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Analysis of students' solutions to geometry questions forming a bundle

Abstract: One of the main goals of mathematical education is to develop the skills for problem solving as well as skills that help carry out mathematical reasoning and argumentation.

Geometric problems play here a special role. These require the person solving them to act with an inquiry attitude and a 'specific vision'. The 'specific vision' is the ability one can manipulate with geometric objects in ones' mind and perceive, separate and focus on the important information only. However, it is not enough to "see" it is also necessary to know how to interpret what is being seen. Although many researchers have dealt with the problem and many establishments have been made in this scope, the question of how to develop the skills of the "specific vision" stays still open.

Herein article presents the research results which aimed at, among others, verification to what degree the combination of geometry problems formed into a bundle helps the secondary school students 'notice' and understand the presented situation and as a consequence to find the answer to few questions about this situation. We wanted to establish whether such an organised educational environment entails students natural thinking over the subsequent bundle of problems solved, or maybe makes them return to questions already solved, or by the usage of knowledge acquired helps students to find the problem solution for the next question or a correction for the committed mistakes.

The analysis was based on some results coming from the survey School of Independent Thinking conducted by the Institute for Educational Research in 2011.

Key words: geometry, mathematical reasoning, upper secondary school.

1 Remarks on specifics of geometry thinking

Many researchers occupied with didactical aspects of teaching and learning emphasise the complexity of functioning in this area of mathematics. Duval reported (1998, pp. 38-39) that geometrical reasoning involves three kinds of cognitive processes which fulfil specific epistemological functions. These cognitive processes are: visualisation process (for the illustration representation of a geometrical statement, for the heuristic exploration of a complex geometrical situation, for a synoptic glance over it or for a subjective verification), construction process (using tools) and reasoning process (particularly discursive processes for the extension of knowledge, for explanation, for proof). Hence, Duval stated (1999) that representation, vision and visualization are very important cognitive functions in mathematical thinking. According to Kurina (2003) three arts (skills, abilities): seeing, drawing and thinking, mutually intertwined in the process of geometrical problem solving are the small modifications of listed functions. However, as noted by Duval (1998, p. 37) "teaching geometry is more complex and often less successful than teaching numerical operations or elementary algebra". Furthermore, the skills needed to solve geometry problems are different than those needed for other domains of mathematics. One of the fundamental differences consists in the fact that geometry questions are a type of problems 'with excess data' – there exist various data and links, but only some of them are relevant from the point of view of the problem being solved. Sometimes the data are presented by means of a diagram which needs to be interpreted. Panek & Pardała (1999, pp. 65-69) wrote that geometry problems require a 'selective vision' (spatial imagination) from the person who solves them. That 'selective vision' consists of manipulating geometric objects in the mind and perceiving, separating and focusing only on the important information. It is not enough to 'see'; it is also necessary to know how to interpret what is being seen. Kurina (1998, p. 73) reported that visual information can be understood differently by different persons. Despite this, the problem of 'selective vision' in the geometry teaching and learning processes was the focus of multiple researchers, some mysteries of the 'vision' phenomenon are yet to be discovered and explained. The issue of how to shape, develop, and diagnose the spatial imagination of students is still being discussed. Ben-Chaim, Lappan & Houang (1989) state that it can be developed by performing appropriate tasks. This includes playful tasks of building geometrical solids out of cubes, drawing constructed plane geometrical solids and acquiring information from such drawings. Very important are the problems for whose solutions students need be able to imagine the transformation, distribution and movement of the figures. What is essential here is geometric intuition for only those of the existing data and the relations existing among them shall be selected which are relevant to the solution of the problem (Jones, 1998a; Fujita, Jones & Yamamoto, 2004; Kurina, 2003; Kunimune, Fujita & Jones, 2010). Fischbein (1987, pp. 43-56) defined intuition as a special type of cognition, characterized by the following properties: self-evidence and immediacy, intrinsic certainty, perseverance, coerciveness, theory status, extrapolativeness, globality and implicitness.

Geometry problems rarely impose a ready pattern of thinking or acting on the student. They can often be solved in many ways, depending on the perceived and selected data and relations between them. Geometry questions offer, therefore, a perfect opportunity for development of students' ability to create strategies of solving problems, carrying out mathematical reasoning and argumentation. They enforce adoption of the attitude of inquiry, analysis of the situation, thorough understanding of the concepts and mathematical theorems used, combination or processing of various elements of knowledge.

Teaching and learning mathematics usually takes place through problem solving. These problems are the source of experience out of which the student's mind builds up its mathematical competences. Geometry is an indispensable part of the mathematics curriculum. Thus, geometry tasks fulfil an important role in the development of the problem-solving skill and formation of mathematical reasoning of students (Duval, 1999; Jones, 1998b, Krygowska, 1977; Kurina, 2003; Swoboda 2008, 2012). By design, the teaching and learning of geometry in school should help students to develop ways of thinking in mathematics.

An analysis of the ways of solving geometry problems is important from the point of view of assessment of the skills possessed by students. It enables seeing their approach to the problem, giving an insight into the reasoning, the method used and its implementation, assessment of the ability to apply knowledge, creativity and inventiveness. In addition, it provides a lot of valuable information on the types of errors made by students and the difficulties met (De Lange 1986, Panek & Pardała 1999, Swoboda 2008). The solution analysis of a certain bundle of geometry problems is the subject of this work. This paper presents the research results which aimed at, among others, verification to what degree the combination of geometry problems formed into a bundle helps the secondary school students 'notice' and understand the presented situation and as a consequence to find the answer to some questions about this situation. We wanted to establish whether such an organised educational environment entails students' natural thinking over the subsequent bundle of problems solved, or maybe makes them return to questions already solved, or by the usage of knowledge acquired helps students to find the problem solution for the next question or a correction for the mistakes committed.

2 Goal and survey questions

The aim of the mathematical part of the survey *School of Independent Thinking* (*Szkoła samodzielnego myślenia*)[1] was to diagnose complex mathematical skills (mathematical modelling, creating a strategy of solving problems, reasoning and argumentation) of students at grade 4 (age 9-10 years old), grade 7 (age 12-13 years old), grade 10 (age 15-16 years) and grade 12 or 13 (18-20 years old). Here we present partial results of that survey. The analysis covered solutions of geometry questions which formed a bundle, submitted by upper secondary school students. By bundle of questions we mean several questions concerning a specific situation, preceded with an introductory text that describes that situation. Subsequent questions from a bundle refer to the same situation. When solving the posed problems, students could, therefore, make use of the experience collected during analysis of the earlier questions from the bundle.

We looked for answers to the following questions:

- What heuristic approach do they use in the course of solving open, dynamic[2] geometry problems?
- When obtaining knowledge related to solving a subsequent problem from a bundle, do they go back to the solutions of problems already solved and correct possible errors?
- When solving questions forming a bundle, to what extent do students use knowledge obtained from solving previous problems?

To answer these questions we carried out an analysis of students' work as follows. We used a two-digit code to assess each solution of the task. The first digit of the code indicates a correct, incorrect or partially correct answer. The second digit of the code indicates a way of solving the problem. Next, we have reviewed the code. We carried out a statistical analysis of the results, too. More information about it, we provide in the further part of our paper.

[1]The survey *Szkoła samodzielnego myślenia* was carried out within the systemic project *Badanie jakości i efektywności edukacji oraz instytucjonalizacja zaplecza badawczego* (*Examination of the quality and efficiency of the education and institutionalisation of the research infrastructure*), implemented from the funds of the European Social Fund within the Human Capital Operational Project, Priority III: High quality of the education system, Submeasure 3.1.1 Creation of conditions and tools for monitoring, evaluation and studying of the education system.

[2]By a dynamic problem we mean a problem which contains a description of a specific action. It presents a situation before and after a certain change. A dynamic problem contains operational verbs and phrases which suggest performance of certain actions, e.g. *was added, was cut off, was replaced, was divided into equal parts, was increased.*

3 Organisation of the survey, characteristics of the studied group

The survey *School of Independent Thinking* was carried out between November 21^{st} and December 20^{th} 2011. The schools were randomly sampled for the survey. Based on the data from the Education Information System, schools from the whole area of Poland were sampled in strata depending on: voivodeship, school type (primary, lower secondary school, general, technical and specialised upper secondary schools, basic vocational school), the population of the locality and the size of the school. Table 1 presents information on the numbers of students at grade 10 and grade 12 or 13 (last grade), whose works are discussed in the paper.

Grade	School type	Number of class sections	Number of students	Total number of students
grade 10	General Upper Secondary School (GUSS)	60	1490	**3489**
	Technical and Specialised Upper Secondary School (T&SUSS)	50	1129	
	Basic Vocational School (BVS)	40	870	
grade 12 or 13	General Upper Secondary School (GUSS)	60	1359	**3004**
	Technical and Specialised Upper Secondary School (T&SUSS)	50	918	
	Basic Vocational School (BVS)	40	727	

Table 1. Numbers of upper secondary students participating in the survey *School of Independent Thinking* by school type. Source: *Own study.*

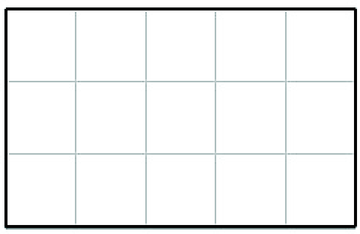
4 Characteristics of the problems, the solutions of which were subject to analysis

The main research tool were the questions included in the test. Solutions to questions, submitted by the students, constituted the research material, which was analysed from many angles, but mainly in quantitative terms. Selected works were analysed also in qualitative terms in order to establish both the students' ways of reasoning, and connections between solutions of subsequent bundled questions. The adopted organisation of the study and the size of groups did not allow individual interviews with the students solving the problems. We are aware of the fact that further qualitative research is needed to enable deeper insight into the way of reasoning and acting of students in the course of solving bundled problems. We treat the results described in this paper as recognition of the problem and a contribution to further research.

The test for upper secondary students contained 16 open-ended questions. They included the following three problems, preceded with an introductory text. The questions are of dynamic nature (in the sense defined above) – students examined the change in the area and perimeter of a shape as a result of a change in the situation. The order of the problems was intentional. Combination of the questions into a bundle gave an insight not only into the way of solving a single question by students, but also determination of the degree to which, examining the situation from various angles and in different boundary conditions, the students were able to use more and more comprehensive information about the situation. Also how the obtained heuristic experience will influence the manner of solving subsequent questions from the bundle or recognition and improvement of previously made errors.

Problems from the bundle "Rectangle"

The rectangle presented in the Figure is built of 15 small squares. The length of the side of the small square equals 1.

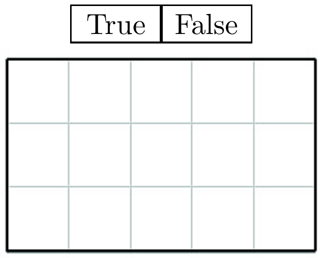


Which of the following sentences are true, and which are false?

Problem 1.

One can cut off such small squares along the sides of the rectangle that the shape obtained after removing them has an area smaller by 2 and the perimeter equal to that of the rectangle.

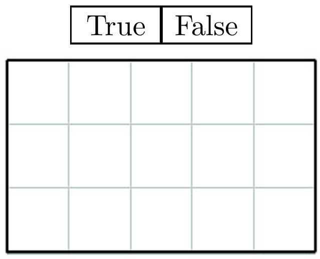
Circle the correct answer. If you circle TRUE, shade those small squares that should be cut off. If you circle FALSE, justify your answer.



Problem 2.

One can cut off such small squares along the sides of the rectangle that the shape obtained after removing them has an area smaller by 2 and a perimeter greater by 2 than that of the rectangle.

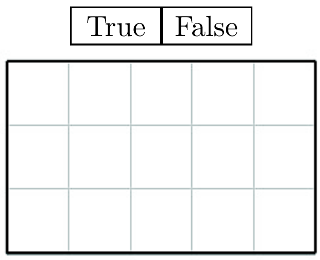
Circle the correct answer. If you circle TRUE, shade those small squares that should be cut off. If you circle FALSE, justify your answer.



Problem 3.

One can cut off such small squares along the sides of the rectangle that the shape obtained after removing them has an area smaller by 2 and a perimeter smaller by 2 than that of the rectangle.

Circle the correct answer. If you circle TRUE, shade those small squares that should be cut off. If you circle FALSE, justify your answer.



To assess similar sentences in terms of truthfulness, a student had to examine, how removal of small squares will affect the area and perimeter of the shape. As a rule, that type of analysis was connected to noticing the fact that the area of the new shape is smaller by exactly that much, as many small squares have been removed (thus, precisely two small squares shall be removed, to make the area of the new shape smaller by two than the area of the rectangle). On the other hand, the choice of the place of removal of squares affects the perimeter of the new shape. The perimeter will not change when two corner squares are removed (e.g. Figures 1. and 2.) or two adjacent (sharing one side) squares, precisely one of which is located at a corner (e.g. Figure 3.).

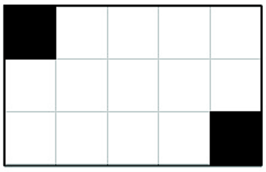


Fig. 1.

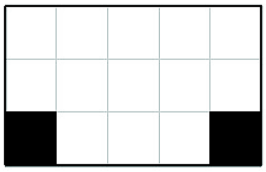


Fig. 2.

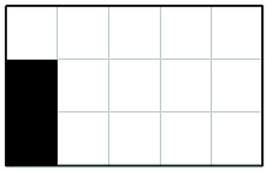


Fig. 3.

Then, the perimeter of the new shape will be greater by 2 than the perimeter of the rectangle, when two not adjacent squares are removed (sharing no side), precisely one of which is located at a corner (e.g. Fig. 4 and 5) or two adjacent (sharing a side) squares, none of which is a corner one (e.g. Fig. 6).

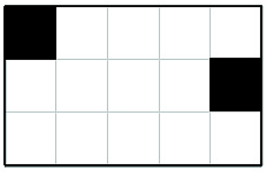


Fig. 4.

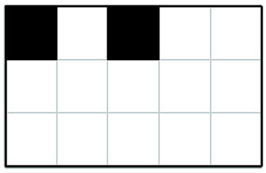


Fig. 5.

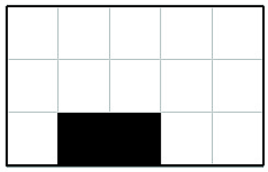


Fig. 6.

In addition, a student could manifest awareness that there exist several equivalent arrangements, obtained by symmetry transformations of the layouts forming solutions to problems 1 and 2.

Consideration of various situations in the first two problems was supposed to lead students to the conclusion that, after removing precisely two side squares, the perimeter of the new shape may not be smaller than the perimeter of the rectangle, and thus the statement found in question 3 is false. There was no intention to obtain a formal proof – it was sufficient for the student to provide a justification of the sort: "Cutting off of a corner square does not change the perimeter of the shape, while cutting off of a non-corner side square, which is not adjacent to another removed square, increases the perimeter of the shape."

5 Results of qualitative analysis

The use of previously solved problems from the bundle

As we wrote earlier, all solutions of the tasks were assessed using two-digital code. Next, for each student separately, we have reviewed the codes allocated for all the answers to the questions forming a bundle. This procedure allowed us to differentiate four characteristic types of behaviour.

Those are:

1. Acquiring knowledge and heuristic experiences while solving subsequent bundled questions.
2. Considering various possibilities of removals of squares in question 1 and using the results of the inquiry in questions 2 and 3.
3. Considering only one case in question 1 and referring to it in subsequent problems.
4. Treating each question as separate, not noticing that they are connected in a bundle.

We will discuss each of the behaviours, illustrating them with students' papers.

Acquiring knowledge and heuristic experiences while solving subsequent bundled questions

This group encompasses solutions, in which one can see that students were flexible in referring to the problems presented in subsequent problems. When solving them, they acquired new knowledge on the relations between the change of the area of the shape and its perimeter. Let us have a look at fragments of the paper of a grade 12 student of a general upper secondary school, presented in the following example.

Example[3] 1.

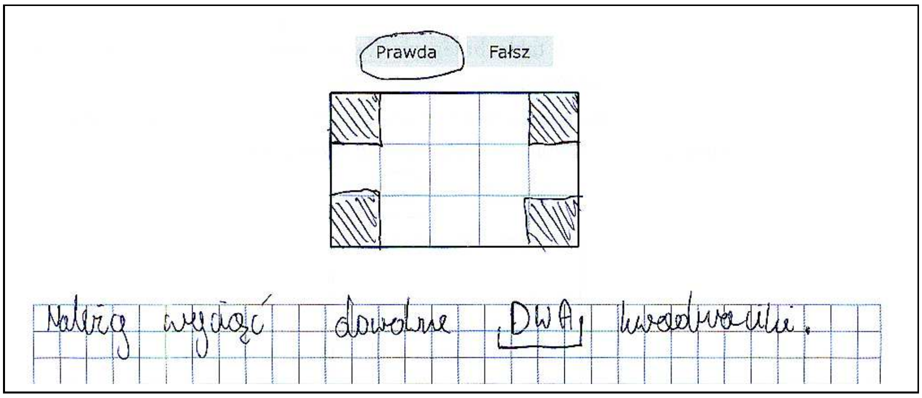


Illustration 1.[4]

In question 1, the student considered only the corner squares. He noticed that they had a specific property – cutting off one "corner" square decreases

[3] In the test booklet, the problems were numbered 13, 14, 15 respectively.

[4] "Any TWO small squares shall be cut off" (translation of the text in Illustration 1, written by the student).

the area by 1 and, at the same time, does not change the perimeter of the shape. Thus, if the area is to be smaller by 2, and the perimeter may not change, two “corner” squares should be cut out.

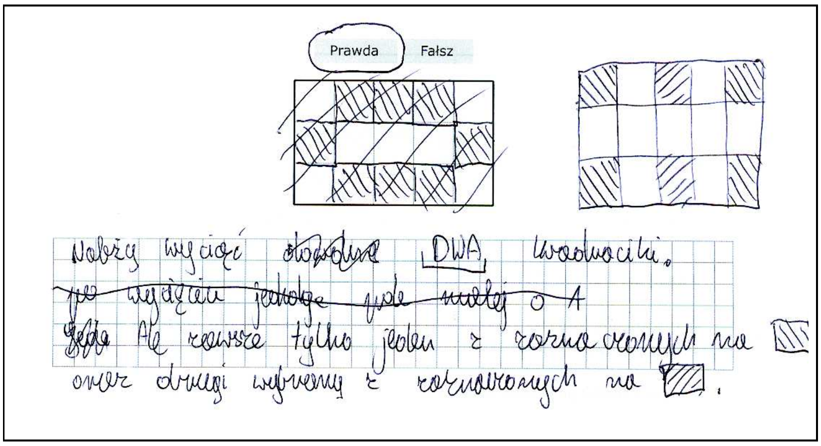


Illustration 2.[5]

In question 2, the student probably used the reasoning carried out in question 1. Focusing on the fact that cutting off of one “corner” square will reduce the area by one and will not change the perimeter of the shape made the student originally suggest in question 2 removal of any two “side” squares, none of which is at a corner. However, there occurred the reflection that such removal of squares must be further qualified, not any two “non-corner” shall be cut off, which is testified to by crossing out the word “any”. The student focused on the fact that cutting off any “corner” square will not change the perimeter of the shape, and cutting off a “side, middle” one – will increase the perimeter by two and finally concluded that one “corner” and one “side, middle” one shall be cut off.

The student noticed that problems 1 and 2 can be solved in many correct ways. However, fixation on the discovered solution to question 1 resulted in failure to examine other possibilities, e.g. he did not consider in question 1 the situation, also meeting the conditions of the question, where two squares, one of which is at a corner, are adjacent.

It also happened that students solving question 2 or 3 noticed an error in the prior problems and corrected it. This is testified to by numerous corrections and deletions in their works.

[5] “TWO squares shall be cut off. But always only one marked with and another one selected of those marked with, ” (translation of the text on Illustration 2, written by the student).

Consideration of the possibility to remove squares in question 1 and using the results for questions 2 and 3

Very few students examined in detail the situation in the course of solving question 1. In subsequent tasks, those students usually marked only the selected answer and provided a brief justification or did not provided any at all.

Example 2.

Zadanie 13. [M_74]
Z prostokąta można wyciąć takie brzegowe kwadraciki, aby otrzymana po ich usunięciu figura miała pole mniejsze o 2 i obwód taki jak prostokąt.

Zakreśl kółkiem poprawną odpowiedź. Jeśli zakreślisz PRAWDA, zamaluj te kwadraciki, które należy wyciąć. Jeśli zakreślisz FAŁSZ, uzasadnij odpowiedź.

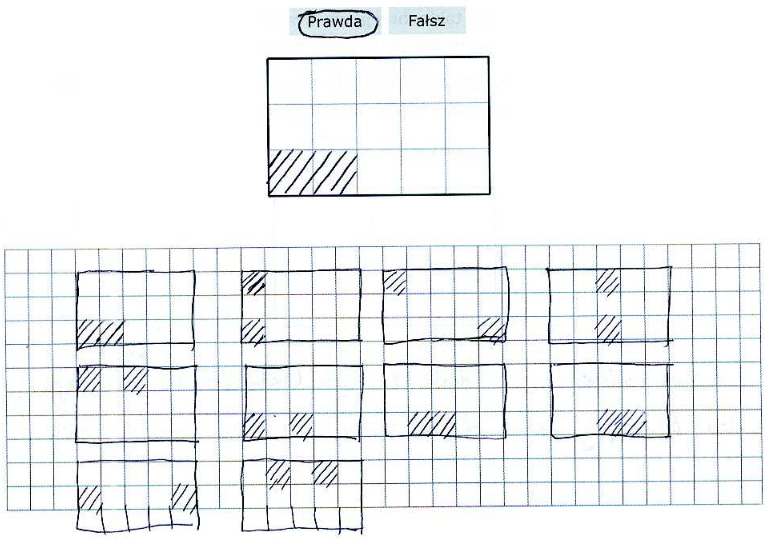
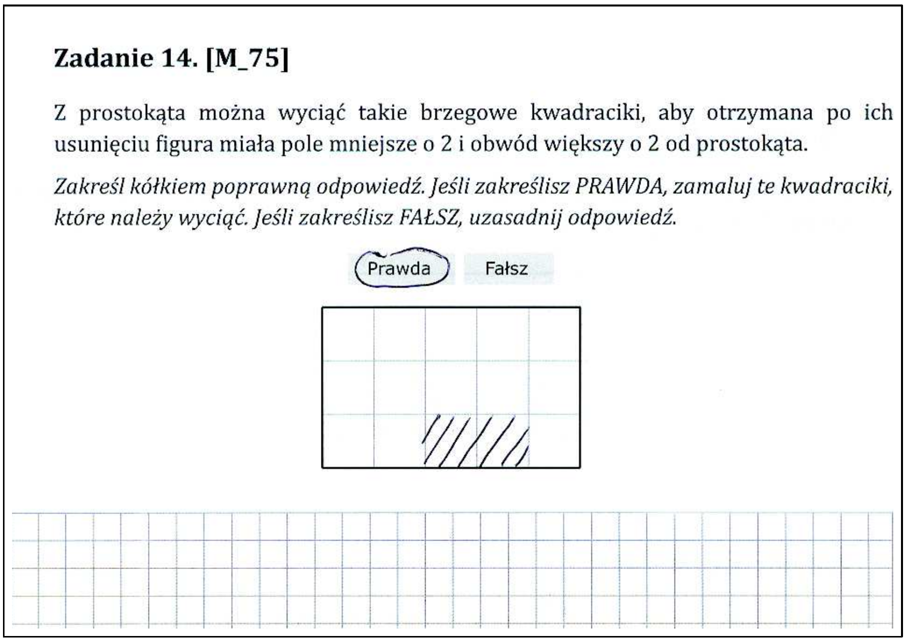


Illustration 3.

A grade 12 student of a general upper secondary school considered many specific cases in question 1. They included both the situation of cutting off only two corner squares, only two middle ones, as well as a corner and a middle one. Then, he used the results of those considerations in the course of solving subsequent problems from the bundle.



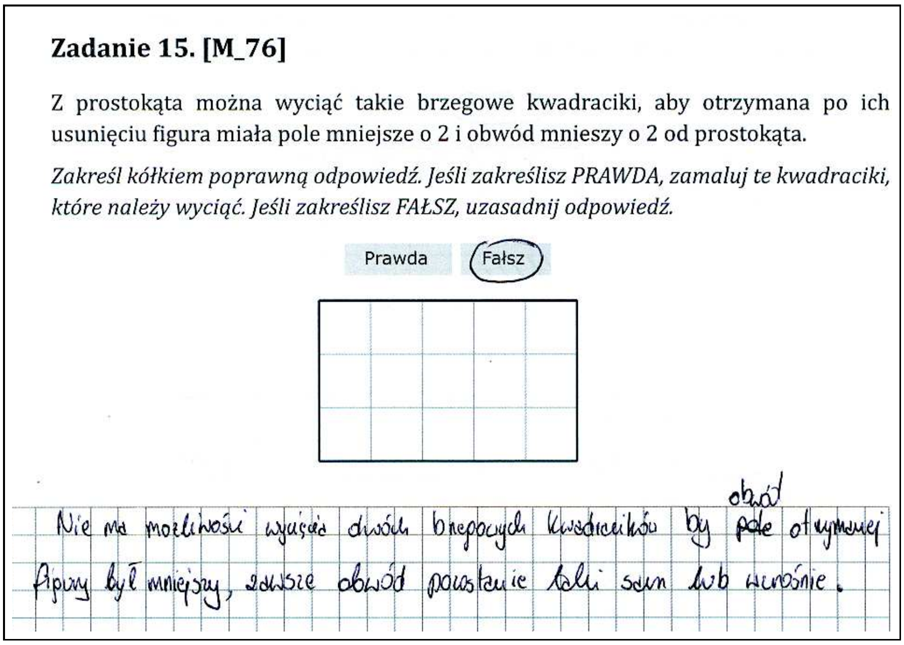
Zadanie 14. [M_75]

Z prostokąta można wyciąć takie brzegowe kwadraciki, aby otrzymana po ich usunięciu figura miała pole mniejsze o 2 i obwód większy o 2 od prostokąta.

Zakreśl kółkiem poprawną odpowiedź. Jeśli zakreślisz PRAWDA, zamaluj te kwadraciki, które należy wyciąć. Jeśli zakreślisz FAŁSZ, uzasadnij odpowiedź.

Prawda Fałsz

Illustration 4a).



Zadanie 15. [M_76]

Z prostokąta można wyciąć takie brzegowe kwadraciki, aby otrzymana po ich usunięciu figura miała pole mniejsze o 2 i obwód mnieszy o 2 od prostokąta.

Zakreśl kółkiem poprawną odpowiedź. Jeśli zakreślisz PRAWDA, zamaluj te kwadraciki, które należy wyciąć. Jeśli zakreślisz FAŁSZ, uzasadnij odpowiedź.

Prawda Fałsz

Nie ma możliwości wyciąć dwóch brzegowych kwadracików by ~~pole~~ obwód otrzymanej figury był mniejszy, zawsze obwód pozostanie taki sam lub wzrośnie.

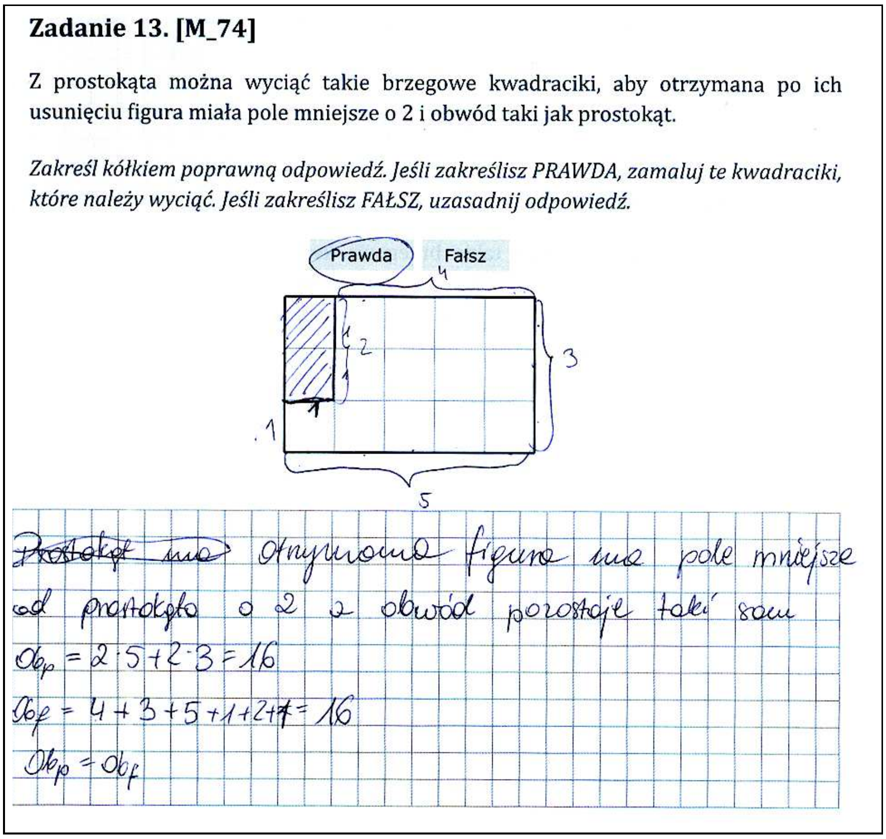
Illustration 4b).

In question 2, he provided only one specific example meeting the set conditions, and wrote in question 3: "It is not possible to cut off two side squares in a way to make the perimeter of the obtained shape smaller, the perimeter will always be the same or greater" (Illustration 4b).

Considering only one case in question 1 and referring to it in subsequent problems

Students assigned to this group considered in question 1 only one case, to which they referred when solving subsequent problems from the bundle. Let us look at the following examples.

Example 3.


Zadanie 13. [M_74]

Z prostokąta można wyciąć takie brzegowe kwadraciki, aby otrzymana po ich usunięciu figura miała pole mniejsze o 2 i obwód taki jak prostokąt.

Zakreśl kółkiem poprawną odpowiedź. Jeśli zakreślisz PRAWDA, zamaluj te kwadraciki, które należy wyciąć. Jeśli zakreślisz FAŁSZ, uzasadnij odpowiedź.

Prawda Fałsz

~~Prostokąt ma~~ Otrzymana figura ma pole mniejsze od prostokąta o 2 a obwód pozostaje taki sam

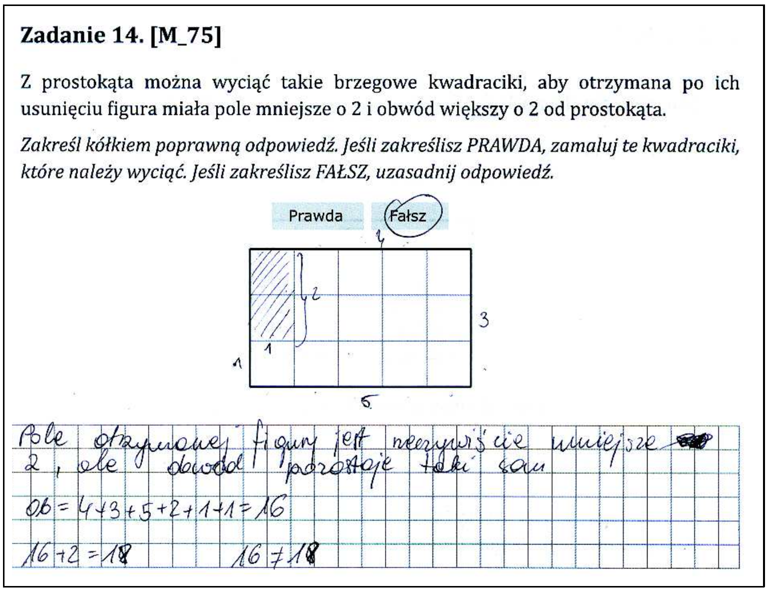
$Ob_p = 2\cdot 5 + 2\cdot 3 = 16$

$Ob_f = 4 + 3 + 5 + 1 + 1 + 2 + 1 = 16$

$Ob_p = Ob_f$

Illustration 5a).[6]

[6] "The obtained shape has the area smaller than the rectangle by 2, and the perimeter remains the same." (translation of the text in Illustration 5a, written by the student).



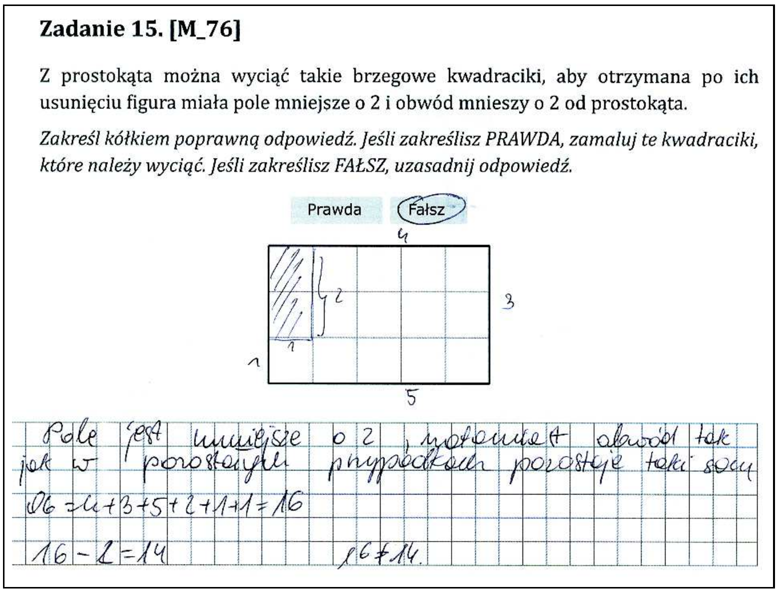
Zadanie 14. [M_75]

Z prostokąta można wyciąć takie brzegowe kwadraciki, aby otrzymana po ich usunięciu figura miała pole mniejsze o 2 i obwód większy o 2 od prostokąta.

Zakreśl kółkiem poprawną odpowiedź. Jeśli zakreślisz PRAWDA, zamaluj te kwadraciki, które należy wyciąć. Jeśli zakreślisz FAŁSZ, uzasadnij odpowiedź.

Prawda Fałsz

Illustration 5b).[7]



Zadanie 15. [M_76]

Z prostokąta można wyciąć takie brzegowe kwadraciki, aby otrzymana po ich usunięciu figura miała pole mniejsze o 2 i obwód mnieszy o 2 od prostokąta.

Zakreśl kółkiem poprawną odpowiedź. Jeśli zakreślisz PRAWDA, zamaluj te kwadraciki, które należy wyciąć. Jeśli zakreślisz FAŁSZ, uzasadnij odpowiedź.

Prawda Fałsz

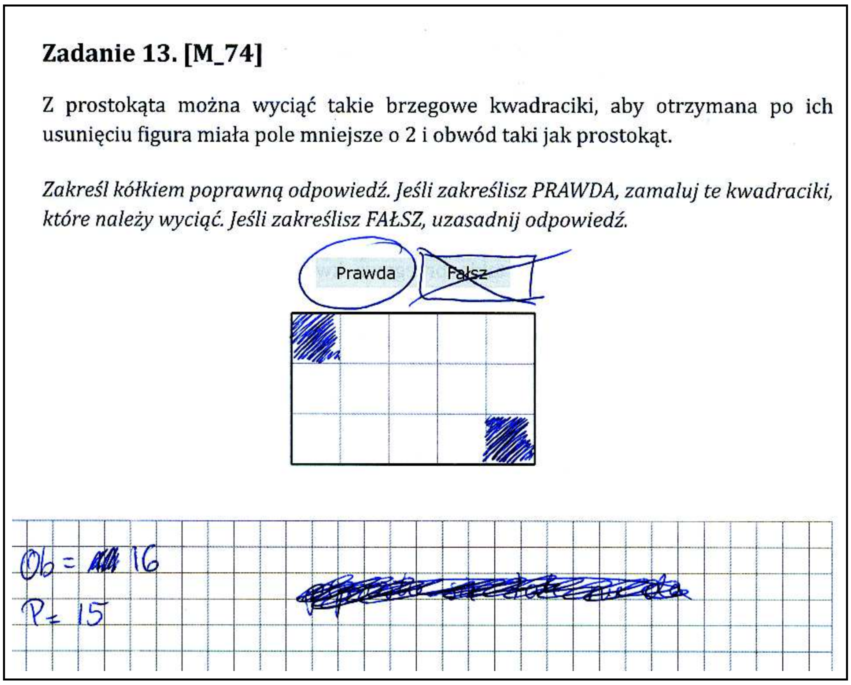
Illustration 5c).[8]

[7] "The area of the obtained shape is actually smaller by 2, and the perimeter remains the same." (translation of the text in Illustration 5b, written by the student).

[8] "The area is smaller by 2, while the perimeter, as in the other cases, remains the same." (translation of the text in Illustration 5c, written by the student).

In question 1, a grade 12 student of a general upper secondary school considered only one of the possibilities of removing two squares (two adjacent squares, one of which is placed at a corner). The selection was correct, which was confirmed by the performed calculations. Then, in questions 2 and 3, without checking other possibilities, he marked the answer "false". In justification, he shaded the same squares and calculated the perimeter, copying the same calculations used in question 1 (obtaining the result 16). To refer to the problem at hand, he artificially created values higher and lower than 16, with no link to the drawing, and wrote that $16 \neq 18$ (question 2) or $16 \neq 14$ (question 3) accordingly. What is more, in question 3, he wrote "The area is lower by 2, while the perimeter, as in the other cases, remains the same." We may conjecture that by "the other cases" the student meant the previous two problems.

EXAMPLE 4.



Zadanie 13. [M_74]

Z prostokąta można wyciąć takie brzegowe kwadraciki, aby otrzymana po ich usunięciu figura miała pole mniejsze o 2 i obwód taki jak prostokąt.

Zakreśl kółkiem poprawną odpowiedź. Jeśli zakreślisz PRAWDA, zamaluj te kwadraciki, które należy wyciąć. Jeśli zakreślisz FAŁSZ, uzasadnij odpowiedź.

Prawda ~~Fałsz~~

Ob = 16

P = 15

Illustration 6.

A grade 12 student of a general upper secondary school, when solving question 1, originally decided that such a removal of small squares was not possible, and the new shape would have an area smaller by 2 and a perimeter the same as that of the rectangle. He could not, however, explain that and wrote "This is simply not possible" (later he carefully painted out that statement). The supplied calculations indicate a simple error in calculations, with good placement of the "cut off" squares. In the next problem, he probably reconsidered the case meeting the set conditions, as there are no new proposals of placing the removed squares in the drawing. The drawing for question 3 looks just the same. Focusing on the discovered solution resulted in the fact that the student marked the answer "false" in the next bundled problems and wrote: "If the 2 squares are removed, the area will be reduced, but the perimeter will remain the same" (question 2) and "The area will change, but the perimeter of the shape will remain the same" (question 3).

Another student of grade 10 of a general upper secondary school marked the answer "true" in question 1 and provided a correct example. In the subsequent problems, he marked the answer "false" and wrote in the justification: "Analogously to question 13".

A similar thing was done by some students who made a mistake in question 1. Let us look at the following examples.

Example 5.

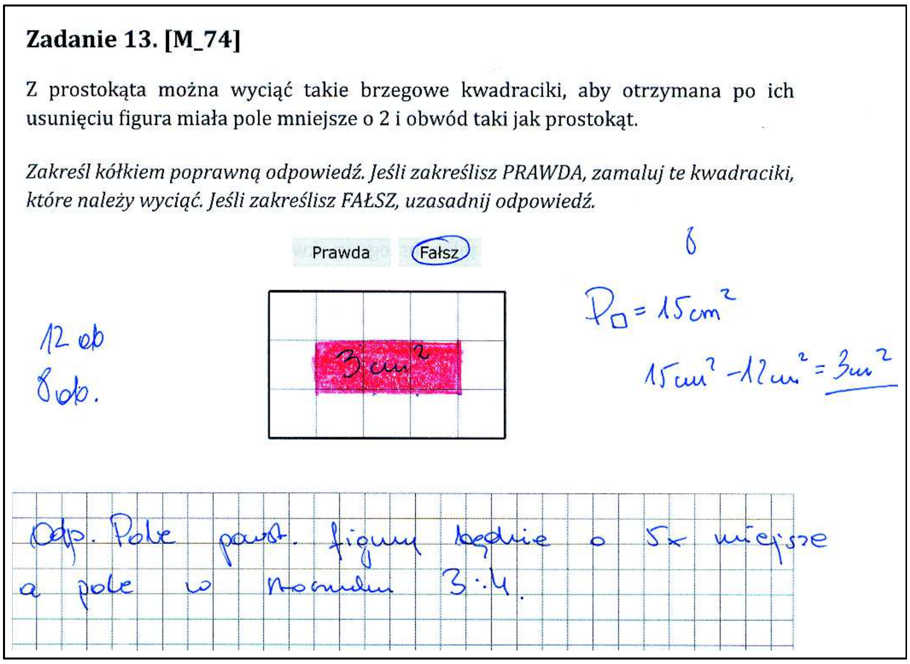
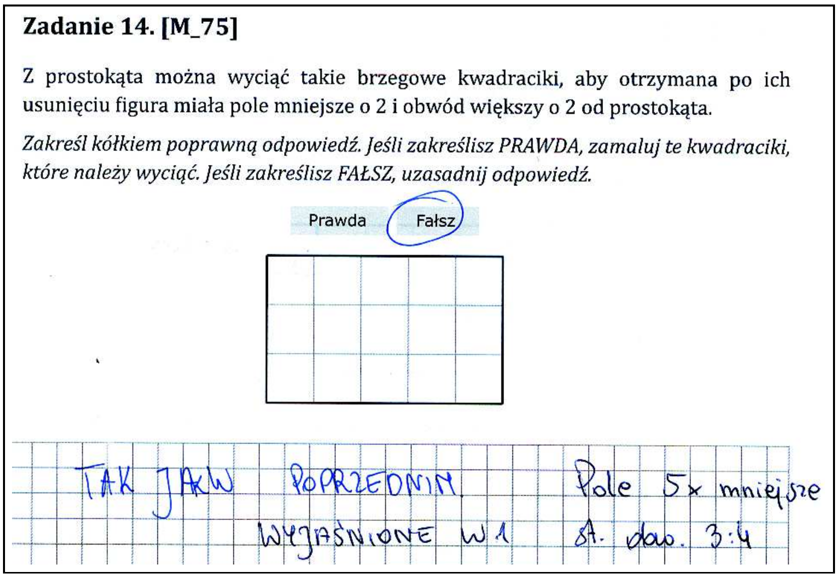


Illustration 7a).[9]

[9] "Answer: The area of the shape will be 5 times smaller, and the area in the ratio of 3 : 4." (translation of the text in Illustration 7a, written by the student).


Zadanie 14. [M_75]

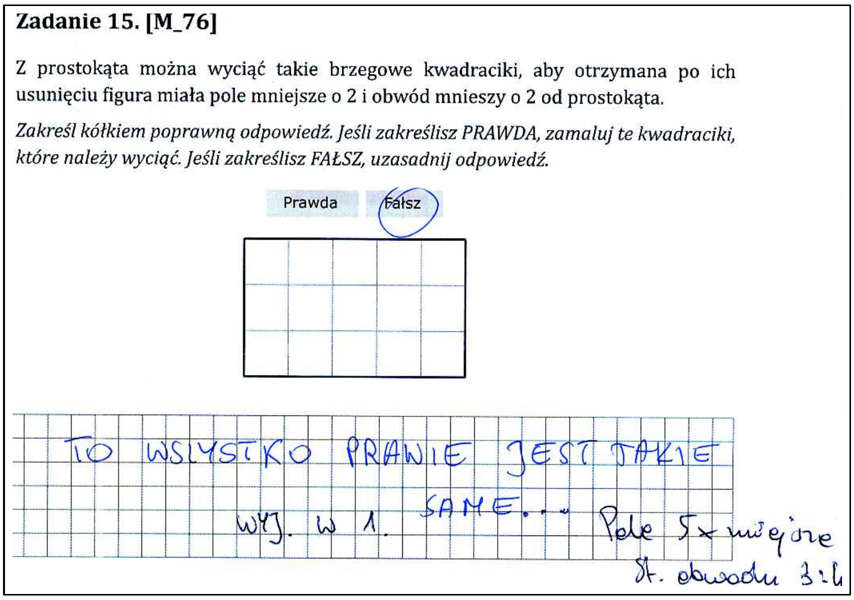
Z prostokąta można wyciąć takie brzegowe kwadraciki, aby otrzymana po ich usunięciu figura miała pole mniejsze o 2 i obwód większy o 2 od prostokąta.

Zakreśl kółkiem poprawną odpowiedź. Jeśli zakreślisz PRAWDA, zamaluj te kwadraciki, które należy wyciąć. Jeśli zakreślisz FAŁSZ, uzasadnij odpowiedź.

Prawda Fałsz

TAK JAKW POPRZEDNIM. Pole 5x mniejsze
WYJAŚNIONE W 1 st. obw. 3:4

Illustration 7b).[10]


Zadanie 15. [M_76]

Z prostokąta można wyciąć takie brzegowe kwadraciki, aby otrzymana po ich usunięciu figura miała pole mniejsze o 2 i obwód mnieszy o 2 od prostokąta.

Zakreśl kółkiem poprawną odpowiedź. Jeśli zakreślisz PRAWDA, zamaluj te kwadraciki, które należy wyciąć. Jeśli zakreślisz FAŁSZ, uzasadnij odpowiedź.

Prawda Fałsz

TO WSZYSTKO PRAWIE JEST TAKIE
WYJ. W 1. SAME... Pole 5x mniejsze
st. obwodu 3:4

Illustration 7c).[11]

[10] "JUST LIKE ABOVE. EXPLAINED IN 1. Area 5 time smaller, and the perimeter in the ratio of 3 : 4." (translation of the text in Illustration 7b, written by the student).

[11] "THIS IS ALL ALMOST THE SAME... EXPLANATION IN 1. Area 5 time smaller, and the perimeter in the ratio of 3 : 4" (translation of the text in Illustration 7c, written by the student).

An analysis of the solution of the first question from the bundle leads to the conclusion that the student did not understand the sense of the question. He removed all side squares and concluded that the obtained rectangle had an area equal to three and a perimeter equal to eight. Then, he calculated the ratios of the areas and perimeters of the two rectangles, making an error when calculating the perimeter of the original rectangle. In the subsequent bundled questions, he referred directly to the first solution, writing: "Just like in the above. Explained in 1" (question 2), "This is all almost the same… Explained in 1" (question 3).

Another student of grade 10 of a technical upper secondary school, when solving question 1, filled in two "adjacent side squares", none of which was situated at a corner, and decided that the sentence was false, since "the perimeter may increase by 2". Then, he referred to that example in the subsequent problems from the bundle, marking the answers "true" in question 2 and "false" in question 3.

Many upper secondary school students followed a similar path. They often copied the justification provided in question 1 in questions 2 and 3, or directly referred to the explanation provided there. For those students, the need to justify the answer in questions 2 and 3 made no sense. It should be also noted that, although they made the right assessment of the problem in question 3, the submitted justification (drawing a general conclusion on the basis of one example) was an incorrect procedure. Pursuant to the adopted coding key, such students received 1 point.

Treating each question as separate, not noticing that they are bundled

The analysed solutions included some, in which it was clearly visible that the student treated each question as separate, without considering it in the context of the whole bundle. For instance, one of the general upper secondary students marked the answer "false" in question 1, justifying that "after removal of two squares the area will be smaller, but the perimeter may not be the same as the perimeter of the rectangle", while in question 3 he also marked the answer "false", claiming that the perimeter "may not be smaller, only the same as the perimeter of the rectangle".

Strategies used by students

The above description does not exhaust all approaches found in the students' works. Below, we present various strategies applied by students in the course of working on the bundled problems.

Finding one solution and sticking to it

A vast majority of students who correctly solved questions 1 or 2 were content with providing only one example that met the conditions of the question. Such procedure is justified in the case of the two problems. Finding one correct example enabled assessment of the sentence and its justification. Some students only filled in the relevant squares, while others, as in the following example, justified additionally, by means of calculations, that the obtained shape met the conditions of the question.

EXAMPLE 6.

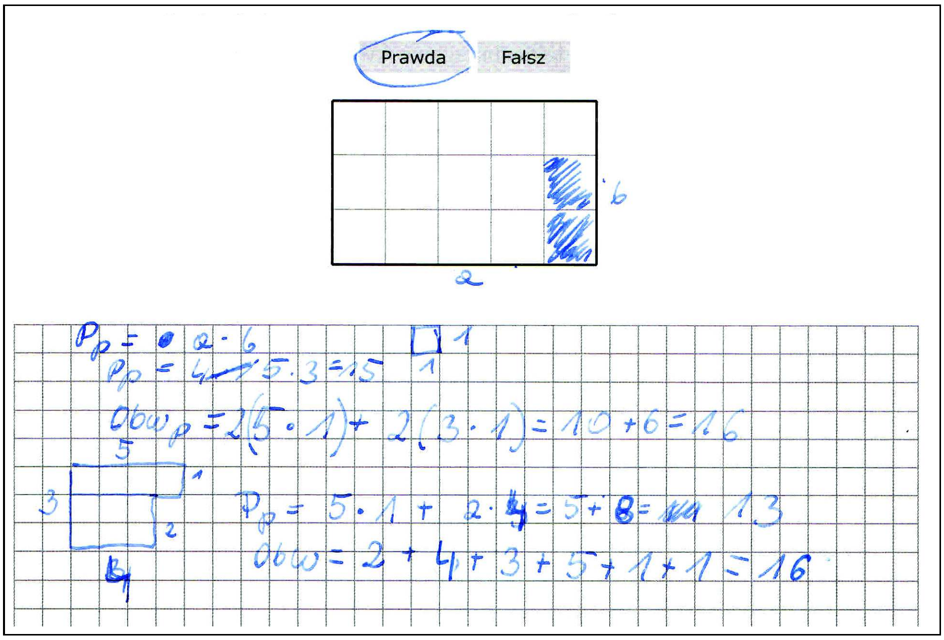


Illustration 8.

A student of grade 12 explained her assessment not only by shading the relevant squares, but also by calculating the area and perimeter of the rectangle and the shape obtained after removal of the two squares. To make the calculations legible, he drew the shape, already without the shaded squares.

However, such procedure could lead students to an answer different than the expected one, if in the first approach they failed to hit upon the right arrangement of squares to be cut off. This did not inspire them to further search, they just gave a wrong answer, as in the following example.

Example 7.

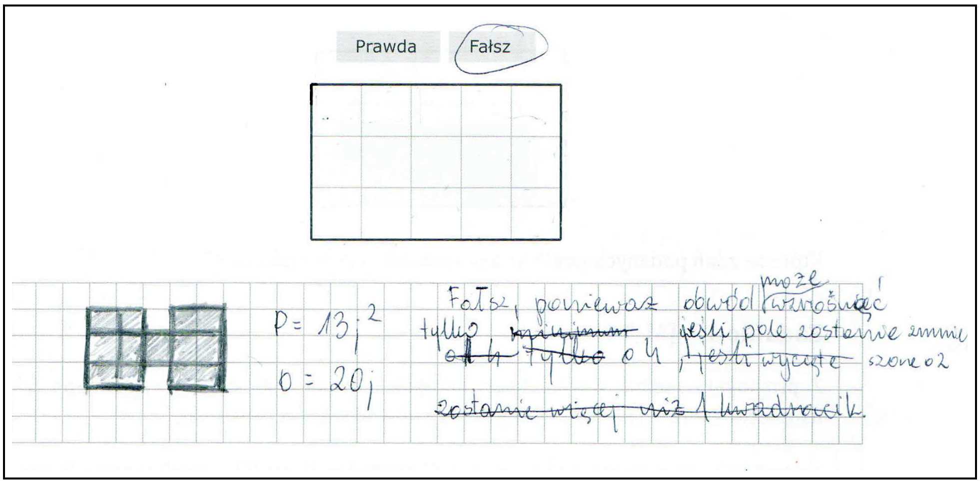


Illustration 9.[12]

In question 2 a student of grade 10 of a general upper secondary school considered only one case and drew on its basis the conclusion that "perimeter may increase only by 4 if the area is reduced by 2".

Also, in question 3, students looking for answers to the posed questions often considered only one example (much more often than several examples) and drew general conclusions on its basis. They mistakenly "justified" a general fact "on the example". It should be noted that they sometimes marked the right answer despite an incorrect explanation. Thus, the correct answer of a student did not always testify to having the ability to carry out mathematical reasoning and argumentation. The problem is discussed in more detail further in the paper, when quantitative results are discussed.

Identification of various ways to solve a question

Some students were aware of the fact that solutions to questions 1 and 2 are not unequivocal and there are many possibilities of removing two squares to obtain an area smaller by 2 and a perimeter of the new shape equal to or greater by 2 than the perimeter of the rectangle, which they tried to express in various ways (see examples 1 and 2).

[12] "False, since the perimeter may increase only by 4 if the area is reduced by 2." (translation of the text in illustration 9, written by the student).

Attempts at generalisation of a question

Attempts at generalisation of a question were perceptible in various ways especially in the solutions to questions 2 or 3. Students noted that removal of any square will reduce the area by 1, a "corner" square will not change the perimeter and one "side middle" square will increase the perimeter by 2 (with proper choices of squares). The students examined, e.g. what smallest number of squares should be removed to make the perimeter smaller than the perimeter of the rectangle, or noted that the perimeter may increase only by an even number. Exemplary solutions to question 3, in which attempts at generalisation are visible, are presented below.

EXAMPLE 8.

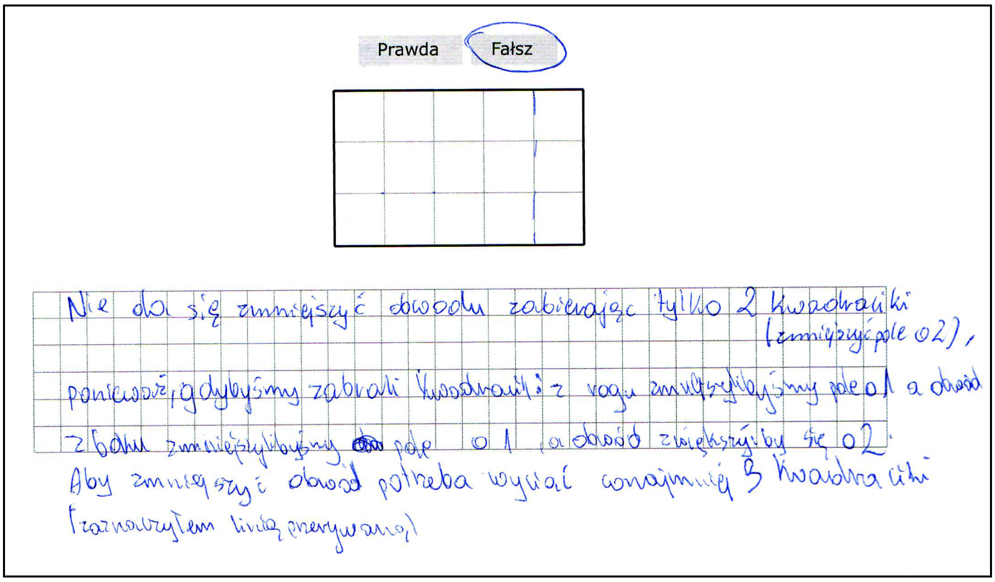


Illustration 10.[13]

The student noted that removal of any square will reduce the area of the shape by 1 and, at the same time, depending on the place from which the square is removed, may retain the perimeter of the shape unchanged or increase it by 2. Reduction of the perimeter is possible only by "cutting off"

[13] "It is not possible to reduce the perimeter by taking out only 2 small squares (reducing the area by 2), for if we took a square from a corner, we would reduce the area by 1, and the perimeter would remain the same, and from the side we would reduce the area by 1, and the perimeter would increase by 2. To reduce the perimeter, it is necessary to count out at least 3 squares (marked with the dotted line)." (translation of the text in Illustration 10, written by the student).

all squares along one of the sides. Since the shorter side of the rectangle has the length of 3, reduction of the perimeter will be possible only in the case of removing at least three squares.

6 Results of quantitative analysis

Tables 2, 3, and 4 specify the percentages of students who did not undertake to solve the question, made a failed attempt at solving a question, or presented a solution that was fully or partially correct. For questions 1 and 2, due to the fact that in each of the examined groups of students, the percentage of students who correctly shaded squares and did not mark any answer was under 1% and that in such a case lack of marking an answer could result only from the students' inattention, the percentages of students who correctly shaded squares and marked the answer "true" and those who correctly shaded squares and did not mark any answer are combined. In question 3, the category "Student marked "false" and provided incorrect justification" also included situations, when the presented explanations in fact was not a justification of the statement, why side squares could not be cut off, so that all conditions of the question would be met. Then, proper solutions included also responses, in which the student did not mark any answer, but gave proper justification why side squares could not be cut off to meet the conditions of the question.

	grade 10				**last grade of upper secondary school**			
	GUSS	T&SUSS	BVS	**total**	GUSS	T&SUSS	BVS	**total**
Student did not undertake to solve the question.	7%	12%	23%	**12%**	4%	7,5%	24%	**8%**
Student made a failed attempt at solving the question.	43%	54%	56%	**49%**	36%	46%	52,5%	**42%**
Student marked the answer "true" and did not shade squares.	4%	7%	11%	**6%**	3%	5%	10,5%	**4%**
Student marked the answer "true" and incorrectly shaded two squares.	2%	3%	4%	**3%**	1%	2%	4%	**2%**
Student correctly shaded two squares.	44%	24%	6%	**30%**	56%	39,5%	9%	**44%**

Table 2. Categories of solutions to question 1, broken down by grade and school type. *Source: Own study based on the results of the survey* School of Independent Thinking. GUSS – general upper secondary school; T&SUSS – technical and specialised upper secondary school; BVS – basic vocational school.

	grade 10				last grade of upper secondary school			
	GUSS	T&SUSS	BVS	**total**	GUSS	T&SUSS	BVS	**total**
Student did not undertake to solve the question.	12%	16,5%	31%	**17,5%**	7%	14%	31%	**12,5%**
Student made a failed attempt at solving the question.	45%	56,5%	50%	**50%**	44%	47%	50%	**46%**
Student marked the answer "true" and did not shade squares.	5%	8%	13%	**7,5%**	2%	5%	12%	**4,5%**
Student marked the answer "true" and incorrectly shaded two squares.	5%	6%	4%	**5%**	5%	5%	4%	**5%**
Student correctly shaded two squares.	33%	13%	2%	**20%**	42%	29%	3%	**32%**

Table 3. Categories of solutions to question 2, broken down by grade and school type. *Source: Own study based on the results of the survey* School of Independent Thinking. GUSS – general upper secondary school; T&SUSS – technical and specialised upper secondary school; BVS – basic vocational school.

	grade 10				last grade of upper secondary school			
	GUSS	T&SUSS	BVS	**total**	GUSS	T&SUSS	BVS	**total**
Student did not undertake to solve the question.	16%	19,5%	34%	**21%**	10%	17%	32%	**15%**
Student made a failed attempt at solving the question.	21%	33,5%	37%	**28%**	17%	24%	38%	**22%**
Student made a failed attempt at solving the question.	20%	23,5%	25%	**22%**	19%	25%	25%	**22%**
Student marked the answer "false" and supplied an incorrect justification.	21%	14,5%	3,5%	**16%**	23%	20%	2%	**19%**
Student gave the right answer and justified it.	22%	9%	0,5%	**13%**	31%	14%	3%	**22%**

Table 4. Categories of solutions to question 3, broken down by grade and school type. *Source: Own study based on the results of the survey* School of Independent Thinking. GUSS – general upper secondary school; T&SUSS – technical and specialised upper secondary school; BVS – basic vocational school.

The difficulty level of particular questions broken down by the grade level and school type is presented in Table 5.

	grade 10				last grade of upper secondary school			
	GUSS	T&SUSS	BVS	**total**	GUSS	T&SUSS	BVS	**total**
Question 1	46%	29%	13%	**34%**	58%	43%	16%	**47%**
Question 2	38%	20%	11%	**26%**	45%	33%	11%	**37%**
Question 3	42%	28%	15%	**32%**	52%	36%	16%	**42%**

Table 5. Difficulty level of questions broken down by grade level and school type. *Source: Own study based on the results of the survey* School of Independent Thinking. GUSS – general upper secondary school; T&SUSS – technical and specialised upper secondary school; BVS – basic vocational school.

It follows from the above specifications that the definite majority of students undertook to solve the questions from the bundle, yet most of the attempts resulted in a failure. Less than half of the students, even in a general upper secondary school, were able to analyse the situation from various points of view, draw conclusions and provide a reasonable justification. Although the solvability of the questions significantly increased from the grade 10 to the last grade of the upper secondary schools, especially in the technical and general schools, the results are unsatisfactory.

It should be noted that the higher difficulty level of the third question than the second question is determined by the way of marking for the questions.

For each of the questions discussed in this paper, the students could obtain 0, 1 or 2 points. In the case of students who gave the correct answer but whose justifications were too brief or incomplete, it was difficult to conclude if they had the skills of justifying or of presenting justifications. It was decided that such students should receive 1 point. Statistical analysis revealed that such a solution was correct only in the case of question 3, which we shall explain below.

To model the probability of providing the correct answer to the questions under consideration, the Graded Response Model (GRM) was applied (Pokropek & Kondratek, 2013).

The characteristic curves in this case are described by the formula:

$$P_x\left(u_i \leqslant x|\theta, a_i, b_{i,x}\right) = \frac{-1}{1+e^{-a_i(\theta-b_{i,x})}}, \quad x \in \{0,1,2\},$$

where θ (theta) – student skill level, and a_i and $b_{i,x}$ are the estimated model parameters: the indicators of discrimination and question threshold.

The curves inform about the probability of obtaining by the students, for a given question, x or less points.

Then, the characteristic curves were plotted to describe the probability of obtaining by a student with a given skill level θ the specific number of points in each category:

- 0 points: $P(u_i = 0|\theta) = P_0(u_i \leqslant 0|\theta)$,
- 1 point: $P(u_i = 1|\theta) = P_1(u_i \leqslant 1|\theta) - P_0(u_i \leqslant 0|\theta)$,
- 2 points: $P(u_i = 2|\theta) = 1 - P_1(u_i \leqslant 1|\theta)$.

Graphs 1a-1f present the characteristic curves for each of the questions in both groups of students (the first and last grade of an upper secondary school).

It follows from the above graphs that the questions rather well differentiated the groups of tested students. The arrangement of the curves reveals that in all questions, for students with low skills ($\theta < -0.5$) the most probable was obtaining zero points.

For questions 1 and 2, the probability of obtaining 1 point was quite small – in question 1, the highest point, both for the students of the grade 10 and the last grades of upper secondary schools did not exceed 0.2; the situation is similar with respect to question 2. On the other hand, for question 3, the probability of obtaining 1 point was the highest for students with skills slightly above average ($0 < \theta \leqslant 1$). Such results may be explained by the types of statements, whose truthfulness the students were to assess. In all of the tasks, the statements have an existential form, but they are true in questions 1 and 2, and false in question 3. Justification of the truthfulness of the statements in questions 1 and 2 requires provision of an appropriate example. Therefore, usually when the students found the right example, they received 2 points, and if not – zero points. The situation of proper assessment of the statement, without justification (provision of an example) was rare. Then, in question 3, the students had first to decide that the statement was false, then formulate (even mentally) its contradiction and prove a general fact. The analysis of the students' papers showed that they often made the correct assessment of the question, but were unable to justify it, obtaining thus one point. Therefore, it is possible that those students had good geometric intuitions, but had not managed to acquire the skills of reasoning and argumentation, or found it difficult to express their own mathematical thoughts.

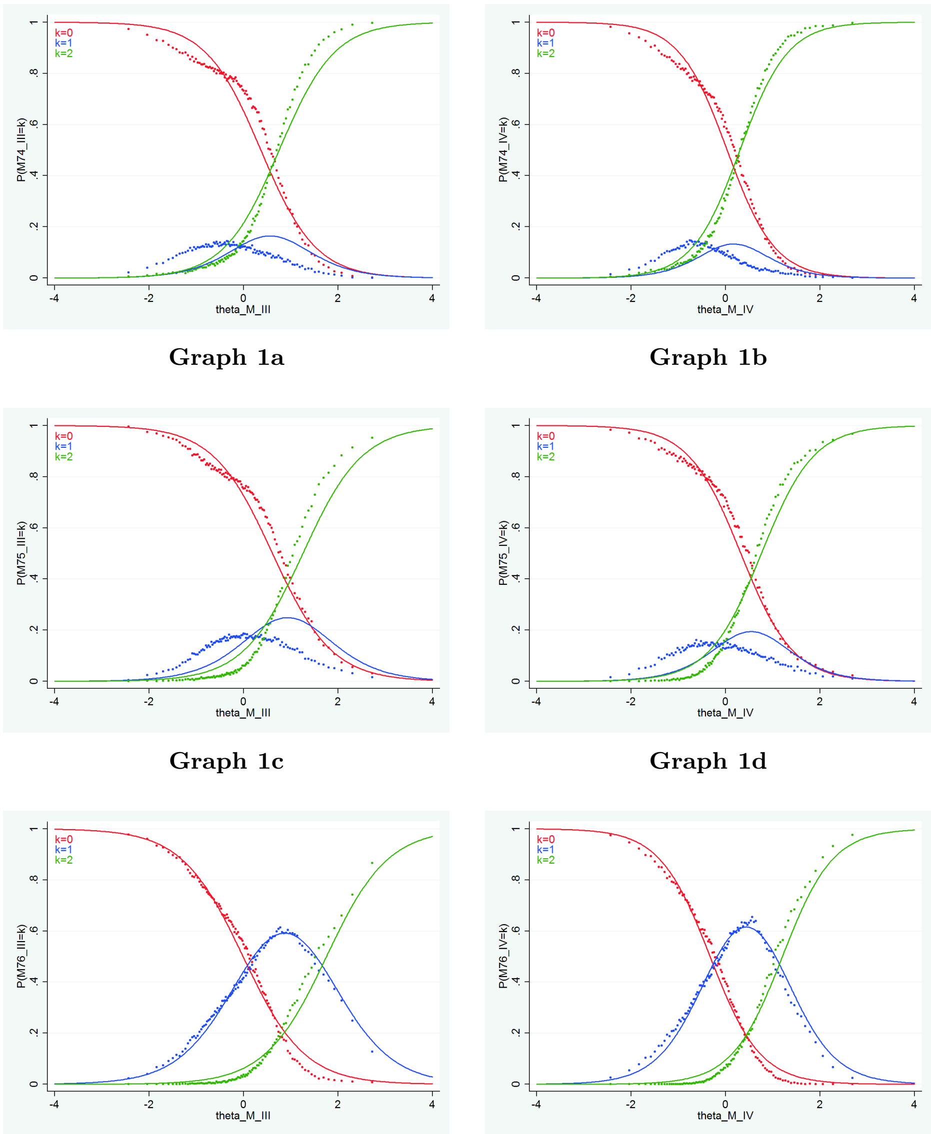
For questions 1 and 2, we can see some maladjustment of the curves to the low level of skills – students in fact were more likely to get 1 point, less to get 0 points than would result from the course of the curves in the GRM model. With such insignificant use of the category of score "1" (the distribution is u-shaped – most of the students obtained either 0 or 2 points, with a small percentage getting 1 point), it seems that the best solution in the case of those questions would be to assess at the 0-1 scale. Such maladjustment may be a signal that there occurs another factor than the skill measured by means of the whole test, which would be responsible for obtaining 1 instead of 0 points by students at a lower level of skill – for example guessing the answer, for which 1 point could be obtained.

Grade 10 | Last grade of an upper secondary school

Question 1

Question 2

Question 3



Graph 1a **Graph 1b**

Graph 1c **Graph 1d**

Graph 1e **Graph 1f**

Graph 1. The curves describing the probability of obtaining by a student the specific number of points k-number of points ($k = i$, $i \in \{0, 1, 2\}$). Development of Graphs 1a–1f: Bartek Kondratek.

7 Conclusions

The studies conducted have led to the following conclusions:

- For many students, arrangements of the questions into a bundle was not helpful in finding solutions. They did not feel "guided" or "directed". There did not occur the natural reflection on subsequent bundled questions, returning to the already solved problems and use of the knowledge of the situation obtained in subsequent questions. For some, the arrangement of the questions into a bundle not only did not help in better, deeper understanding of the situation and perception of possible errors, but just the opposite – it made the purpose of later questions unclear, and the request for justification – nonsensical.
- Many times, the students either considered only one way of cutting off squares and referred to it in subsequent tasks, or treated each task as a separate problem and did not use in subsequent tasks the already possessed knowledge of the situation. Thus, it seems justified to claim that students had rarely had the opportunity to experiment, to examine one situation from various angles and to modify it.
- What is disconcerting is the lack of ability to carry out mathematical reasoning and understand the sense of a mathematical proof by many students of upper secondary schools. They frequently were satisfied to check one case when justifying a general fact.
- A student's erroneous answer did not always result from his or her lack of knowledge or mathematical skills, but sometimes from misunderstanding of the situation described in the question. Those students did not understand the meaning of the specific words or expressions used in the text and gave them their own meanings. They would sometimes impose additional conditions. However, the problem-solving strategy they developed or the reasoning they carried out were mathematically correct with the interpretation of the situation consistent with the students' understanding.

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Analiza uczniowskich rozwiązań zadań geometrycznych tworzących wiązkę

S t r e s z c z e n i e

Jednym z głównych celów edukacji matematycznej jest kształcenie umiejętności rozwiązywania problemów, przeprowadzania rozumowań matematycznych i argumentowania. Szczególną rolę pełnią tutaj zadania geometryczne, które wymagają od osoby je rozwiązującej przyjęcia postawy badawczej i „specyficznego widzenia". To „specyficzne widzenie" polega na manipulowaniu obiektami geometrycznymi w umyśle oraz dostrzeżeniu, wydzieleniu i skupieniu uwagi tylko na istotnych informacjach. Nie wystarczy bowiem „patrzeć", trzeba jeszcze wiedzieć jak zinterpretować to, co się zobaczyło. Pomimo, że wielu badaczy zajmowało się omawianym problemem do dziś otwarte jest pytanie o to, w jaki sposób rozwijać umiejętności tego „specyficznego widzenia".

W niniejszym artykule przedstawiamy wyniki badań, których celem było sprawdzenie, w jakim stopniu połączenie zadań geometrycznych w wiązkę pomaga uczniom szkółponadgimnazjanych w „zobaczeniu" i zrozumieniu przedstawionej sytuacji, a następnie znalezieniu odpowiedzi na kilka pytań dotyczących tej sytuacji. Chciałyśmy ustalić, czy w takiej sytuacji pojawi się naturalna refleksja uczniów nad kolejnymi rozwiązanymi zadaniami wiązki, powroty do zadań już rozwiązanych i wykorzystywanie swoich spostrzeżeń i nowo nabytej wiedzy do poszukiwania odpowiedzi na kolejne pytania lub do korekty popełnionych błędów.

W prowadzonych przez nas analizach wykorzystałyśmy częściowe wyniki badania *Szkoła samodzielnego myślenia* przeprowadzonego przez Instytut Badań Edukacyjnych w 2011 roku.

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