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A Résumé on Interval Runge-Kutta Methods

Abstract The paper presents explicit and implicit interval methods of Runge-Kutta type. Such methods introduce the errors of methods. It means that this kind of errors are included in the interval solutions obtained. Applying these methods for solving the initial value problem in floating-point interval arithmetic we can obtain solutions in the form of intervals which contain all possible numerical errors. Numerical examples are presented.

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1. Introduction. Many scientific and engineering problems are described in the form of ordinary differential equations. If such equations cannot be solved analytically, we use approximate methods to solve them, usually providing all calculations in floating-point arithmetic.

One of the most popular one-step methods to solve the initial value problem in ordinary differential equations are the methods of Runge-Kutta. These methods yield approximate solutions of the problem considered. From the well-known theorems we can evaluate the local truncation error of the solution obtained, although it can not be easy. Moreover, the floating-point arithmetic causes two kinds of errors: representation errors and rounding errors. It means that the computed results could be inaccurate, not only because of the method applied, but also because to the arithmetic used.

In interval methods of Runge-Kutta types the errors of the methods are included in the interval solutions obtained. Applying such methods in floating-point interval arithmetic (see e.g. [7] or [22]) we can obtain solutions in the form of intervals which contain all possible numerical errors.

In this paper, after recalling conventional Runge-Kutta methods (Section 2), we present explicit (Section 3) and implicit (Section 4) interval methods of Runge-Kutta types. For both types of interval methods we quote relevant theorems on the inclusion of the exact solution in interval solutions and on estimations of the widths of interval solutions obtained. The second kind of theorems present also the orders of the interval methods considered. Since the paper is a résumé on interval Runge-Kutta methods, we present these theorems without the proofs (the proofs can be found in the given references). In the last section some numerical examples with interval methods considered are presented.

2. Conventional Runge-Kutta Methods. For the initial value problem

$$y' = f(t, y(t)), \qquad y(0) = y_0, \tag{2.1}$$

where $t \in [0, a]$, $y \in \mathbf{R}^N$ and $f\colon [0, a] \times \mathbf{R}^N \to \mathbf{R}^N$, the well-known explicit m-stage Runge-Kutta methods are of the form (see e.g. [2–4, 8–10, 15, 16, 25, 27])

$$y_{k+1} = y_k + h \sum_{i=1}^{m} w_i \kappa_{ij}, \tag{2.2}$$

where

$$\begin{aligned} \kappa_{1k} &= f(t_k, y_k), \\ \kappa_{ij} &= f\Big(t_k + c_i h, y_k + h \sum_{j=1}^{i-1} a_{ij}\kappa_{ij}\Big), \qquad i = 2, 3, \ldots, m, \end{aligned} \tag{2.3}$$

and where the coefficients w_i, c_i and a_{ij} are some parameters,

$$c_i = \sum_{j=1}^{i-1} a_{ij}, \qquad i > 1, \qquad c_1 = 0,$$

$h = t_{k+1} - t_k$ and y_k is an approximation for the exact value $y(t_k)$, $k = 1, 2, \ldots$. It is convenient to present the coefficients in a form of an array, called the Butcher table:

$$\begin{array}{c|ccccc} 0 & & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ \vdots & \vdots & \vdots & & & \\ c_m & a_{m1} & a_{m2} & \cdots & a_{m,m-1} & \\ \hline & w_1 & w_2 & \cdots & w_{m-1} & w_m \end{array}$$

If we do not assume that $c_1 = 0$, then we can get more general, implicit m-stage Runge-Kutta methods in which [2–4, 8–10, 15, 16, 25, 27]

$$K_{ik} = f\Big(t_k + c_i h, y_k + h \sum_{j=1}^{m} a_{ij} K_{jk}\Big), \qquad i = 1, 2, \ldots, m, \tag{2.4}$$

where

$$c_i = \sum_{j=1}^{m} a_{ij}.$$

The local truncation error of step $k+1$ for a Runge-Kutta method (explicit and implicit) of order p can be written in the form

$$\begin{aligned} r_{k+1}(h) &= \psi(t_k, y(t_k))h^{p+1} + O(h^{p+2}) = \\ &= r_{k+1}^{(p+1)}(0)\frac{h^{p+1}}{(p+1)!} + r_{k+2}^{(p+1)}(\theta h)\frac{h^{p+2}}{(p+2)!}, \qquad 0 < \theta < 1. \end{aligned} \tag{2.5}$$

The function $\psi(t,y) = \psi(t,y(t))$, occurring in the above equation, depends on coefficients w_i, c_i, a_{ij} and on partial derivatives of $f(t,y) = f(t,y(t))$. The form of $\psi(t,y)$ is very complicated and cannot be written in a general form for an arbitrary p. But this form is very important from the point of view of interval methods developed.

From the conditions $r_{k+1}^{(l)}(0) = 0$ (for $l = 1,\ 2,\ \ldots,\ p$) follow the equations for determining the coefficients w_i, c_i and a_{ij}. Unfortunately, there are fewer equations than the number of unknowns, and usually we consider some special cases. In order to reduce the number of equations for coefficients in the case of implicit Runge-Kutta methods, one can consider semi-implicit, diagonally implicit, symmetric, symplectic etc. methods.

It can be proved that if $p_{\max}(m)$ denotes the maximum order of the m-stage explicit Runge–Kutta method, then we have

$$\begin{aligned}
&p_{\max}(m) = m, && m = 1,\ 2,\ 3,\ 4,\\
&p_{\max}(m) = m - 1, && m = 5,\ 6,\ 7,\\
&p_{\max}(m) = m - 2, && m = 8,\ 9,\\
&p_{\max}(m) = leqm - 2, && m \geqslant 10.
\end{aligned}$$

In the case of implicit Runge–Kutta methods for each m there exists a method with maximum order $p_{\max}(m) = 2m$.

3. Explicit Interval Methods. The basis of interval computations one can find e.g. in [1, 7, 22–24, 26].

Let us denote

- Δ_t and Δ_y — bounded sets in which the function $f(t,y)$ occurring in (2.1), is defined, i.e.

$$\Delta_t = \{t \in \mathbf{R} : 0 \leqslant t \leqslant a\},$$
$$\Delta_y = \{y = (y_1, y_2, \ldots, y_N)^T \in \mathbf{R}^N : \underline{b_i} \leqslant y_i \leqslant \overline{b_i},\ i = 1,\ 2,\ \ldots,\ N\},$$

- $F(T,Y)$ — an interval extension of $f(t,y)$, where an interval extension of the function

$$f\colon \mathbf{R} \times \mathbf{R}^N \supset \Delta_t \times \Delta_y \to \mathbf{R}^N$$

we call a function

$$F\colon \mathbf{IR} \times \mathbf{IR}^N \supset \mathbf{I}\Delta_t \times \mathbf{I}\Delta_y \to \mathbf{IR}^N$$

such that

$$(t,y) \in (T,Y) \implies f(t,y) \in F(T,Y),$$

and where $\mathbf{IR}$ and $\mathbf{IR}^N$ denote the space of real intervals, and the space of N-dimensional real interval vectors, respectively,

- $\Psi(T,Y)$ — an interval extension of $\psi(t,y)$ (see (2.5)),

and let us assume that:

- the function $F(T,Y)$ is defined and continuous for all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
- the function $F(T,Y)$ is monotonic with respect to inclusion, i.e.

$$T_1 \subset T_2 \wedge Y_1 \subset Y_2 \implies F(T_1,Y_1) \subset F(T_2,Y_2).$$

- for each $T \subset \Delta_t$ and for $Y \subset \Delta_y$ there exists a constant $\Lambda > 0$ such that

$$w(F(T,Y)) \leqslant \Lambda(w(T) + w(Y)). \tag{3.1}$$

where $w(A)$ denotes the width of the interval A (if $A = (A_1, A_2, \dots, A_N)^T$ then the number $w(A)$ is defined by $w(A) = \max_{i=1,2,\dots,N} w(A_i)$),

- the function $\Psi(T, Y)$ is defined for all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
- the function $\Psi(T, Y)$ is monotonic with respect to inclusion.

For $t_0 = 0$ and $y_o \in Y(0) = Y_0$ where the interval Y_0 is given, the explicit m-stage interval method of Runge–Kutta type, introduced by Shokin et al. [14,26], is defined as follows:

$$Y_{k+1} = Y_k + h \sum_{i=1}^{m} w_i K_{ik} + (\Psi(T_k, Y_k) + [-\alpha, \alpha]) h^{p+1}, \qquad k = 0,\ 1,\ \dots, n-1, \tag{3.2}$$

where $Y_k = Y(t_k)$ and Y_k depends also on n, $K_{ik} = K_{ik}(h)$.

$$\begin{aligned} K_{1k} &= F(T_k, Y_k), \\ K_{ik} &= F\left(T_k + c_i h, Y_k + h \sum_{j=1}^{i-1} a_{ji} K_{jk}\right), \qquad i = 2,\ 3,\ \dots,\ m, \end{aligned} \tag{3.3}$$

α is a constant such that $\alpha = M h_0$, where h_0 is a given initial step size, p denotes the order of adequate conventional method, and (see (2.5))

$$\left| \frac{r_{k+1}^{(p+2)}(\theta h)}{(p+2)!} \right| \leqslant M, \qquad 0 < \theta < 1. \tag{3.4}$$

The step size h of the methods (3.2)–(3.3), which fulfills the condition $0 < h \leqslant h_0$, is given by

$$h = \frac{\xi_m^*}{n}, \tag{3.5}$$

where $\xi_m^* = \min\{\xi_0, \xi_2, \dots, \xi_m\}$, and where for $Y_0 \subset \Delta_y$ and $y_0 \in Y_0$ the numbers $\xi_2 > 0$, $\xi_3 > 0$, $\dots$, $\xi_m > 0$ are such that

$$Y_0 + \xi c_i F(\Delta_t, \Delta_y) \subset \Delta_y, \qquad i = 2,\ 3,\ \dots,\ m,$$

and the number $\xi_0 > 0$ fulfills the condition

$$Y_0 + \xi_0 \sum_{i=1}^{m} w_i F(\Delta_t, \Delta_y) + (\Psi(\Delta_t, \Delta_y) + [-\alpha, \alpha]) h_0^p \subset \Delta_y.$$

We divide the interval $[0, \xi_m^*]$ into n parts by the points $t_k = kh$ ($k = 0,\ 1,\ \dots,\ n$), whereas the intervals T_k, which appear in the methods (3.2)–(3.3), are selected in such a way that

$$t_k = kh \in T_k \subset [0, \xi_m^*].$$

On the basis of (3.2)–(3.3) we can present interval methods corresponding with some well-known conventional explicit Runge–Kutta methods:

- the interval version of Euler's method

$$Y_{k+1} = Y_k + hF(T_k, Y_k) + (\Psi(T_k, Y_k) + [-\alpha, \alpha]) h^2, \qquad k = 0,\ 1,\ \dots, n-1, \tag{3.6}$$

- the interval version of Euler's improved method

$$\begin{gathered} Y_{k+1} = Y_k + hK_{2k} + (\Psi(T_k, Y_k) + [-\alpha, \alpha]) h^3, \\ K_{1k} = F(T_k, Y_k),\ K_{2k} = F\left(T_k + \frac{h}{2}, Y_k + \frac{h}{2} K_{1k}\right), \quad k = 0,\ 1,\ \dots, n-1, \end{gathered} \tag{3.7}$$

- the interval version of the Euler-Cauchy method

$$\begin{gathered} Y_{k+1} = Y_k + \tfrac{h}{2}(K_{1k} + K_{2k}) + (\Psi(T_k, Y_k) + [-\alpha, \alpha])h^3, \\ K_{1k} = F(T_k, Y_k),\ K_{2k} = F\left(T_k + h, Y_k + hK_{1k}\right), \quad k = 0,\ 1,\ \ldots, n-1, \end{gathered} \tag{3.8}$$

- the interval version of the Runge–Kutta method

$$\begin{gathered} Y_{k+1} = Y_k + \tfrac{h}{6}(K_{1k} + 2K_{2k} + 2K_{3k} + K_{4k}) + (\Psi(T_k, Y_k) + [-\alpha, \alpha])h^5, \\ K_{1k} = F(T_k, Y_k), \qquad K_{2k} = F\left(T_k + \frac{h}{2}, Y_k + \frac{h}{2}K_{1k}\right), \\ K_{3k} = F\left(T_k + \frac{h}{2}, Y_k + \frac{h}{2}K_{2k}\right), \qquad K_{4k} = F(T_k + h, Y_k + hK_{3k}) \\ k = 0,\ 1,\ \ldots, n-1, \end{gathered} \tag{3.9}$$

In (3.6) the function $\Psi(T, Y)$ is an interval extension of the function $\psi(t, y)$ occurring in (2.5) for $p = 1$, in (3.7) and (3.8) for $p = 2$, while in (3.9) for $p = 4$ (the function $\psi(t, y)$ is different for the different values of p). Note that in (3.6), (3.7) and (3.9) the values of α are also different (see the definition of the constant α and (3.4)).

For the explicit m-stage interval method of Runge–Kutta type we can prove

THEOREM 3.1 *For the exact solution $y(t)$ of the initial value problem* (2.1) *we have $y(t_k) \in Y_k$ $(k = 0\ ,\ 1,\ \ldots,\ n)$, where Y_k are obtained from the method* (3.2)–(3.3).

The proof of the above theorem (the mathematical induction with respect to k) can be found in [5, 17, 22].

We also have

THEOREM 3.2 *If Y_k $(k = 1,\ 2,\ \ldots,\ n)$ are obtained from* (3.2)–(3.3) *, then*

$$w(Y_k) \leqslant Qh^p + Rw(Y_0) + S \max_{l=1,2,\ldots,n} w(T_l),$$

where Q, R and S denote some nonnegative constants.

In the proof, which can be found in [5, 7, 22], we use the following estimation:

$$w(K_{ik}) \leqslant \Lambda(w(T_k) + w(Y_k)) \sum_{j=0}^{i} \mu_{ij}(h\Lambda)^j.$$

where μ_{ij} denote some nonnegative constants and Λ is given by (3.1).

4. Implicit Interval Methods. Let $F(T, Y)$ and $\Psi(T, Y)$ be interval extensions of $f(t, y)$ and $\psi(t, y)$, respectively, and fulfill the same assumptions as in Section 3. For $t_0 = 0$ and $y_0 \in Y_0$, where the interval Y_0 is given, an implicit m-stage interval method of Runge–Kutta type, which solves the initial value problem (2.1), is given by [5, 6, 17, 22]

$$Y_{k+1} = Y_k + h\sum_{i=1}^{m} w_i K_{ik} + (\Psi(T_k, Y_k) + [-\alpha, \alpha])h^{p+1}, \qquad k = 0,\ 1,\ \ldots, n-1, \tag{4.1}$$

where

$$K_{ik} = F\Big(T_k + c_i h, Y_k + h\sum_{j=1}^{m} a_{ij}K_{jk}\Big),$$
$$a = Mh_0, \qquad 0 < h \leqslant h_0, \tag{4.2}$$

and where h_0 denotes a given number (initial value of step size) and M is given by (3.4).

To find h we apply the following formulas:

$$h = \frac{\eta_m^*}{n} \qquad \eta_m^* = \min\{\eta_0, \eta_1, \ldots, \eta_m\}, \tag{4.3}$$

where the numbers $\eta_1 > 0$, $\eta_2 > 0$, ... $\eta_m > 0$ should be evaluated in such a way that

$$Y_0 + \eta_i c_i F(\Delta_t, \Delta_y) \subset \Delta_y, \qquad i = 1,\ 2,\ \ldots,\ m,$$

and the number $\eta_0 > 0$, should fulfill the following condition:

$$Y_0 + \eta_0 \sum_{i=1}^{m} w_i F(\Delta_t, \Delta_y) + (\Psi(\Delta_t, \Delta_y) + [-\alpha, \alpha])h_0^p \subset \Delta_y,$$

and where $Y_0 \subset \Delta_y$ and $y_0 \in Y_0$ The interval $[0, \eta_m^*]$ is then divided into n parts by the points $t_k = kh$, where $k = 0,\ 1,\ \ldots,\ n$, and the intervals T_k occurring in the method should be chosen in such a way that

$$t_k = kh \in T_k \subset [0, \eta_m^*].$$

From (4.2) it follows that in each step k we have to solve a nonlinear equation of the form

$$X = G(T, X),$$

where

$$T \in \mathbf{I}\Delta_t \subset \mathbf{IR}, \quad X = (X_1, X_2, \ldots, X_N)^T \in \mathbf{I}\Delta_y \subset \mathbf{IR}^N, \quad G\colon \mathbf{I}\Delta_t \times \mathbf{I}\Delta_y \to \mathbf{IR}^N.$$

If we assume that G is a contraction mapping, then the well-known fixed-point theorem implies that the iteration

$$X^{(l+1)} = G(T, X^{(l)}), \qquad l = 0,\ 1,\ \ldots, \tag{4.4}$$

is convergent to X^* i.e. $\lim_{l\to\infty} X^{(l)} = X^*$, for an arbitrary choice of $X^{(0)} \in \mathbf{I}\Delta_y$. Let us recall that G is called a contraction mapping if

$$d(G(T, X_{(1)}), G(T, X_{(2)})) \leqslant \beta d(X_{(1)}, X_{(2)}),$$

where d is a metric and $\beta < 1$ denotes a constant.

For the equation (4.2) the process (4.4) is of the form

$$K_{ik}^{(l+1)} = F\Big(T_k + c_i h, Y_k + h\sum_{j=1}^{m} a_{ij}K_{jk}^{(l)}\Big),$$
$$i = 1,\ 2,\ \ldots,\ m, \qquad k = 0,\ 1,\ \ldots,\ n-1, \qquad l = 0,\ 1, \ldots, \tag{4.5}$$

where $K_{ik}^{(0)} = F(T_k + c_i h, Y_k)$ The process (4.5) may be modified to the following form:

$$K_{ik}^{(l+1)} = F\Big[T_k + c_i h, Y_k + h\Big(\sum_{j=1}^{i-1} a_{ij} K_{jk}^{(l+1)} + \sum_{j=1}^{m} a_{ij} K_{jk}^{(l)}\Big)\Big], \tag{4.6}$$

which should reduce the number of iterations.

Interval methods corresponding with some well-known conventional implicit Runge–Kutta methods are as follows:

- the interval version of the implicit midpoint rule

$$\begin{aligned} Y_{k+1} &= Y_k + hK_{1k} + (\Psi(T_k, Y_k) + [-\alpha, \alpha])h^3, \\ K_{1k} &= F\Big(T_k + \frac{h}{2}, Y_k + \frac{h}{2}K_{1k}\Big), \qquad k = 0,\ 1,\ \ldots, n-1, \end{aligned} \tag{4.7}$$

- the interval version of the Hammer–Hollingsworth method

$$\begin{aligned} Y_{k+1} &= Y_k + h(K_{1k} + K_{2k}) + (\Psi(T_k, Y_k) + [-\alpha, \alpha])h^5, \\ K_{1k} &= F\Big[T_k + \Big(\frac{1}{2} \mp \frac{\sqrt{3}}{6}\Big)h, Y_k + \frac{h}{4}K_{1k} + \Big(\frac{1}{4} \mp \frac{\sqrt{3}}{6}\Big)hK_{2k}\Big], \\ K_{2k} &= F\left[T_k + \left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right)h, Y_k + \left(\frac{1}{4} \pm \frac{\sqrt{3}}{6}\right)hK_{1k} + \frac{h}{4}K_{2k}\right], \\ &\qquad k = 0,\ 1,\ \ldots, n-1, \end{aligned} \tag{4.8}$$

- the interval version of the Butcher semi-implicit method

$$\begin{aligned} Y_{k+1} &= Y_k + \frac{h}{6}(K_{1k} + 4K_{2k} + K_{3k}) + (\Psi(T_k, Y_k) + [-\alpha, \alpha])h^5, \\ K_{1k} &= F(T_k, Y_k), \qquad K_{2k} = F\left[T_k + \frac{h}{2}, Y_k + \frac{h}{4}(K_{1k} + K_{2k})\right], \\ K_{3k} &= F(T_k, Y_k + hK_{2k}), \qquad k = 0,\ 1,\ \ldots, n-1, \end{aligned} \tag{4.9}$$

- the interval versions of Alexander's diagonally implicit methods

$$\begin{aligned} Y_{k+1} &= Y_k + h\left[\frac{1}{8\zeta^2}K_{1k} + \left(1 - \frac{1}{4\zeta^2}\right)K_{2k} + \frac{1}{8\zeta^2}K_{3k}\right] + (\Psi(T_k, Y_k) + [-\alpha, \alpha])h^5, \\ K_{1k} &= F\left[T_k + \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\zeta\right)h, Y_k + \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\zeta\right)hK_{1k}\right], \\ K_{2k} &= F\left[T_k + \frac{h}{2}, Y_k - \frac{\sqrt{3}}{3}\zeta hK_{1k} + \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\zeta\right)hK_{2k}\right], \\ K_{3k} &= F\left[T_k + \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\zeta\right)h,\right. \\ &\qquad \left. Y_k + \left(\frac{1}{2} + \frac{2\sqrt{3}}{3}\zeta\right)hK_{1k} - \left(\frac{1}{2} + \frac{4\sqrt{3}}{3}\zeta\right)hK_{2k} + \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\zeta\right)hK_{3k}\right], \end{aligned} \tag{4.10}$$

where $k = 0,\ 1,\ \ldots,\ n = 1$, and $\zeta = \cos 10^\circ,\ -\cos 50^\circ$ or $-\cos 70^\circ$.

In each of the above formulas the form of $\Psi(T, Y)$ depends on the order p and the number of stages m.

As in the case of the explicit interval methods we can prove the following

THEOREM 4.1 *For the exact solution $y(t)$ of the initial value problem* (2.1) *we have $y(t_k) \in Y_k$ $(k = 0, 1, \ldots, n)$, where Y_k are obtained from the method* (4.1)–(4.2).

In order to estimate the widths of interval solutions obtained by the implicit methods we have to consider these methods for each m separately. It follows from the fact that for K_{ik} from (4.2) and properties of the function F we get

$$w(K_{ik}) \leqslant \Lambda[w(T_k) + w(Y_k)] + h\Lambda \sum_{j=1}^{m} |a_{ij}| w(K_{jk}),$$

where $i = 1$ for $m = 1$, $i = 1, 2$ for $m = 2$, etc., Λ is given by (3.1), and it is impossible to write a general solution of these inequalities (with respect to $w(K_{ik})$) for an arbitrary value of m.

For $m = 1$ and $m = 2$ we have

THEOREM 4.2 *If Y_k $(k = 1, 2, \ldots, n)$ are obtained from* (4.1)–(4.2) *with $m = 1$, then for $h_0 < 2/\Lambda$ we have*

$$w(Y_k) \leqslant Q_1 h^2 + R_1 w(Y_0) + S_1 \max_{l=1,2,\ldots,n} w(T_l),$$

where Q_1, R_1, S_1 denote some nonnegative constants.

THEOREM 4.3 *If Y_k $(k = 1, 2, \ldots, n)$ are obtained on the basis of the method* (4.1)–(4.2) *with $m = 2$, then for h_0 such that*

$$h_0 \leqslant \min\left\{1, \frac{1}{\Lambda|a_{11}|}, \frac{1}{\Lambda|a_{22}|}, \frac{1}{\Lambda(|a_{11}| + |a_{22}|) + \Lambda^2|a_{12}||a_{21}|}, \right\}$$

we have

$$w(Y_k) \leqslant Q_2 h^p + R_2 w(Y_0) + S_2 \max_{l=1,2,\ldots,n} w(T_l),$$

where $p \leqslant 4$ and Q_2, R_2, S_2 denote some nonnegative constants.

The proofs of the above theorems one can find in [17, 22]. In [5] and [22] there are also similar theorems for $m = 3$ and $m = 4$. In [18] and [22] it is presented an algorithm to find the maximum integration intervals in floating-point interval arithmetic for both, explicit and implicit interval methods of Runge–Kutta types.

5. Numerical Examples. Below we present some numerical experiments that confirm the theoretical justifications given in the previous sections. All calculations have been performed using the unit called *IntervalArithmetic* written in the Delphi Pascal programming language and presented in [22].

EXAMPLE 5.1 Let us solve, by a number of interval Runge–Kutta type methods, the initial value problem

$$y' = 0.5y, \qquad y(0) = 1. \tag{5.1}$$

This problem has the exact solution of the form $y = \exp(0.5t)$, from which it follows that $y(1.0) \approx 1.648721270700128$.

Let us consider the interval explicit methods (3.6), (3.8), (3.9), and implicit ones (4.7), (4.8) and (4.9). Let us assume that in each of these methods the input data are as follows:

$$\Delta_t = [0, 10], \quad \Delta_y = [\underline{0.9}, 149], \quad h_0 = 0.001, \quad T_0 = [0, 0], \quad Y_0 = [1, 1],$$

where $\underline{x}$ denotes the largest machine number less or equal to x (similarly, further $\overline{x}$ will denote the smallest machine number greater or equal to x). The maximum integration interval, i.e. the value of ξ_m^* given by (3.5) or the value of η_m^* given by (4.3), is approximately equal to 1.985. Taking $[0, 1]$ as the integration interval and partitioning this interval into $n = 2000$ parts, we have $h = 0.0005$. The obtained interval results at

$$T = [9.9999999999999999\text{E}{-0001},\ 1.0000000000000001\text{E}{+0000}]$$

are presented in Table 5.1. In implicit interval methods considered we assumed the accuracy 10^{-18} in both iterations (4.5) and (4.6), and we have obtained the same results (in each method the number of iterations was equal to 5 or 6).

Table 5.1: The interval solutions of the problem (5.1)

Method	Y	*Width of* Y
(3.6)	[1.6487210675225932E+0000, 1.6487214567067031E+0000]	$\approx 3.89 \cdot 10^{-7}$
(3.8)	[1.6487212706768890E+0000, 1.6487212707222942E+0000]	$\approx 4.54 \cdot 10^{-11}$
(3.9)	[1.6487212707001280E+0000, 1.6487212707001283E+0000]	$\approx 2.78 \cdot 10^{-16}$
(4.7)	[1.6487212706784987E+0000, 1.6487212707239042E+0000	$\approx 4.54 \cdot 10^{-11}$
(4.8)	[1.6487212707001280E+0000, 1.6487212707001286E+0000]	$\approx 5.61 \cdot 10^{-16}$
(4.9)	[1.6487212707001277E+0000, 1.6487212707001283E+0000]	$\approx 5.60 \cdot 10^{-16}$

In Table 5.2 we present the solutions obtained by conventional methods corresponding to the interval ones. In implicit methods we assumed the accuracy 10^{-18}. One can observe that only the fourth order methods of Runge–Kutta, Hammer-Hollingsworth and Butcher give solutions that belongs to the interval solutions obtained.

Table 5.2: The solutions of the problem (5.1) at $t = 1.0$ obtained by conventional methods

Method	*Order*	y	*Error*
Euler's	1	1.6486182460106884E+0000	$\approx 1.03 \cdot 10^{-4}$
Euler-Cauchy	2	1.6487212621146481E+0000	$\approx 8.59 \cdot 10^{-9}$
Runge–Kutta	4	1.6487212707001281E+0000	$\approx 3.69 \cdot 10^{-17}$
Implicit midpoint rule	2	1.6487212749936732E+0000	$\approx 4.29 \cdot 10^{-9}$
Hammer-Hollingsworth	4	1.6487212707001281E+0000	$\approx 1.53 \cdot 10^{-17}$
Butcher's semi-implicit	4	1.6487212707001282E+0000	$\approx 4.34 \cdot 10^{-18}$

Example 5.2 Let us take into account the two-dimensional problem

$$\frac{dy_l}{dt} = y_{l+2}, \quad l = 1,\ 2, \quad \frac{dy_3}{dt} = -\frac{y_1}{r^3}, \quad \frac{dy_4}{dt} = -y_2, \quad r = \sqrt{y_1^2 + y_2^2}, \tag{5.2}$$
$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad y_4(0) = 1,$$

the exact solution of which is given by

$$y_1 = \cos t, \qquad y_2 = \sin t, \qquad y_3 = -\sin t, \qquad y_4 = \cos t.$$

Let us try to solve this problem by the following explicit interval methods:

• (3.8), i.e. by the interval version of the Euler–Cauchy method which is a two-stage method ($m = 2$) and of the second order ($p = 2$),

• (3.9), i.e. by the interval version of the Runge–Kutta method ($m = p = 4$), and by the two-stage ($m = 2$) implicit interval method of the Hammer-Hollingsworth ($p = 4$).

Let us assume that

$$\Delta_t = [0, 1], \qquad \Delta_{y_1} - \Delta_{y_4} = [\underline{0.8}, \overline{1.2}], \qquad \Delta_{y_2} = [\underline{-0.2}, \overline{0.2}],$$
$$T_0 = [0, 0], \qquad Y_1(0) = Y_4(0) = [1, 1], \qquad Y_2(0) = Y_3(0) = [0, 0].$$

For these intervals and $h_0 = 0.01$ we obtain for each of the method the maximum integration interval approximately equal to 0.085. Taking $[0, 0.05]$ as the integration interval and splitting this interval into 10 parts, we get at

$$T = [4.9999999999999999E{-}0002, 5.0000000000000000E{-}0002]$$

the interval solutions presented in Table 5.3. For the implicit method (4.8) in the iteration (4.6) we assumed the accuracy 10^{-18} and we obtained the results after 8–9 iterations. It should be noted that in each case the exact solution

$$y_1(0.05) = y_4(0.05) \approx 0.998750260394966,$$
$$y_2(0.05) = -y_3(0.05) \approx 0.049979169270678$$

is included in the interval solution obtained.

Table 5.3: The interval solutions of the problem (5.2)

Method	Y_s	*Width of* Y_s
(3.8)	$Y_1 =$ [9.9875024733284893E-0001, 9.9875027293211718E-0001]	$\approx 2.56 \cdot 10^{-8}$
	$Y_2 =$ [4.9979156577987591E−0002, 4.9979182150246763E−0002]	$\approx 2.56 \cdot 10^{-8}$
	$Y_3 =$ [−4.9979183082789349E−0002,−4.9979155738949369E−0002]	$\approx 2.73 \cdot 10^{-8}$
	$Y_4 =$ [9.9875025200135706E−0001, 9.9875027763381159E−0001]	$\approx 2.56 \cdot 10^{-8}$
(3.9)	$Y_1 =$ [9.9875026039496204E−0001, 9.9875026039496845E−0001]	$\approx 6.40 \cdot 10^{-15}$
	$Y_2 =$ = [4.9979169270675078E−0002, 4.9979169270681472E−0002]	$\approx 6.39 \cdot 10^{-15}$
	$Y_3 =$ [−4.9979169270681769E−0002,−4.9979169270674931E−0002]	$\approx 6.84 \cdot 10^{-15}$
	$Y_4 =$ [9.9875026039496203E−0001, 9.9875026039496845E−0001]	$\approx 6.41 \cdot 10^{-15}$
(4.8)	$Y_1 =$ [9.9875026039496244E−0001, 9.9875026039496886E−0001]	$\approx 6.40 \cdot 10^{-15}$
	$Y_2 =$ [4.9979169270675104E−0002, 4.9979169270681498E−0002]	$\approx 6.39 \cdot 10^{-15}$
	$Y_3 =$ [−4.9979169270681762E−0002,−4.9979169270674925E−0002]	$\approx 6.84 \cdot 10^{-15}$
	$Y_4 =$ [9.9875026039496244E−0001, 9.9875026039496886E−0001]	$\approx 6.41 \cdot 10^{-15}$

In the problem considered the maximum integration intervals are very small and are approximately equal to merely 1.35% of the orbit period. Of course, we can take the intervals obtained at the last time interval and start any of the methods again. But a better approach is to take the interval solutions obtained (by any of these methods) as starting intervals for multistep interval algorithms (see e.g. [11–13, 19–22]).

EXAMPLE 5.3 Finally, let us consider the motion of a simple pendulum given by the initial value problem

$$\begin{gathered} y_1' = -\frac{g}{L} y_2, \quad y_2' = y_1 \\ y_1(0) = 0, \quad y_2(0) = \varphi_0, \end{gathered} \tag{5.3}$$

where g is the gravitational acceleration at the Earth's surface, L denotes the pendulum length and φ_0 is an initial angle.

Let us try to solve it by the interval version of Alexander's diagonally implicit method with $\zeta = -\cos 50°$ (see (4.10)). Assuming $L = 1$ and

$$\Delta_t = [0, 2], \quad \Delta_{y_1} = [-2.5, 2.5], \quad \Delta_{y_2} = [-1, 1], \quad h_0 = 0.01,$$

$$T_0 = [0, 0], \quad Y_1(0) = [0, 0], \; Y_2(0) = [\underline{0.52359877559829887}, \overline{0.52359877559829888}],$$

and taking the intervals

$$[\underline{9.80665}, \overline{9.80665}] \text{ and } [\underline{-0.642787609686539327}, \overline{-0.642787609686539326}]$$

to represent the gravitational acceleration g at Earth's surface and the value of ζ, respectively, we obtained the maximum integration interval approximately equal to 0.19. This means that the method considered could be applied to about 10% of the period. Taking the iteration (4.6) with accuracy 10^{-18} for finding K_{ik} ($i = 1, 2, 3$), the interval $[0, 0.1]$ as the integration interval and splitting it into 20 parts, we obtained the interval solution shown in Table 5.4. The number of iterations in the process (4.6) was equal to 8 and, of course, the exact solution belongs to the intervals obtained.

Table 5.4: The interval solution of the problem (5.3) obtained by the method (4.10)

T	Y_s	*Width of* Y_s
[4.9999999999999999E−0002, 5.0000000000000001E−0002]	Y_1 = [−2.5568972570054137E−0001, −2.5568972569288790E−0001]	$\approx 7.65 \cdot 10^{-12}$
	Y_2 = [5.1719344066940931E−0001, 5.1719344067582207E−0001]	$\approx 6.41 \cdot 10^{-12}$
[9.9999999999999999E−0002, 1.0000000000000001E−0001]	Y_1 = [−5.0512359899636122E−0001, −5.0512359897780303E−0001]	$\approx 1.86 \cdot 10^{-12}$
	Y_2 = [4.9813415251025603E−0001, 4.9813415252354455E−0001]	$\approx 1.33 \cdot 10^{-12}$

All the examples presented in this section, like the others performed by the author, confirm the theoretical studies developed in the previous sections. Explicit and implicit interval methods of Runge–Kutta type can be used to find interval solutions (in floating-point interval arithmetic) for one- and multidimensional problems. In each case the exact solution belongs to the interval solutions obtained.

It is obvious that higher order methods give interval solutions with smaller widths. One should consider implicit interval methods which with small number of stages give solutions with greater order, e.g. the interval version of the Hammer–Hollingworth method (4.8).

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Przegląd metod przedziałowych typu Rungego-Kutty

Streszczenie. W artykule przedstawiono jawne i niejawne metody przedziałowe typu Rungego-Kutty. Metody takie zawierają w sobie błędy metod, co oznacza, że ten rodzaj błędów jest uwzględniony w otrzymywanych rozwiązaniach przedziałowych. Stosując te metody do rozwiązywania zagadnienia początkowego w zmiennopozycyjnej arytmetyce przedziałowej otrzymujemy zatem rozwiązania w postaci przedziałów, które zawierają wszystkie możliwe błędy numeryczne. W artykule przedstawiono także przykłady numeryczne.

Słowa kluczowe: metody przedziałowe dla równań różniczkowych zwyczajnych, metody Rungego-Kutty, zmiennopozycyjna arytmetyka przedziałowa

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