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On Finding Optimal Partitions of a Measurable Space

Abstract We present an algorithm for finding almost optimal partitions of the unit interval $[0,1)$ according to given nonatomic measures $\mu_1, \mu_2, \ldots, \mu_n$. This algorithm is based on the idea of Riemann integral and the linear programming method. We also discuss the number of cuts needed for finding the optimal partitions.

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1. Introduction Let $\mu_1, \mu_2, \ldots, \mu_n$ denote nonatomic probability measures defined on a measurable space $(\mathcal{X}, \mathcal{B})$ and let $I = \{1, 2, ..., n\}$ be a set of numbered players. By a partition $P = \{A_i\}_{i=1}^n$ of this space among the players $i \in I$ we mean a collection of $\mathcal{B}$-measurable disjoint subsets $A_1, \ldots, A_n$ of $\mathcal{X}$ whose union is equal to $\mathcal{X}$. Let $\mathscr{P}$ stand for the set of all measurable partitions $P = \{A_i\}_{i=1}^n$ of $\mathcal{X}$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote a vector with positive coordinates satisfying $\sum_{i=1}^n \alpha_i = 1$.

Definition 1.1 A partition $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}$ is said to be α-optimal if

$$v^\alpha(\vec{\mu}) := \min_{i \in I} \left[\frac{\mu_i(A_i^*)}{\alpha_i} \right] = \sup_{P \in \mathscr{P}} \min_{i \in I} \left[\frac{\mu_i(A_i)}{\alpha_i} \right], \tag{1}$$

where the number $v^\alpha(\vec{\mu})$ denotes the best value achievable for the vector measure $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ of partitioning of $\mathcal{X}$ proportional to α.

The number $v^\alpha(\vec{\mu})$ (or v^α in short) will be called the *α-optimal value* for the problem of α-partitioning of a measurable space. We say that a partition $P^e = \{A_i^e\}_{i=1}^n \in \mathscr{P}$ is *equitable optimal* if it is *α-optimal* for $\alpha = (1/n, 1/n, \ldots, 1/n)$.
The existence of α-optimal partition $P^* = \{A_i^*\}_{i=1}^n$ follows from the theorem of Dvoretzky et al. [7]:

Theorem 1.2 *If $\mu_1, \mu_2, \ldots, \mu_n$ are nonatomic finite measures defined on the measurable space $(\mathcal{X}, \mathcal{B})$ then the range $\vec{\mu}\,(\mathscr{P})$ of the mapping $\vec{\mu}\colon \mathscr{P} \to \mathbb{R}^n$ defined by*

$$\vec{\mu}\,(P) = (\mu_1(A_1), \ldots, \mu_n(A_n))\,, P = \{A_i\}_{i=1}^n \in \mathscr{P},$$

is convex and compact in $\mathbb{R}^n$.

The problem of α-optimal partitioning of a measurable space $(\mathcal{X}, \mathcal{B})$ can be viewed as a problem of fair division of an object $\mathcal{X}$ (e.g. a cake). Suppose a group $I = \{1, 2, ..., n\}$ of numbered players are interested in fair division of a cake in such way, that each of them receive at least $1/n^{th}$ value of the cake according to his own estimation. Here each measure μ_i, $i \in I$, represents the individual evaluation of sets from $\mathcal{B}$ for the i-th player. A partition $P = \{A_i\}_{i=1}^n$ of the cake $\mathcal{X}$ is called *equitable fair*, if $\mu_i(A_i) \geq \frac{1}{n}$ for all $i \in I$. This problem can be generalized if we assume that players do not have the same position in the game, but they have to divide the cake according to the individual shares $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $\sum_{i \in I} \alpha_i = 1$. In this case a partition $P = \{A_i\}_{i=1}^n$ is $\alpha-$*fair*, if $\mu_i(A_i) \geq \alpha_i$ for all $i \in I$.
There are known many algorithms of obtaining equitable partitions. A simple method of realizing the fair division for two players is "for one to cut, the other to choose". Banach and Knaster [11] found an extension of this procedure to arbitrary n. Their result was modified by Dubins and Spanier [5]. In turn Fink [9] gave an algorithm in which the number of players may be unknown. Brams and Taylor [4] found an interesting method of getting an envy free partition for which nobody would be better off with someone else's piece of cake.

Most of these procedures may be generalized so that the resulting algorithms generate α-fair partition for arbitrary sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ of positive rationals summing to one.

In the fair division theory each player is interested in getting the biggest possible piece of cake according to his own evaluation. It means that we need to find the α-optimal value v^α defined by (1) and effective methods of finding α-optimal partitions. The first estimation of the number v^α was given by Elton et al. [8] and further by Legut ([12]). An interesting algorithm for finding the bounds for the α-optimal value was found by Dall'Aglio and Di Luca [1]. In the literature of the fair division field there are known only few results concerning effective methods of finding optimal partitions. Legut and Wilczynski [13] showed how to obtain α-optimal partitions for two players. Dall'Aglio and Di Luca [2] found an algorithm for computing approximately optimal partition by construction of some maximin allocation in games of fair division.

The problem of α-optimal partitioning of a measurable space $(\mathcal{X}, \mathcal{B})$ is also considered in the classification problem (cf. [10]). Assume that a $\mathcal{X}$-

valued random variable Y has an unknown distribution μ which belongs to a given set $\{\mu_1, \mu_2, \dots, \mu_n\}$ of nonatomic probability measures defined on the space $(\mathcal{X}, \mathcal{B})$. After observing a single value y of Y it is to be decided which is the true distribution of Y. A decision rule is a measurable partition $P = \{A_i\}_{i=1}^n$ of $\mathcal{X}$ with the understanding that if Y falls in A_i then $\mu = \mu_i$ is guessed. If the guess is correct and the true value of μ is μ_i then the gain equals α_i^{-1}, otherwise it is equal to zero. It is easy to see that the α-optimal partition is the minimax solution of the problem of finding a partition which maximizes the expected risk R associated with the classification over all partitions $P = \{A_i\}_{i=1}^n \in \mathscr{P}$ and given by

$$R = \min_{i \in I} \left[\alpha_i^{-1} \mu_i(A_i)\right] = \min_{i \in I} \left[\alpha_i^{-1} P\left(Y \in A_i \mid \mu = \mu_i\right)\right].$$

2. A general form of α-optimal partitions Without loss of generality assume here and throughout the rest of the paper that $(\mathcal{X}, \mathcal{B}) = \big([0,1), \mathcal{B}_{[0,1)}\big)$, where $\mathcal{B}_{[0,1)}$ is the Borel σ-field on the unit interval $[0,1)$.

A general form of the α-optimal partition could be helpful in some cases for finding constructive methods of optimal partitioning of a measurable space. Let $S = \{s = (s_1, \dots, s_n) \in \mathbb{R}^n, s_i > 0,\ i \in I,\ \sum_{i=1}^n s_i = 1\}$ be $(n-1)$-dimensional open simplex and $\overline{S}$ be the closure of S in $\mathbb{R}^n$. We can assume that all nonatomic measures $\mu_1, \mu_2, \dots, \mu_n$ are absolutely continuous with respect to the same measure v (e.g. $\nu = \sum_{i=1}^n \mu_i$). Denote by $f_i = d\mu_i/d\nu$ the Radon-Nikodym derivatives for all $i = 1, 2, ..., n$, i.e.

$$\mu_i(A) = \int_A f_i \, d\nu, \ \text{for } A \in \mathcal{B}_{[0,1)} \text{ and } i \in I.$$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in S$, $p = (p_1, ..., p_n) \in \overline{S}$ and $i \in I$ define the following measurable sets:

$$B_i(p) = \bigcap_{j=1, j \neq i}^{n} \left\{x \in [0,1) : p_i \alpha_i^{-1} f_i(x) > p_j \alpha_j^{-1} f_j(x)\right\}$$

$$C_i(p) = \bigcap_{j=1}^{n} \left\{x \in [0,1) : p_i \alpha_i^{-1} f_i(x) \geq p_j \alpha_j^{-1} f_j(x)\right\}$$

Legut and Wilczyński [14] proved the following theorem using a minmax theorem of Sion (cf. [3]):

THEOREM 2.1 *For any $\alpha \in S$ there exists a point $p^* \in \overline{S}$ and a corresponding α-optimal partition $P^* = \{A_i^*\}_{i=1}^n$ satisfying*

(i) $B_i(p^*) \subset A_i^* \subset C_i(p^*)$,

(ii) $\dfrac{\mu_1(A_1^*)}{\alpha_1} = \dfrac{\mu_1(A_2^*)}{\alpha_n} = \ldots = \dfrac{\mu_n(A_n^*)}{\alpha_n}$

Moreover, any partition $P^* = \{A_i^*\}_{i=1}^n$ *which satisfies (i) and (ii) is* α*-optimal.*

The above theorem gives a general form of α-optimal partition but unfortunately in general case of the densities f_i, $i \in I$ finding the numbers $p_1^*, ..., p_n^*$ is not easy.

Assume now that for all $i, j \in I, i \neq j$ and for any $\gamma \in \mathbb{R}$

$$\lambda\left(\{x : x \in [0,1),\ f_i(x) - \gamma f_j(x) = 0\}\right) = 0, \tag{2}$$

where λ denotes the Lebesgue measure defined on measurable subsets of the interval $[0,1)$. Suppose we know the α-optimal value $v^\alpha(\overrightarrow{\mu})$ for densities satisfying (2) . Then we get from the Theorem 2

Corollary 2.2 *Let the densities* f_i, $i \in I$ *satisfy (2) for all* $i, j \in I, i \neq j$ *and* v^α *be the* α*-optimal value. Then a partition* $P^* = \{A_i^*\}_{i=1}^n$ *is* α*-optimal if and only if there exist numbers* $\gamma_1, ..., \gamma_n$ *such that*

$$\frac{\mu_1(A_1^*)}{\alpha_1} = \frac{\mu_2(A_2^*)}{\alpha_2} = \ldots = \frac{\mu_n(A_n^*)}{\alpha_n} = v^\alpha(\overrightarrow{\mu}),$$

where $A_i^* = \bigcap_{j=1, j\neq i}^n \{x \in [0,1) : \gamma_i f_i(x) > \gamma_j f_j(x)\}$.

If the accurate number v^α or at least its lower bound is known then the Corollary 1 can be used in some cases to obtain a partition $P^* = \{A_i^*\}_{i=1}^n$ satisfying

$$\min_{i \in I}\left[\frac{\mu_i(A_i^*)}{\alpha_i}\right] \geq v^\alpha$$

Example 1. Consider the example given by Dall'Aglio and Di Luca [1] for the density functions

$$f_1(x) \equiv 1, \quad f_1(x) = 2x, \quad f_3(x) = 30x(1-x)^4, \quad x \in [0,1].$$

It is assumed here that $\alpha = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. By using a subgradient algorithm Dall'Aglio and Di Luca [1] achieved the lower bound for the α-optimal value

$$v^\alpha \geq 1.48768$$

It follows from the Corollary 2 that the only way to get an almost α-optimal partition is to divide the interval $[0,1)$ into subintervals

$$[0, t_1), [t_1, t_2), [t_2, t_3), [t_3, 1),$$

where the numbers $0 < t_1 < t_2 < t_3 < 1$ satisfy

$$\mu_1([0, t_1) \cup [t_2, t_3)) \geq 0.495893,$$

$$\mu_2([t_3,1)) \geq 0.495893,$$
$$\mu_3([t_1,t_2)) \geq 0.495893.$$

At first we compute t_3 from the equation:

$$\mu_2([t_3,1)) = \int_{t_3}^{1} 2x\,dx = 0.495893,$$

and we get $t_3 = 0.710005$. Hence we have

$$t_1 + t_3 - t_2 = 0.495893, \text{ and } \quad t_2 - t_1 = 0.214112$$

To calculate the number t_1 we solve the equation

$$\int_{t_1}^{t_2} 30x(1-x)^4\,dx = \int_{t_1}^{t_1+0.214112} 30x(1-x)^4\,dx = 0.495893 \qquad (3)$$

Because the number 1.48768 is only a lower bound for the v^α we obtain two solutions of the equation (3)

$$t_1^{(1)} = 0.104532 \quad \text{and} \quad t_1^{(2)} = 0.110127$$

Finally we have two almost α-optimal partitions
$A_1 = [0, 0.104532) \cup [0.318644, 0.710005)$,
$A_2 = [0.710005, 1)$,
$A_3 = [0.104532, 0.318644)$
and
$B_1 = [0, 0.110127) \cup [0.324239, 0.710005)$,
$B_2 = [0.710005, 1)$,
$B_3 = [0.0.110127, 0.324239)$.
□

The construction of an α-optimal partition can be simplified when the densities f_i, $i \in I$, have the *monotone likelihood ratio property* defined below (cf. [15]). This property is often used in testing statistical hypothesis to find the uniformly most powerful test.

Definition 2.3 The family of densities f_i, $i \in I$, is said to have *monotone likelihood ratio* (MLR) if there exists a real-valued function $t(x)$ such that for any $1 \leq k < j \leq n$ the densities f_k and f_j are distinct and the ratio

$$\frac{f_k(x)}{f_j(x)}$$

is a nondecreasing function of $t(x)$.

The following well known result (cf. Hill and Tong (1989) [10], Theorem 1.6) is an immediate consequence of above property.

PROPOSITION 2.4 *Assume that family f_i, $i \in I$ has the MLR property with respect to a given function $t(x)$. Then for any $\alpha \in S$ there exist numbers $-\infty = t_0^* < t_1^* < \ldots < t_{n-1}^* < t_n^* = \infty$ such that $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}$ is an α-optimal partition, where*

$$A_i^* = \{x \in \mathcal{X} : t_{i-1}^* < t(x) \leq t_i^*\} \text{ for all }, i \in I.$$

Example 2.
Consider three nonatomic measures μ_1, μ_2, μ_3 defined on the unit interval $[0,1]$ with the probability Lebesgue density functions $f_1(x) = 2x$, $f_2(x) = 3x^2$, $f_3(x) = 4x^3$ and with the corresponding distribution functions $F_1(x) = x^2$, $F_2(x) = x^3$ and $F_3(x) = x^4$. Clearly, the family f_1, f_2, f_3 has the MLR property with respect to the function $t(x) = x$. Then for any $\alpha \in S$ the α-optimal partition has the form $A_1^* = [0, x_1^\alpha)$, $A_2^* = [x_1^\alpha, x_2^\alpha)$ and $A_3^* = (x_2^\alpha, 1]$, where the numbers $0 < x_1^\alpha < x_2^\alpha < 1$ satisfy the following condition

$$\frac{F_1(x_1^\alpha)}{\alpha_1} = \frac{F_2(x_2^\alpha) - F_2(x_1^\alpha)}{\alpha_2} = \frac{1 - F_3(x_2^\alpha)}{\alpha_3}.$$

In the case, where $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, we have $x_1^\alpha = 0.640496$, $x_2^\alpha = 0.876334$ and $v^\alpha = 1.230707$.

3. Finding almost optimal partitions We show in this section how to obtain almost α-optimal partition by approximation of the continuous densities by simple and piecewise linear functions.

Assume we are given n simple functions $h_i(x) = \sum_{j=1}^m h_{ij} I_{[a_j, a_{j+1})}(x)$ were $\{[a_j, a_{j+1})\}_{j=1}^m$ is a partition of the interval $[0,1)$ such that

$$[0,1) = \bigcup_{j=1}^m [a_j, a_{j+1}), \;\; a_1 = 0, \, a_{m+1} = 1, \;\; a_{j+1} > a_j \;\; j = 1, 2, ..., m \qquad (4)$$

and h_{ij}, $i = 1, ..., n$, $j = 1, ..., m$ are nonegative real numbers with $\int_0^1 h_i dx > 0$ for $i = 1, 2, ..., n$. Define measures $\nu_1, \nu_2, ..., \nu_n$ by

$$\nu_i(A) = \int_A h_i dx, \text{ for } A \in \mathcal{B},\, i = 1, 2, ..., n$$

These measures are not necessarily to be probabilistic but we can also define α-optimal partition $P^* = \{A_i^*\}_{i=1}^n$ and the α-optimal value $v^\alpha(\overrightarrow{\nu})$ similarly to (1). For any natural number $k \geq n-1$ denote by $\mathscr{P}(k)$ the collection of all partitions of the unit interval $[0,1)$ which are obtained by using at most k cuts.
Now we prove

PROPOSITION 3.1 *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$. Then for the measures $\nu_1, \nu_2, ..., \nu_n$ there exists an α-optimal partition $P_A^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}(mn-1)$.*

Proof. Define an $n \times m$ matrix $\mathbf{H} = [h_{ij}]_{n\times m}$ for $i = 1, 2, ..., n$, $j = 1, 2, ..., m$. Let a stochastic matrix $\mathbf{X}_H^* = [x_{ij}^*]_{n\times m}$ be a solution of the following linear programming (LP) problem:

$$\max z \tag{5}$$

with constraints

$$z = \frac{1}{\alpha_i}\sum_{j=1}^{m} x_{ij}h_{ij}(a_{j+1} - a_j),\ i = 1, 2, ..., n;$$

$$\sum_{i=1}^{n} x_{ij} = 1,\ j = 1, 2, ..., m;$$

$$x_{ij} \geq 0,\ \forall i = 1, 2, ..., n;\ j = 1, 2, ..., m.$$

Denote by $v_{LP}(\mathbf{H})$ the optimal value of the objective function z. We construct a partition $P_A = \{A_i^*\}_{i=1}^n$ of the interval $[0, 1)$ such that

$$\frac{\nu_i(A_i^*)}{\alpha_i} = v_{LP}(\mathbf{H}) \quad \text{for} \quad i = 1, 2, ..., n.$$

For $j = 1, 2, ..., m$ we can find subpartitions $\{[b_i^j, b_{i+1}^j)\}_{i=1}^n$ of intervals $[a_j, a_{j+1})$ with

$$[a_j, a_{j+1}) = \bigcup_{i=1}^{n} [b_i^j, b_{i+1}^j)$$

where $b_i^j \in [a_j, a_{j+1})$ are numbers satisfying the following conditions

$$\frac{b_{i+1}^j - b_i^j}{a_{j+1} - a_j} = x_{ij}^*,\ i = 1, 2, ..., n$$

and $b_1^j = a_j, b_{n+1}^j = a_{j+1}$. If $x_{ij}^* = 0$ for some $i = 1, 2, ..., n$ then $b_{i+1}^j = b_i^j$ and we put $[b_i^j, b_{i+1}^j) = \emptyset$ in this case. Define a partition $P_A = \{A_i^*\}_{i=1}^n$ by

$$A_i^* = \bigcup_{j=1}^{m} [b_i^j, b_{i+1}^j),\ i = 1, 2, ..., n$$

Hence we have

$$\frac{\nu_i(A_i^*)}{\alpha_i} = \frac{1}{\alpha_i}\sum_{j=1}^{m} h_{ij}(b_{i+1}^j - b_i^j) = \frac{1}{\alpha_i}\sum_{j=1}^{m} x_{ij}^* h_{ij}(a_{j+1} - a_j),\ i = 1, 2, ..., n;$$

Now we prove that the partition $P_A = \{A_i^*\}_{i=1}^n$ is $\alpha-$optimal. Suppose that a partition $P_B = \{B_i\}_{i=1}^n$ is $\alpha-$optimal for the measures $\nu_1, \nu_2, ..., \nu_n$ and

$$\min_i \left\{\frac{\nu_i(B_i)}{\alpha_i}\right\} > \frac{\nu_i(A_i^*)}{\alpha_i} = v_{LP}(\mathbf{H}) \quad \text{for} \quad i = 1, 2, ..., n.$$

If $\nu_i([a_j,a_{j+1}))>0$ we denote

$$y_{ij}=\frac{\nu_i(B_i\cap[a_j,a_{j+1}))}{\nu_i([a_j,a_{j+1}))}=\frac{\nu_i(B_i\cap[a_j,a_{j+1}))}{h_{ij}(a_{j+1}-a_j)},\ i=1,2,...,n,\ j=1,2,...,m$$

otherwise we put $y_{ij}=0$. Now we show that $\mathbf{Y}=[y_{ij}]$ is a stochastic matrix. Simple functions h_i are constant on intervals $[a_j,a_{j+1})$, then we have $\nu_i(B_i\cap[a_j,a_{j+1}))=h_{ij}\lambda(B_i\cap[a_j,a_{j+1}))$, where λ is the Lebesgue measure. Therefore we obtain

$$\sum_{i=1}^{n}y_{ij}=\sum_{i=1}^{n}\frac{\nu_i(B_i\cap[a_j,a_{j+1}))}{\nu_i([a_j,a_{j+1}))}=\sum_{i=1}^{n}\frac{h_{ij}\lambda(B_i\cap[a_j,a_{j+1}))}{h_{ij}(a_{j+1}-a_j)}=$$

$$=\frac{1}{a_{j+1}-a_j}\sum_{i=1}^{n}\lambda(B_i\cap[a_j,a_{j+1}))=1$$

Finally for $i=1,2,...,n$ we have

$$\frac{\nu_i(B_i)}{\alpha_i}=\frac{1}{\alpha_i}\sum_{j=1}^{m}\nu_i(B_i\cap[a_j,a_{j+1}))=\frac{1}{\alpha_i}\sum_{j=1}^{m}y_{ij}h_{ij}(a_{j+1}-a_j)>\frac{\nu_i(A_i^*)}{\alpha_i}=$$

$$=\frac{1}{\alpha_i}\sum_{j=1}^{m}x_{ij}^*h_{ij}(a_{j+1}-a_j).$$

The above inequality contradicts that the stochastic matrix $\mathbf{X}_H^*=[x_{ij}^*]_{n\times m}$ is a solution of the linear programming problem (5). Thus constructed partition $P_A=\{A_i^*\}_{i=1}^n$ is $\alpha-$optimal. Moreover it is easy to verify that the partition P_A can be obtained using at most $m(n-1)+m-1=mn-1$ cuts. $\square$

The above constructive method of finding α-optimal partition for simple functions is based on an interesting method found by Demko and Hill [6] for optimal randomized solution of equitable distribution of indivisible objects.

Consider now measurable nonnegative bounded functions f_i, $i\in I$ defined on the unit interval $[0,1)$ and corresponding finite measures $\nu_1,\nu_2,...,\nu_n$ defined by

$$\nu_i(A)=\int_A f_i d\lambda,\ \text{for } A\in\mathcal{B},\ i=1,2,...,n$$

Assume that functions f_i, $i\in I$ are piecewise linear (PWL). It means that there exists a partition $\{[a_j,a_{j+1})\}_{j=1}^m$ of the interval $[0,1)$ satisfying (4) and numbers $c_{ij},d_{ij}\in\mathbb{R}$ satisfying

$$f_i(x)=c_{ij}x+d_{ij},\text{for } x\in[a_j,a_{j+1}),\ i=1,2,...,n,\ j=1,2,...,m$$

For PWL functions given above define an $n\times m$ matrix $\mathbf{A}=[a_{ij}]_{n\times m}$ by

$$a_{ij}=\nu_i([a_j,a_{j+1}))=\int_{a_j}^{a_{j+1}}f_i d\lambda\ \ i=1,2,...,n,\ j=1,2,...,m \tag{6}$$

Let the stochastic matrix $\mathbf{X}_A^* = [x_{ij}^*]_{n\times m}$ be a solution of the following linear programming problem:

$$\max z \tag{7}$$

with constraints

$$z = \frac{1}{\alpha_i}\sum_{j=1}^{m} x_{ij}a_{ij},\ \ i = 1, 2, ..., n;$$

$$\sum_{i=1}^{n} x_{ij} = 1,\ \ j = 1, 2, ..., m;$$

$$x_{ij} \geq 0,\ \ \forall i = 1, 2, ..., n;\ \ j = 1, 2, ..., m.$$

Denote by $v_{LP}(\mathbf{A})$ the optimal value of the objective function z.
Now we prove:

Proposition 3.2 *For given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in S$ there exists a partition $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}(m(2n-1)-1)$ such that*

$$\frac{\nu_i(A_i^*)}{\alpha_i} = v_{LP}(\mathbf{A}),\ i = 1, 2, ..., n$$

Moreover each A_i^* can be chosen to be a union of at most $2m$ intervals.

Before proving the above Proposition we need to show the following

Lemma 3.3 *Let $g_1, g_2, ..., g_n$ be nonnegative linear functions defined on the interval $[a, b)$ such that*

$$g_i(x) = c_i x + d_i,\ \ \forall\ i = 1, 2, ..., n.$$

for some numbers c_i, d_i, $i = 1, ..., n$. Then for arbitrary nonnegative numbers $\beta_1, \beta_2, ..., \beta_n$ with $\sum_{i=1}^n \beta_i = 1$ there exists a partition $\{B_i\}_{i=1}^n$ of the interval $[a, b)$ such that

$$\int_{B_i} g_i d\lambda = \beta_i \int_a^b g_i d\lambda = \beta_i g_i\left(\frac{a+b}{2}\right)(b-a),\ \ \forall\ i = 1, 2, ..., n, \tag{8}$$

where λ denotes the Lebesgue measure. Moreover for all $i = 1, 2, ..., n$ each set B_i can be chosen as a union of at most two subintervals.

Proof. If $\beta_i = 0$ we put $B_i = \emptyset$. Without loss of generality we may assume that $\beta_i > 0$ for $i = 1, 2, ..., k \leq n$. Define by recursion two increasing finite sequences of numbers $\{s_i\}_{i=1}^k$,$\{t_i\}_{i=1}^k$ as follows

$$s_1 = a,\ \ t_k = b,$$

and

$$s_{i+1} = s_i + \frac{\beta_i(b-a)}{2}, \quad t_{k-i} = t_{k-i+1} - \frac{\beta_i(b-a)}{2}$$

for all $i = 1, 2, ..., k-1$. For $i = 1, 2, ..., k-1$ we define the set B_i by

$$B_i = [s_i, s_{i+1}) \cup [t_{k-i}, t_{k-i+1})$$

and for $i = k$ we put $B_k = [s_k, t_k)$. It is easy to verify that the partition $\{B_i\}_{i=1}^n$ satisfies (8). The proof of Lemma 1 is complete. □

Proof of Proposition 3
For fixed $1 \leq j \leq m$ we make use of the Lemma 2 for $g_i = f_i$, $i = 1, 2, ..., n$, $[a, b) = [a_j, a_{j+1})$ and $\beta_i = x_{ij}^*$, $i = 1, 2, ..., n$. As in proof of Lemma 2 we construct a partition $\{A_{ij}\}_{i=1}^n$ of the subinterval $[a_j, a_{j+1})$ satisfying

$$\int_{A_{ij}} f_i d\lambda = \nu_i(A_{ij}) = x_{ij}^* a_{ij}, \tag{9}$$

where $[x_{ij}^*]_{n\times m}$ is a stochastic matrix being solution of the LP problem (5). Define a partition $P^* = \{A_i^*\}_{i=1}^n$ by

$$A_i^* = \bigcup_{j=1}^m A_{ij}, \; i \in I. \tag{10}$$

We have

$$\frac{\nu_i(A_i^*)}{\alpha_i} = \frac{1}{\alpha_i}\sum_{j=1}^m \nu_i(A_{ij}) = \frac{1}{\alpha_i}\sum_{j=1}^m x_{ij}^* a_{ij} = v_{LP}(\mathbf{A}), \; i = 1, 2, ..., n$$

Moreover it follows from the Lemma 2 that the partition $P^* = \{A_i^*\}_{i=1}^n$ can be obtained by at most $m(2n-1)-1$ cuts of the unit interval $[0, 1)$. Unfortunately the partition P^* constructed in the proof of Proposition 3 does not need to be the $\alpha-$optimal, but it can be used to obtain almost $\alpha-$optimal partitions by approximation of the densities $\{f_i\}_{i=1}^n$ by PWL functions.

Now using the idea of the Riemann integration we present a method of obtaining almost α-optimal partition for arbitrary continuous densities $\{f_i\}_{i=1}^n$.

Definition 3.4 For a given $\varepsilon > 0$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in S$ a partition $P_\varepsilon = \{A_i^\varepsilon\}_{i=1}^n$ is said to be $\varepsilon - \alpha$-optimal if the following inequality holds

$$v^\alpha(\varepsilon) := \min_{i\in I}\left[\frac{\mu_i(A_i^\varepsilon)}{\alpha_i}\right] \geq \sup_{P\in\mathscr{P}} \min_{i\in I}\left[\frac{\mu_i(A_i)}{\alpha_i}\right] - \varepsilon$$

Suppose we are given arbitrary continuous densities $\{f_i\}_{i=1}^n$ defined on the unit interval $[0,1)$. To get $\varepsilon - \alpha$-optimal partitions for these functions we can use approximation by simple functions. Increasing the number of subintervals dividing $[0,1)$ we get the desired accuracy. For any $\varepsilon > 0$ we can find a partition $\{[a_j, a_{j+1})\}_{j=1}^m$ of the interval $[0,1)$ and two collections of nonnegative simple functions

$$h_i^L(x) = \sum_{j=1}^m h_{ij}^L I_{[a_j,a_{j+1})}(x) \text{ and } h_i^U(x) = \sum_{j=1}^m h_{ij}^U I_{[a_j,a_{j+1})}(x)$$

such that:

1. $h_i^L \leq f_i \leq h_i^U$ for $i \in I$,
2. $\max_{i\in I} \left\{ \frac{1}{\alpha_i} \int_0^1 (h_i^U - h_i^L)\, d\lambda \right\} < \varepsilon$

Denote $\mathbf{H}_L = [h_{ij}^L]_{n\times m}$ and $\mathbf{H}_U = [h_{ij}^U]_{n\times m}$. For these matrices we can solve suitable LP problems (cf. Proposition 1) and obtain the lower and upper bounds for the number v^α:

$$v_{LP}(\mathbf{H}_L) \leq v^\alpha \leq v_{LP}(\mathbf{H}_U).$$

where $(v_{LP}(\mathbf{H}_U) - v_{LP}(\mathbf{H}_L)) < \varepsilon$. Moreover, as in Proposition 1, we can also construct a partition $P_\varepsilon = \{A_i^\varepsilon\}_{i=1}^n$ satisfying

$$\min_{i\in I} \left[\frac{\mu_i(A_i^\varepsilon)}{\alpha_i} \right] \geq v_{LP}(\mathbf{L}_U) - \varepsilon$$

Now we present an example illustrating described methods of finding almost $\alpha-$ optimal partitions.

Example 3.

Consider three nonatomic measures μ_1, μ_2, μ_3 defined on the unit interval $[0,1)$ with the density functions

$$f_1(x) = \frac{6}{5}\, x^2 + \frac{3}{5}, \quad f_2(x) = \frac{3}{2}\,(1-x^2), \quad f_3(x) = \frac{7}{6}\,(1-x^6).$$

The first function f_1 is convex while the functions f_2 and f_3 are concave. We find almost α-optimal partition and almost α-optimal value v^α for $\alpha = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ by dividing equally the interval $[0,1)$ into $m = 12$ subintervals

$$\left\{ \left[\frac{j-1}{m}, \frac{j}{m} \right) \right\}_{j=1}^m .$$

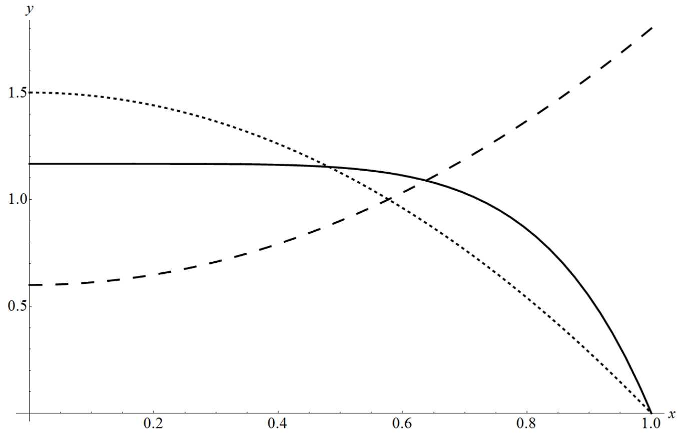

Figure 1: The density functions considered in Example 3, f_1 : tiny dashing, f_2 : large dashing, f_3 : continuous line.

For finding the lower bound of v^α to get better estimation we use approximation of the functions f_i, $i = 1, 2, 3$ by PWL functions g_i, $i = 1, 2, 3$.
Let g_1 be any PWL function satisfying

$$g_1 \leq f_1 \text{ and } g_1\left(\frac{2(j-1)+1}{m}\right) = f_1\left(\frac{2(j-1)+1}{m}\right), \; j = 1, 2, ..., m$$

For $i = 1, 2$ we put

$$g_i(x) = m\left[f_i\left(\frac{j}{m}\right) - f_i\left(\frac{j-1}{m}\right)\right]x + jf_i\left(\frac{j-1}{m}\right) - (j-1)f_i\left(\frac{j}{m}\right), \; j = 1, 2, ..., m$$

Define matrix $\mathbf{A}_L = [a^L_{ij}]_{n\times m}$ by

$a^L_{1j} = \dfrac{1}{m} f_1\left(\dfrac{2(j-1)+1}{m}\right)$, for $j = 1, ..., m$,

$a^L_{ij} = \dfrac{f_i\left(\frac{j-1}{m}\right) + f_i\left(\frac{j}{m}\right)}{2m}$, for $i = 2, 3$ and $j = 1, ..., m$.

For finding the upper bound we define a matrix $\mathbf{A}_U = [a^U_{ij}]_{n\times m}$ by

$a^U_{1j} = \dfrac{f_1\left(\frac{j-1}{m}\right) + f_i\left(\frac{j}{m}\right)}{2m}$, for $j = 1, ..., m$,

$a^U_{ij} = \dfrac{1}{m} f_i\left(\dfrac{2(j-1)+1}{m}\right)$, for $i = 2, 3$ and $j = 1, ..., m$.

Solving the LP problem (5) for $\alpha = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and matrices $\mathbf{A}_L, \mathbf{A}_U$ we get

$$v_{LP}(\mathbf{A}_L) = 1.32916 \quad \text{and} \quad v_{LP}(\mathbf{A}_U) = 1.33164.$$

Hence we have the following estimation of an α-optimal value v^α

$$1.32916 \leq v^\alpha \leq 1.33164. \tag{11}$$

We also obtain the stochastic matrix

$$\mathbf{X_L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.61869 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0.66842 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.33158 & 1 & 1 & 1 & 1 & 0.38131 & 0 & 0 & 0 \end{bmatrix}.$$

Now we are ready to construct the almost equitable optimal partition $P_L = \{A_i^L\}_{i=1}^n$ by analysing the matrix $\mathbf{X_L}$. The first player receives the interval $\left[\frac{3}{4}, 1\right)$, the second player $\left[0, \frac{1}{4}\right)$ and the third $\left[\frac{1}{3}, \frac{2}{3}\right)$. The remaining two intervals $\left[\frac{1}{4}, \frac{1}{3}\right)$ and $\left[\frac{2}{3}, \frac{3}{4}\right)$ are divided using the method given in the proof of the Lemma 2. The interval $\left[\frac{1}{4}, \frac{1}{3}\right)$ is to be divided among the second and the third player according to the shares $x_{24}^L = 0.66842$ and $x_{34}^L = 0.33158$ respectively. Similarly we divide the interval $\left[\frac{2}{3}, \frac{3}{4}\right)$ among the first and the third player according to the shares $x_{19}^L = 0.61869$ and $x_{39}^L = 0.38131$ respectively. Finally we obtain the almost equitable optimal partition $P_L = \{A_i^L\}_{i=1}^n$ as follows

$$A_1^L = [0.66667, 0.69245) \cup [0.72422, 1),$$

$$A_2^L = [0, 0.27785) \cup [0.30548, 0.33333),$$

$$A_3^L = [0.27785, 0.30548) \cup [0.33333, 0.66667) \cup [0.69245, 0.72422).$$

Now we check the measures of obtained partition for each player

$$3 \cdot \mu_1(A_1^L) = 1.32985, \quad 3 \cdot \mu_2(A_2^L) = 1,33068, \quad 3 \cdot \mu_3(A_3^L) = 1,33140.$$

Then it turns out that we have a little better improvement of the lower bound for the α-optimal value v^α

$$v^\alpha \geq 1.32985 > v_{LP}(\mathbf{A}_L) = 1.32916.$$

We can increase the accuracy of the estimation (11) of the number v^α using a larger number m in the LP problem. For $m = 100$ we get estimation

$$1.3336228 \leq v^\alpha \leq 1.3336580. \tag{12}$$

Looking at the form of the obtained almost equitable optimal partition $P_L = \{A_i^L\}_{i=1}^n$ we could consider the hypothesis that the equitable optimal partition can be effected by using only two cuts $0 < t_1 < t_2 < 1$ of the unit interval $[0, 1)$. Based on the estimation (12) assume that $v_\varepsilon^\alpha = 1.333623$. Similarly to the methods given in Example 1 we solve the following equations to find t_1 and t_2

$$3\int_{t_2}^{1} f_1(x)\,dx = 1.333623,$$

and

$$3\int_{0}^{t_1} f_2(x)\,dx = 1.333623.$$

We get the solutions $t_1 = 0.305904$ and $t_2 = 0.698531$. I turns out that

$$3\int_{t_1}^{t_2} f_3(x)\,dx = 1.333743$$

and then we obtain much better partition

$$P_L^\varepsilon = \{[0.698531, 1), [0, 0.305904), [0.305904, 0.698531)\}$$

which is effected by using only two cuts.

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O uzyskaniu optymalnych podziałów przestrzeni mierzalnej

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Streszczenie W pracy zaprezentowano algorytm uzyskania prawie optymalnego podziału odcinka jednostkowego $[0,1)$ według danych probabilistycznych miar bezatomowych $\mu_1, \mu_2, ..., \mu_n$. Algorytm ten oparty jest na idei całki Riemanna oraz wykorzystuje metodę programowania liniowego. Ponadto autorzy podają wystarczającą liczbę cięć potrzebnych do uzyskania podziałów optymalnych.

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