag{46}$$

Since A and $A + BL$ are stable then by Lemma 5 there exist $0 < \mu < 1$ and $c > 0$ such that

$$\left\|\prod_{i=j}^{k} A_i\right\| < c\mu^{k-j+1}, \quad \left\|\prod_{i=j}^{k}(A_i + B_iL)\right\| < c\mu^{k-j+1}, \quad \bigwedge k > j, \quad \bigwedge j \geq 0. \tag{47}$$

If, for some k, $t_k = \infty$, then the assertion of this lemma is trivial. Assume that $t_k < \infty$. From (36) and (37), it is easy to see that for $j \in [1, \tau_{k+1} - t_k]$

$$x_{t_k+j} = \Big(\prod_{i=t_k}^{t_k+j-1} A_i\Big)x_{t_k} + \sum_{i=0}^{j-1}\Big(\prod_{l=t_k+i+1}^{t_k+j-1} A_l\Big)(B_{t_k+i}v_{t_k+i} + w_{t_k+i+1}),$$

and

$$\|x_{t_k+j}\| \leq c\mu^j \|x_{t_k}\| + c\sum_{i=0}^{j-1} \mu^{j-i-1} \|B_{t_k+i}v_{t_k+i} + w_{t_k+i+1}\|. \tag{48}$$

For real numbers a, b we have

$$(a+b)^2 \leq 2a^2 + 2b^2.$$

Using this inequality we can bound the right hand side of (48) as follows

$$\|x_{t_k+j}\|^2 \leq 2c^2\mu^{2j} \|x_{t_k}\|^2 + 2c^2\Big(\sum_{i=0}^{j-1} \mu^{j-i-1} \|B_{t_k+i}v_{t_k+i} + w_{t_k+i+1}\|\Big)^2. \tag{49}$$

From (49) we can obtain (42) as in the proof of Lemma 3.1 in [1]. To proof (43) we observe that by (34) and (36) we can estimate r_k as follows

$$\begin{aligned} r_k &\leq \sum_{i=0}^{k-1} \|x_i\|^2 \left\|K_i^0\right\|^2 + 2\sum_{i=0}^{k-1} \left\|K_i^0\right\| |x_i v_i| + \sum_{i=0}^{k-1} \|v_i\|^2 \\ &\leq \sum_{i=0}^{k-1} \|x_i\|^2 \left\|K_i^0\right\|^2 + 2\sqrt{\sum_{i=0}^{k-1} \|K_i^0\|^2 |x_i|^2 \sum_{i=0}^{k-1} \|v_i\|^2} + \sum_{i=0}^{k-1} \|v_i\|^2 \\ &\quad \ln^{\varepsilon} k \sum_{i=0}^{k-1} \|x_i\|^2 + 2\sqrt{\ln^{\varepsilon} k \sum_{i=0}^{k-1} |x_i|^2 \sum_{i=0}^{k-1} \|v_i\|^2} + \sum_{i=0}^{k-1} \|v_i\|^2, \end{aligned} \tag{50}$$

where the Schwartz inequality has been used. Hence by (42) and (45), (43) follows. ∎

THEOREM 8. *Under the adaptive control defined by* (36)–(39), *the estimate* θ_k *is strongly consistent with the rate of convergence*

$$\|\theta_k - \theta\| = o(n^{-\gamma}) \ a.s \bigwedge \gamma \in \left[0, \frac{1}{2} - \alpha\right),$$

where

$$\frac{\delta + 3\varepsilon}{1+\varepsilon} < \alpha < \frac{1}{2}.$$

Proof. The conclusion of the theorem will follow from Theorem 4 if we can show that

$$\lim_{k\to\infty} r_k = \infty, \tag{51}$$

and

$$\lambda_k < \frac{c_0}{r_k^{1-\alpha}} \quad \text{with } c_0 > 0, \tag{52}$$

because the choice of ε and (43) guarantee that assumption (20) holds. The proof of these two facts follows the line of reasoningt for the time invariant case in [1]. ∎

V. Optimality of adaptive control. In this section, we show that the control law given by (36)–(39) is optimal. Theorem 8 establish the convergence $(\widehat{A}_k, \widehat{B}_k)$ to (A, B) as $k \to \infty$, so by assumption (A4), $(\widehat{A}_k, \widehat{B}_k)$ is controllable and $(\widehat{A}_k, \sqrt{Q})$ is observable for all sufficiently large k and by Lemma 4.9 of [1]

$$L_k \to L \quad \text{as } k \to \infty \quad \text{a.s.}, \tag{53}$$

where L_k and L are given by (5) and (33) respectively. Hence, by definition (34) we see that

$$K_k = L_k$$

for all sufficiently large k.

LEMMA 9. *Under assumptions* (A1)–(A7) *there exists* $N > 0$ *such that*

$$u_k = L_k x_k + v_k \quad \textit{a.s. for all } k \geq N.$$

Proof. It suffices to prove that for any fixed ω there is an integer k such that $t_k = \infty$. If the opposite were true, then $t_k < \infty$, for all k. In this case, as shown in the proof of Lemma 3.1 in [1] we have $\tau_k < \infty$, for all k. Using the state equation (1) we have

$$\sum_{i=\tau_k+1}^{t_k} \|x_i\|^2 = \sum_{i=1}^{t_k-\tau_k} \|x_{\tau_k+i}\|^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{t_k-\tau_k} \Big\|\Big(\prod_{s=\tau_k}^{\tau_k+i-1} (A_s + B_s L)\Big) x_{\tau_k} \\
&\quad + \sum_{r=1}^{i} \Big(\prod_{s=\tau_k+r}^{\tau_k+i-1} (A_s + B_s L)\Big) x_{\tau_k+r} - \sum_{r=0}^{i-1} \Big(\prod_{s=\tau_k+r}^{\tau_k+i-1} (A_s + B_s L)\Big) x_{\tau_k+r} \|^2 \\
&= \sum_{i=1}^{t_k-\tau_k} \Big\|\Big(\prod_{s=\tau_k}^{\tau_k+i-1} (A_s + B_s L)\Big) x_{\tau_k} + \sum_{r=\tau_k+1}^{\tau_k+i} \Big(\prod_{s=r}^{\tau_k+i-1} (A_s + B_s L)\Big) x_r \\
&\quad - \sum_{r=\tau_k+1}^{\tau_k+i} \Big(\prod_{s=r-1}^{\tau_k+i-1} (A_s + B_s L)\Big) x_{r-1} \Big\|^2 \\
&= \sum_{i=1}^{t_k-\tau_k} \Big\|\Big(\prod_{s=\tau_k}^{\tau_k+i-1} (A_s + B_s L)\Big) x_{\tau_k} \\
&\quad + \sum_{r=\tau_k+1}^{\tau_k+i} \Big(\prod_{s=r}^{\tau_k+i-1} (A_s + B_s L)\Big) (x_r - (A_{r-1} + B_{r-1} L) x_{r-1}) \Big\|^2 \\
&= \sum_{i=1}^{t_k-\tau_k} \Big\|\Big(\prod_{s=\tau_k}^{\tau_k+i-1} (A_s + B_s L)\Big) x_{\tau_k} \\
&\quad + \sum_{r=\tau_k+1}^{\tau_k+i} \Big(\prod_{s=r-1}^{\tau_k+i-1} (A_s + B_s L)\Big) (x_r - A_{r-1} x_{r-1} - B_{r-1} L x_{r-1}) \Big\|^2 \\
&= \sum_{i=1}^{t_k-\tau_k} \Big\|\Big(\prod_{s=\tau_k}^{\tau_k+i-1} (A_s + B_s L)\Big) x_{\tau_k} \\
&\quad + \sum_{r=\tau_k+1}^{\tau_k+i} \Big(\prod_{s=r-1}^{\tau_k+i-1} (A_s + B_s L)\Big) \\
&\quad \times (B_{r-1} L_{r-1} x_{r-1} + B_{r-1} v_{r-1} + w_r - B_{r-1} L x_{r-1}) \Big\|^2 \\
&= \sum_{i=1}^{t_k-\tau_k} \Big\|\Big(\prod_{s=\tau_k}^{\tau_k+i-1} (A_s + B_s L)\Big) x_{\tau_k} \\
&\quad + \sum_{r=\tau_k+1}^{\tau_k+i} \Big(\prod_{s=r-1}^{\tau_k+i-1} (A_s + B_s L)\Big) B_{r-1} (L_{r-1} - L) x_{r-1} \\
&\quad + \sum_{r=\tau_k+1}^{\tau_k+i} \Big(\prod_{s=r-1}^{\tau_k+i-1} (A_s + B_s L)\Big) (B_{r-1} v_{r-1} + w_r) \Big\|^2,
\end{aligned}
$$

so by (47) we can find constants c_1, c_2, and c_3 such that for sufficiently large τ_k

$$\sum_{i=\tau_k}^{t_k} \|x_i\|^2 = \|x_{\tau_k}\|^2 + \sum_{i=\tau_k+1}^{t_k} \|x_i\|^2$$

$$\leq c_1 \|x_{\tau_k}\|^2 + c_2 \max_{r \geq \tau_k} \|L_{r-1} - L\| \sum_{i=\tau_k}^{t_k} \|x_i\|^2$$

$$+ c_3 \sum_{r=\tau_k+1}^{t_k} \|B_{r-1} v_{r-1} + w_r\|^2$$

and from here we have

$$\sum_{i=\tau_k}^{t_k} \|x_i\|^2 \leq \frac{1}{1 - c_2 \max_{r \geq \tau_k} \|L_{r-1} - L\|}$$

$$\times \left(c_1 \|x_{\tau_k}\|^2 + c_3 \sum_{r=\tau_k+1}^{t_k} \|B_{r-1} v_{r-1} + w_r\|^2 \right). \tag{54}$$

Furthermore, by (46), (53) and the fact that

$$\lim_{k \to \infty} \tau_k = \lim_{k \to \infty} t_k = \infty$$

we conclude that

$$\sum_{i=\tau_k}^{t_k} \|x_i\|^2 \leq \|x_{\tau_k}\|^2 \ln^{\varepsilon} \tau_k + t_k^{1+\frac{\varepsilon}{2}}$$

if k is sufficiently large. On the other hand by definition (38) we have

$$\sum_{i=\tau_k}^{t_k} \|x_i\|^2 > \|x_{\tau_k}\|^2 \ln^{\varepsilon} \tau_k + t_k^{1+\frac{\varepsilon}{2}}.$$

The contradiction proves the lemma. ∎

LEMMA 10. *Under assumptions* (A1)–(A7), *the controlled system* (1) *has the following properties*:

$$\limsup_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \|x_i\|^2 < \infty \quad a.s. \tag{55}$$

$$\limsup_{k \to \infty} \frac{\|x_k\|^2}{k} = 0 \quad a.s. \tag{56}$$

It follows the way in the proof of Lemma 4.2 in [1].

THEOREM 11. *For the system described by* (1), *if the assumptions* (A1)–(A7) *are satisfied then the adaptive control law given by* (36)–(39) *is* *optimal i.e.*

$$\lim_{N\to\infty}\frac{1}{N}E\sum_{k=0}^{N}(\langle Q_k x_k, x_k\rangle + \langle H_k u_k, u_k\rangle) = tr\sum P \ a.s.$$

Proof. Define

$$\begin{aligned}\xi_{k+1} = {} & \langle Q_k x_k, x_k\rangle + \langle H_k u_k, u_k\rangle \\ & - tr\sum P + \langle P x_{k+1}, x_{k+1}\rangle - \langle P x_k, x_k\rangle, \quad k \ge 0.\end{aligned}$$

By (5) we have

$$-(H + B'PB)L = B'PA,$$

and then P can also be written as

$$P = (A+BL)'P(A+BL) + LHL + Q. \tag{57}$$

Then by (1) with

$$u_k = L_k x_k + v_k$$

and by (57) for $k \ge N$ with N defined as in Lemma 9 we have

$$\begin{aligned}\xi_{k+1} = {} & \langle (Q_k + L_k' H_k L_k + (A_k + B_k L_k)'P(A_k + B_k L_k) - P)x_k, x_k\rangle \\ & + \langle P(B_k v_k + w_{k+1}), B_k v_k + w_{k+1}\rangle + \langle H_k v_k, v_k\rangle - tr\sum P \\ & + 2\langle R_k v_k, L_k x_k\rangle + 2\langle P(B_k v_k + w_{k+1}), (A_k + B_k L_k)x_k\rangle \\ = {} & \langle (Q_k - Q + L_k' H_k L_k + (A_k + B_k L_k)'P(A_k + B_k L_k) \\ & - (A+BL)'P(A+BL) - LHL)x_k, x_k\rangle \\ & + \langle P(B_k v_k + w_{k+1}), B_k v_k + w_{k+1}\rangle + \langle H_k v_k, v_k\rangle - tr\sum P \\ & + 2\langle H_k v_k, L_k x_k\rangle + 2\langle P(B_k v_k + w_{k+1}), (A_k + B_k L_k)x_k\rangle. \end{aligned} \tag{58}$$

By assumption (A1) and (53) we know that

$$\begin{aligned}\lim_{k\to\infty} & (Q_k - Q + L_k' H_k L_k + (A_k + B_k L_k)'P(A_k + B_k L_k) \\ & -(A+BL)'P(A+BL) - LHL) = 0.\end{aligned}$$

Hence by assumption (A1) and (55) it is clear that

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1} & \langle Q_k - Q + L_k' H_k L_k + (A_k + B_k L_k)'P(A_k + B_k L_k) \\ & -(A+BL)'P(A+BL) - LHL)x_k, x_k\rangle = 0. \end{aligned} \tag{59}$$

By assumption (A1), (45) and (55) it follows that

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\langle P(B_i v_i + w_{i+1}), B_i v_i + w_{i+1}\rangle = tr\sum P, \tag{60}$$

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\langle H_i v_i, v_i\rangle = 0 \tag{61}$$

and

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1} 2\langle H_i v_i, L_i x_i\rangle + 2\langle P(B_i v_i + w_{i+1}), (A_i + B_i L_i)x_i\rangle = 0. \tag{62}$$

By Lemma 6 and (55), we have also

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\langle P w_{i+1}, (A_i + B_i L_i)x_i\rangle$$

$$= O\left(\frac{1}{k}\sqrt{\sum_{i=0}^{k-1}\|x_i\|^2 \ln \sum_{i=0}^{k-1}\|x_i\|^2}\right) = o(1). \tag{63}$$

Hence by (59)–(63) from (58) we obtain

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\xi_{i+1} = 0.$$

By the definition of ξ_{i+1} we conclude that

$$\lim_{N\to\infty}\frac{1}{N}E\sum_{k=0}^{N}(\langle Q_k x_k, x_k\rangle + \langle H_k u_k, u_k\rangle)$$

$$= tr\sum P + \frac{1}{k}\sum_{i=0}^{k-1}\xi_{i+1} + \frac{1}{k}\sum_{i=0}^{k-1}(\langle P x_i, x_i\rangle - \langle P x_{i+1}, x_{i+1}\rangle)$$

$$= tr\sum P + o(1) + \frac{1}{k}(((\langle P x_0, x_0\rangle - \langle P x_k, x_k\rangle)) \stackrel{k\to\infty}{\to} tr\sum P$$

where (57) is used for the last limit tracking. ■

VI. Conclusions. In this paper we have extended the results of [2] for the time varying system. An open question is follows: are the assumptions (A3) about the stability of the control free system and (A7) about the rate of convergence necessary for the solution of the adaptive control problem.

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