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Irrational images – the visualization of abstract mathematical terms

Abstract In this article we would like to draw attention to the cognitive potential hidden in an image and in the art which employs it. We will focus on the visualization of basic mathematical objects e.g. irrational numbers. Our starting point will be the easy and intuitive case of the square root of two, as it is observed in the diagonal of a square. Next we will move over to the golden ratio hidden in a regular pentagon. To visualize this irrational number φ we will use a looped, endless animation. Finally, we will have a closer look at the famous number π and we will suggest an attempt to represent it in a clearly visual way. In the last section of the article we will consider the possibility of representing rational and irrational real numbers by dimensionless points on a straight line. We will also try to present a straight line on a flat surface which – as we know has length – but has no width. The above issues will enable us to see the extent to which mathematics may be inspirational for art, as well as how art may familiarize us with mathematical issues and explain them.

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1. Introduction Information about the external world reaches our minds – to a large extent – through our vision. Aristotle had already noticed this in the first sentences of *Metaphysics* where the writes that „of all the senses sight best helps us to know things, and reveals many distinctions" [1, 980 a]. Therefore images constitute a significant, if not the main element, of our cognition. This pertains not only to the cognition colloquially known as 'common sense'. Images also plays an important role in scientific cognition. In the opinion of many, science is not a completely separate path of intellectual human development but is an extension of common sense [11].

Even in the case of an abstract science such as mathematics, images are often the primary pretext, and a starting point for scientific reflection. At first concrete cases were examined, which had their direct representations in nature. Not until later were generalizations and conclusions made which went beyond observed reality.

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It is such mathematical structures detached as they are from our physical experience that are the topic of this article. We will consider the possibility of visualizing and depicting terms which as a result of using abstract notation have lost their direct relationship with images.

2. To see the irrational Irrational numbers are basic mathematical structures. Despite this fact, the common opinion of them is that they are a difficult issue which escapes intuitive understanding and which is not easily visualized. However they were not discovered in the form we see them in most often these days – in abstract, non-terminating and non-repeating decimals. Irrational numbers appeared before the ancient Greeks in the form of incommensurable segments – segments for which a common measure cannot be found but which are present in basic geometrical figures.

Every elementary school student should have at least come across a visual representation of the irrational square root of two in the shape of a diagonal of a square whose sides have length 1.

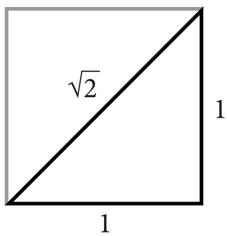

Figure 1: A diagonal of a square whose sides have length 1 has the irrational length of $\sqrt{2}$.

When a student learns Pythagoras theorem he or she has access to a method of visualizing the roots of any natural number. Starting with a square whose sides have length 1 the student may construct a rectangle with height equal to 1 and base of length $\sqrt{2}$. The diagonal of that rectangle obviously has length $\sqrt{3}$ (Fig. 2). Consequently, with the use of a compass and a

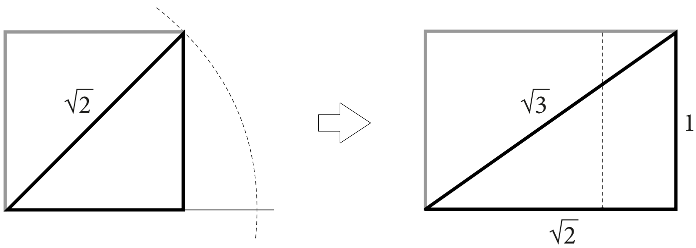

Figure 2: Geometrical construction of $\sqrt{3}$.

straightedge, it is possible construct segments corresponding to the roots of each natural number in succession (Fig. 3) a significant part of which are irrational numbers.

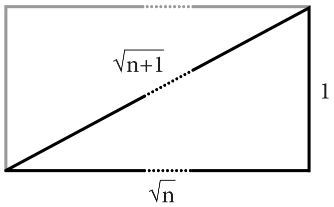

Figure 3: A diagonal of a rectangle with base of length $\sqrt{n}$ and height of 1 has a length which equals $\sqrt{n+1}$.

This example explicitly proves that the visual presentation of irrational numbers is not a difficult task. It seems however, that in the school educational process the withdrawal from geometrical visualization towards symbolic notation and arithmetic operations happens too fast [4]. As a result many people often have problems with fully understanding the essence of irrational numbers.

3. The golden number The ratio of the length of a side of a square to the length of a diagonal was probably not the first irrational ratio to be discovered. Currently it is claimed that the existence of incommensurable segments was noticed for the first time by Hippasus of Metapontum regarding the length of the diagonal of a regular pentagon [13]. In volume VI of *Elements* by Euclid we find a strict definition of the irrational ratio referred to as the *golden ratio*: „A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less" [6, p. 211]. The segment presented in the drawing above (Fig. 4)

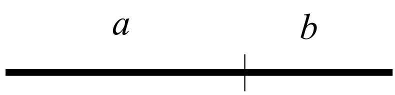

Figure 4: A segment divided into two parts whose lengths are in *golden ratio* to one another.

was divided in compliance with the above definition. Therefore the equation below is true:

$$\frac{a+b}{a} = \frac{a}{b}.$$

The ratio of the total length $a+b$, to the length of the longer segment a, equals the ratio of the length of the longer segment a to the length of the shorter segment b. Therefore, if we denote the golden proportion – the ratio of the length of the longer segment to the shorter one – as $\varphi = \frac{a}{b}$, the above equation may be written down in the form of a quadratic equation:

$$\varphi^2 - \varphi - 1 = 0.$$

This equation has two solutions but we are only interested in the nonnegative solution:

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

This irrational number φ, which is approximately 1.61803398 was the measure of perfect proportion according to the ancients. They saw it in nature and tried to recreate it in architecture and art[1].

In order to visualize the mathematical essence of the golden ratio and follow in the footsteps of the ancient geometers, let us take a closer look at the regular pentagon. The side and diagonal of the figure (Fig. 5.a) are not the only segments which correspond to that proportion. This also pertains to the internal segments (Fig. 5.b,c). The animated sequence shows more

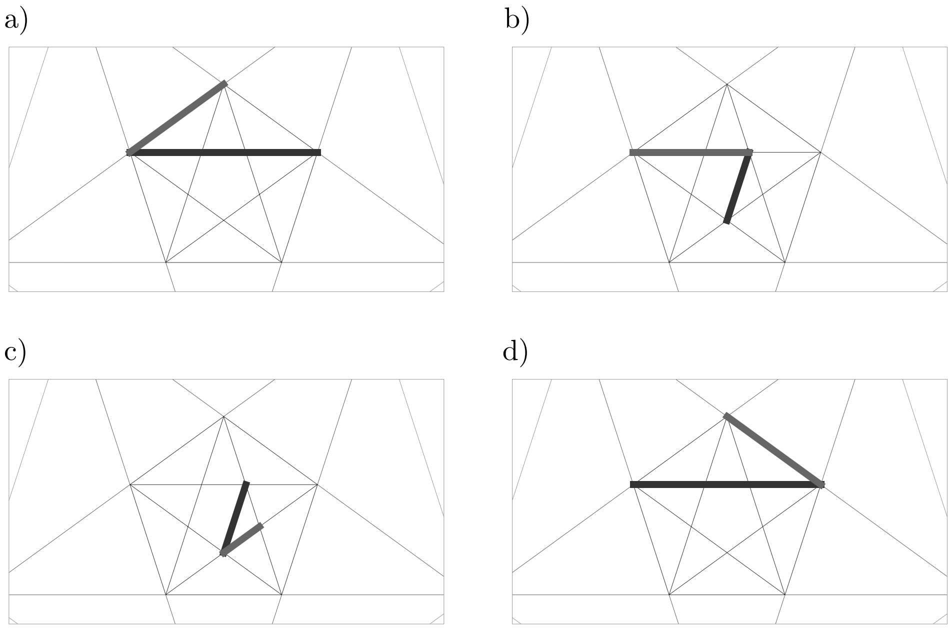

Figure 5: Frames from the animated loop *The Remainder...*(J.Jernajczyk, 2014, https://vimeo.com/93545836).

pairs of segments such as those which are created as a result of the successive subtraction of shorter segments from the longer ones. This procedure is the famous *Euclidean algorithm*, the aim of which is to determine a common measure for two segments [5]. In the case of commensurable segments, after a finite number of steps, the difference beetwen the lengths of two successive segments will finally amount to 0, hence their rational ratio can be found. Whereas in terms of incommensurable segments, such as the side and

[1]The golden proportion is one of the focus points of reflection on the relationship of mathematics with nature and art. A wide study on this issue may be found in already considered to be classic masterpiece (c.f. [7]).

diagonal of the regular pentagon, the Euclidian algorithm will never end – there will always be a nonzero remainder. The intersecting diagonals of a regular pentagon create a pentagram inside which another regular pentagon is created. The segments highlighted in the drawing (Fig. 5.c) constitute the side and the diagonal of the smaller pentagon. After rotating and magnifying the figure (Fig. 5.d) we obtain a figure which is a symmetrical reflection of the initial figure and repeating these steps leads us to the arrangement from which we had started (Fig. 5.a). Thanks to the animated loop we may observe how the irrational ratio of length of a side to length of diagonal is renewed at the successive steps. This moving picture of the golden proportion does not have a finite character, but develops in time endlessly. At the same time, in a clear and transparent way it depicts the unbreakable relationship between irrationality and infinity, which in the algebraic record is expressed in continued fraction:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

4. A Portrait of *Pi* Amongst the irrational numbers probably the most famous is *Pi*. It is commonly associated with the formula for the area of the circle $S = \pi r^2$. This simple formula is a starting point for a graphical representation of the number *Pi*. The aim is to find an image, which would convey the mathematical sense of this number and is not only its conventional representation which is the Greek symbol "π".

The number *Pi* may be calculated using a simple transformation of the formula for the area ot the circle: $\pi = S/r^2$. Let us take a look at the right hand side of the equation. S is the area of the circle. Its graphical equivalent is the disc: Let us emphasize though, that this is not the conventional rep-

resentation of the circle's area. The surface area of a disc is in a strict sense the same as the circle's area[2].

And what is the r^2 in the denominator? It is the area of a square with sides of length equal to the circle's radius. We present it in the form of a

[2] It has to be remembered at this point that each material representation of a circle is only an imperfect approximation of an ideal geometric figure.

square with the appropriate dimensions: We are now able to present the

entire right hand side of the equation $\frac{S}{r^2}$ in graphical form:

This image is then a visual equivalent of the number *Pi*. In this representation we were able to eliminate symbols almost completely. The circle's area is the circle's area, and the square's area is the square's area; and we do not need the conventional symbols S and r^2 to express these concepts. The only element which has remained from algebraic notation is the horizontal line symbolizing division. In order to get rid of it a legible graphical representation of dividing two amounts has to be found. The illustration below entitled *The Portrait of Pi* is an attempt to represent it (Fig. 6), in which dividing a circle by a square is expressed by the interpenetration of these figures. It is worth noting that in the case of a circle with radius 1, the area of the corresponding square is also 1. Such a circle alone is the representation of the number *Pi* and it does not have to be supplemented with the appropriate square[3]. However, if the length of the radius is not known, *Pi* may be represented only in the form of a general proportion between the area of a circle and the area of a square.

This proposal of visualizing the number *Pi* may seem obvious. It has to be remembered though that for many people not involved in mathematics or sciences, understanding this example is in a sense – a discovery. What is more, what turns out after conversations with the audience of the *Portrait of Pi*, even the banal fact that r^2 corresponds to the area of a square with sides of

[3] The author would like to thank an anonymous reviewer for pointing out this seemingly obvious fact which was, nevertheless omitted in the first version of the article.

Figure 6: *The Portrait of Pi* (J. Jernajczyk, digital print, 100 x 100 cm, 2011).

length equaling r is also often a discovery. This is evidence of significant gaps in basic mathematical education even amongst people with higher education. The reasons for this state of affairs may be looked for in among other things an educational program which forces a reduction in the stress placed on visual explanations which influence the students' imagination to the benefit of effective but not necessarily eye-catching symbolic notation.

5. Grasping the dimensionless point Irrational numbers along with rational numbers constitute the set of real numbers – a *continuum*, the geometrical representation of which is a straight line. Each real number – rational and irrational – corresponds with a point on that straight line. It is known however that „A point is that which has no parts" [6, p. xviii]. The question which appears here is this: whether there is a method of represent such dimensionless points? Because how can you indicate something which takes up no space? What pointer should be used? The solution of this seemingly insolvable problem turns out to be surprisingly obvious, and at the same time intuitive and pictorial. A dimensionless point may be graphically represented by cutting a straight line! Let us imagine the prosaic activity of cutting. When we cut a piece of matter of any kind its parts will end up on both sides of the blade whereas along the line of the cut nothing remains. By implementing such an abstract cut of a straight line, we represent a dimensionless point in which nothing remains because everything is either on the left or on the right of the cut. The intuition described here is a simplified, visual equivalent of the formal method of defining real numbers called *Dedekind cuts* [3]. The points represented in this way correspond to real numbers, both rational and irrational[4].

Similar visual representation was used by the Polish artist Wiesław Gołuch in one of his works. He became interested in a problem which is at a level higher – instead of speculating about the possibility of representing points, he asked the following question: how can a line which has length but does not have width be represented on the paper. In response to this question he drew a dashed line and a scissors symbol and entitled his work *Cut out this line* (Fig. 7). And indeed! Cutting the surface creates a straight line which does not have any width, similar to cutting a straight line this allows one to ilustrate a dimensionless point.

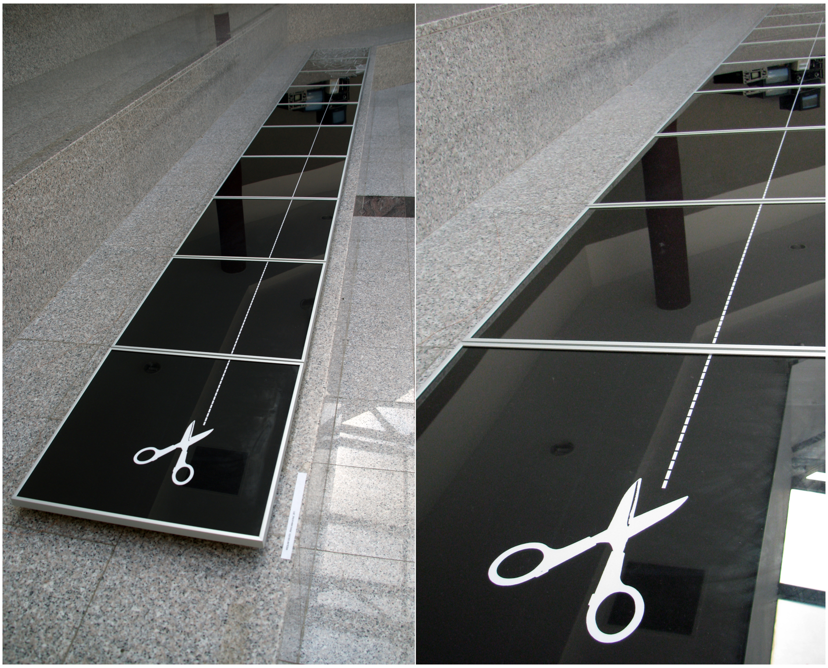

Figure 7: *Cut out this line* (W.Gołuch, digital print, 10 x 50 x 50 cm, 2011).

The artist did not refer to any advanced issues or theorems of algebra logic, topology or number theory. For him, the theoretical basis was only the commonly known definition: „A line is length without breadth" [6, p. xviii]. The rest was an act of intuition and visual imagination, which in this case could be described, as *visual thinking*, as in Arnheim [2].

The work of Gołuch is conceptual in character. Its essence is not what we see directly. The picture presented to the recipient is only a pretext and guideline for further reflection. In this particular case we are not looking at the act of cutting a line but only at graphical simbols – a dotted line and

[4] A separate problem is in what way the points indicated by those cuts exist and in what sense do they correspond with real numbers (for more see [9]).

the scissors symbol which suggest the cut. This way of creating art and its reception brings to mind abstract thinking which we customarily relate to the world of science. In mathematics we often deal with objects, which may exist only in the mind on the basis of some visual premises. For instance we can visualize an ideal circle or an infinite straight line although we cannot observe these objects in nature. We are also able to imagine dimensionless points although „no one has ever seen or touched a point" [12, p. 119].

It has to also be remembered that although experience often constitutes a basis for our abstract understanding, it does not mean that we should identify theoretical mathematical structures with phenomena which take place in the real world in a simple way. Such direct references may lead to problematic conclusions and generalizations, one example of which may be Pythagorean metaphysics postulating a world created out of numbers, or the famous paradoxes of Zeno of Elea which entangled the physical phenomena of change and movement into the traps of mathematical infinity. In order to avoid false conclusions, it is worth bearing in mind that „the real mathematical *continuum* is quite different from that of the physicists and from that of the metaphysicians" [10, p. 18].

6. Conclusions The cognitive potential hidden in images creates a wide spectrum for activities which are educational and popularize science. This article has presented examples of artistic work inspired by mathematical problems. Art, which not only derives inspiration from mathematics but also brings the recipients closer to the issues discussed and explains them, may contribute to the popularization of mathematics in a significant way[5].

Visual imagination lets us 'observe' more than we actually see. In the process of visual abstraction, among other things, we may come closer to the core of many non-trivial mathematical issues. In this way we are able to 'observe' irrational numbers hidden in the proportions of geometrical values. When we treat a picture as a starting point for further visual-intellectual reflection, we are able to imagine dimensionless points and straight lines which do not have width. Even greater possibilities, significantly widening our cognitive competences, are offered by a moving picture. A looped animation presenting the endless renewal of irrational proportions, explicitly shows the intrinsic relationship of irrationality with infinity.

In conceptual approach, art and mathematics come explicitly close. This closeness does not only refer to the similarity of the problems examined but mostly the similarity of the method used. Modeled on a scientist creating abstract problems, a visual artist discovers a way to images which may not be seen or implemented in the material world. An artist bases his work on visual imagination, which, although created under the influence of sensual data, also allows us to grasp phenomena which are far beyond sensual reality.

[5] More on the cognitive role of art may be found in [8].

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Obrazy niewymierne – wizualizacja abstrakcyjnych pojęć matematycznych

Jakub Jernajczyk

Streszczenie W niniejszym artykule chcemy zwrócić uwagę na potencjał poznawczy drzemiący w obrazie oraz w posługującej się nim sztuce. Skupimy się tu na wizualizacji podstawowych obiektów matematycznych, jakimi są liczby niewymierne. Punktem wyjścia będzie dla nas łatwy i intuicyjny przypadek pierwiastka kwadratowego z dwóch, dostrzeżony w przekątnej kwadratu. Następnie przyjrzymy się *złotej proporcji*, ukrytej w pięciokącie foremnym. Przy wizualizacji niewymiernej liczby φ posłużymy się zapętloną, niekończąca się animacją. W końcu pochylimy się nad słynną liczbą π i zaproponujemy próbę jej czysto wizualnego przedstawienia. W ostatniej części artykułu zastanowimy się nad możliwością wskazywania wymiernych i niewymiernych liczb rzeczywistych, reprezentowanych przez bezwymiarowe punkty na prostej. Spróbujemy również przedstawić na płaszczyźnie prostą, która jak wiadomo ma długość, lecz nie ma szerokości. Powyższe zagadnienia pozwolą nam dostrzec, w jakim stopniu matematyka może być inspiracją dla sztuki, a także w jaki sposób sztuka może przybliżać i wyjaśniać zagadnienia matematyki.

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