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Relate before calculate: Students' ways of experiencing relationships between quantities

Abstract: The aim of this article is to contribute with findings concerning students' ways of experiencing general mathematical structures and, in particular, relationships in additive structures. When students discern relationships in additive structures, it may lead to positive consequences for students' future ability to perform calculations in addition and subtraction tasks. In the study, semi-structured interviews were conducted with students in grades 3, 8, and 9. An illustration showing a set of different quantities was the starting point during the interviews, together with an opening question regarding how the diverse quantities could be equalised. After the students' discussions, they were asked if this could be described mathematically using written symbols. The students' expressions concerning the phenomenon "relationships between quantities" were analyzed using phenomenography as an analytical tool. According to phenomenography, there are a limited number of ways in which a phenomenon can be experienced. Further, it is not about exploring how many individuals hold a specific experience that is of interest. In the case of this article, it is about capturing qualitatively different ways of experiencing the phenomenon relationships between quantities. Despite no specific numbers being presented, many students attributed specific numbers and values when expressing relationships between quantities. The students

Key words: general mathematical structures, relationships, part-whole, additive structures, quantities, phenomenography, ways of experiencing, critical aspect.

*Tuominen is the first author of this article, though Tuominen and Andersson have had main responsibility concerning the design of the study, data production, data processing, analysis, and much of the writing process. The final analysis as well as the final written text is the product of all four authors.

expressed general mathematical structures only to a limited extent and, in those cases, mostly only after encouragement from the interviewer. Following the phenomenographical analysis, the students' ways of experiencing "relationships between quantities" are: as something that has to be calculated, or as something that has to be related. The first of these was most common in all grades. In this study, one critical aspect was identified, namely, that quantities are related to each other, additively. Instead of introducing mathematics with a focus on answer-oriented tasks, it is essential to introduce mathematics based on general structures such as additive structures. Even if the students are not familiar with such a mathematical "culture", it is worth it. This was confirmed in our study.

1 Introduction

This article concerns students' ways of experiencing general mathematical structures and, in particular, relationships in additive structures. Essential here is a notion of a focus of mathematics as "know why" and not solely as "know how."

1.1 Know how or know why

An issue frequently addressed by researchers with different theoretical perspectives is that mathematics teaching and learning (often concerning arithmetic) in the early grades tends to focus on answer-oriented tasks and not on mathematical ideas that go "beyond" the tasks (Kilpatrick, Swafford, Findell, 2001, pp. 271–272). Mason, Graham and Johnston-Wilder (2005, p. 135) stress that if students focus on particular tasks merely in order to solve them, it may obscure their possibilities to discern general aspects such as mathematical structures (see also Ca, Knuth, 2011, p. ix). Further, the consequences and effects of mathematics teaching based mainly on specific numbers in tasks, formulas, procedures, and rules, are that it may lead to "the learned" becoming the forgotten (Chevallard, 2015, p. 176).

Mellin-Olsen (1981, p. 351) and Skemp (1978, p. 14, 2006, p. 90) describe mathematics focusing on specific numbers in tasks (e.g., $31 - 29 = \_\_$), formulas, procedures, and rules, as *instrumental understanding*. Chevallard (2005, pp. 23–24) describes this type of knowing as *know how* (praxis). Mathematics teaching mainly focusing on arithmetical operations, the instrumental understanding or know how, can also be termed as *arithmetic teaching* (van Oers, 2001, p. 62). What van Oers terms as arithmetic teaching should not be confused with teaching arithmetic, which is about the four basic operations.

Although the mathematics is actually about arithmetic, teachers can invite students to focus on mathematical structures. Mason, Stephens and Watson (2009, pp. 17–18) emphasize that teaching with a focus on identifying general mathematical structures is an important part of mathematics teaching. This is in line with what Brousseau (1979) points out: "Knowing mathematics is not simply learning definitions and theorems in order to recognize when to use and apply them" (p. 22). When mathematics teaching is based on *relational understanding*, the difference is that there are not a lot of different and separate rules to remember (Mellin-Olsen, 1981, p. 351; Skemp, 1978, p. 13, 2006, p. 92). Chevallard (2005, pp. 26–27) stresses that explaining why techniques apply, concerns *know why* (logos).

Mathematics teaching based on know why can also be understood as *algebraic teaching* (Davydov, 2008, p. 121; van Oers, 2001, pp. 62–63). In parallel with the above, algebraic teaching should not be confused with the teaching of algebra. A paradox may arise since focusing on the *how* in order to find a correct answer instead of the *why* will probably lead to a correct answer more quickly in a short-term perspective, but in the long run, it may lead to a restricted understanding (Skemp, 1978, pp. 12–13, 2006, p. 93). However, relational understanding, which means focusing on why, may prepare students to meet and grasp new, unknown, "problems" in the future (cf. Skemp, 2006, p. 92). Students who participate in mathematics teaching focusing on relationships based on general mathematical structures show good results already in early grades in compulsory school, regarding solving mathematical tasks (Kinard, Kozulin, 2012/2008, pp. 76–77; Schmittau, 2005, pp. 19–21; Slovin, Dougherty, 2004, pp. 212–215; Zuckerman, 2004, pp. 15–16).

The literature above has identified that there are several problems reported concerning mathematics teaching based merely on *know how*. In order to qualify teaching based on *know why*, the aim of the study presented in this article was to explore students' ways of experiencing mathematical structures. The research question is:

- What different ways of experiencing the phenomenon relationships between quantities can be discerned in student interviews?

The article is structured as follows: First, we briefly add to the argumentation for the mathematical and theoretical aspects used in the study reported here. Second, some methodological considerations are given. Third, the students' ways of experiencing relationships between quantities are presented. Fourth, we ultimately discuss the findings and provide some conclusions.

1.2 Additive structures

Vergnaud (1982) emphasizes that, "[a]dditive structures are a difficult conceptual field, more difficult than most mathematics teachers expect" (p. 58). When students encounter inverse relationships, for example $31 - 29 = \_\_$ and its inverse $29 + \_\_ = 31$, they may experience "a basic component in the architecture of mathematical structures" (Greer, 2011, p. 431). Further, Polotskaia (2014, pp. 39–41, 2017, pp. 166–170) claims that when teaching enables students to focus on relationships, there is no need for calculations *initially*. Instead, this approach enables students to discern and describe additive structures, for example, and how numbers are related to each other, and *thereafter* to choose an appropriate operation. One way of introducing additive structures is to explore relationships between quantities such as length, area, or volume, for example, instead of relationships between numbers (Davydov, 1975b, p. 131). Further, Davydov (2008, p. 128) advocates an early introduction of relationships between quantities where concepts and signs such as "equal to," "greater than," and "less than," have a mediating function, enabling students to reflect on and grasp the concept of number. Teaching concerning relationships between quantities is thus advocated to precede the concept of number. Before handling specific numbers, Davydov (2008, p. 128) argues that the students need to explore and identify quantities with general symbols, such as a, b, and c. The relationship between a, b, and c can be described with a "part-whole structure" (Carpenter, Moser, 1982, pp. 17–20; Ng, Lee, 2009, pp. 283–286; Schmittau, 2005, pp. 19–20). In order to visualize a structure, thus enabling students to discern relationships as a "part-whole structure," graphical representations can function as mediating tools: as learning models (Davydov, 2008, p. 95, p. 151). A learning model, which should not be confounded with mathematical models, is a visual (and sometimes tactile) model that captures structural, but abstract properties that students need to discern. Further, a learning model functions as a communicative tool for collective exploration of the phenomenon and its abstract properties (cf. Eriksson, 2017, p. 77; Davydov, 2008, pp. 94–95; Gorbov, Chudinova, 2000, pp. 1–4).

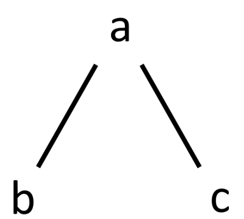

Figure 1: An example of a learning model depicting a part-whole structure.

For example, in the above depicted learning model (Davydov, 2008, pp. 126–127; Schmittau, Morris, 2004, p. 68), a can be described as the "whole", which can be built up by the "parts" (b and c). The *same* relationship can then be described as $a - b = c$; $b + c = a$; $c + b = a$; and $a - c = b$. These four different ways of describing the same relationship, Schmittau (2004, pp. 27–28) termed as the four members of the *fact family*. In the fact family, the additive structure and the inverses of addition and subtraction appear clearly. $b + c = a$, and $c + b = a$, describe the situation where the two parts build up the whole. Thus, $a - b = c$ and $a - c = b$, instead, describe that, in taking away one part from the whole, the other part is what remains. In order to formulate all four members of the fact family, the students need to abstract and to go beyond the original situation, instead focusing on the relationship between the quantities. Also Davydov (1975b, p. 131, 2008, p. 128) and Schmittau and Morris (2004, pp. 67–70) discuss the four possible ways, describing the same relationship, although without using the concept fact family.

This article adopts additive structures as described above, and we will return to the learning model (Figure 1) in the conclusion. In the analysis and the findings presented below, the focus is both on know why in the sense of relationships between quantities in additive structures, as presented in relation to Figure 1 above, and on the extent to which the students carry out abstractions, mainly in the sense of going beyond the original situation.

2 Methodological considerations

In this section the phenomenographical approach, the sample, the interviews, the analysis, and ethical considerations are described.

2.1 A phenomenographical approach with regards to student interviews

In order to find qualitatively different ways of experiencing relationships between quantities, phenomenography was chosen as a methodological approach (Marton, 2015, p. 99), where the chosen method for data production was student interviews. A basic assumption in phenomenography is, first, that people experience a phenomenon in qualitatively different ways, which depends on their individual backgrounds, and second, that there are a limited number of ways in which a phenomenon can be experienced. Thus, when people relate to a phenomenon, as in an interview situation for example, their previous experiences will be the basis for what they express. When analysing interview data, the aim is to describe qualitatively different categories of ways of experiencing a phenomenon (Eriksson, 1999, pp. 35–36; Marton, 2015, p. 106). Further, according to phenomenography, it is not how many individuals express a specific way of experiencing a phenomenon that is of interest. Moreover, in an interview, a single person can express one or several ways of experiencing a phenomenon. Therefore, the findings are presented as, for example, "some students' ways of experiencing the phenomenon are as..." or "several students' ways of experiencing the phenomenon are as..." A phenomenographical analysis requires a comparative reading of the transcribed interviews, while at the same time trying to capture what experiences may lie behind the spoken words.

In order to explore students' ways of experiencing relationships between quantities, semi-structured interviews were conducted with pairs of students, totalling thirty, in Sweden. One researcher (Tuominen) interviewed eight pairs in grade 3 (9 year olds) and one researcher (Andersson) interviewed seven pairs of students in grades 8 and 9. The students in grade 3 attended the same school, and the students in grade 8 and 9 attended three different schools. Further, despite their different ages, the students had, according to their teachers, similarly limited experiences of teaching based on general mathematical structures (know why).

The sample of students was obtained by direct and goal-oriented (purposive sampling) methods, and thus, not randomly (Bryman, 2011/2002, p. 393). The intention was to interview students who, according to their teachers, had not previously met general mathematical structures as addressed in this article. The teachers chose and identified students based on their demonstrated abilities to solve mathematical tasks. The students were selected from different performance levels: those who solved regular tasks in a relatively straightforward manner, those who struggled with the mathematical tasks, and those who were somewhere between these two groups. The intention of these three groups was to find as varied ways of experiencing the phenomenon as possible.

The students in each pair knew each other well. Each interview was 30–60 minutes long and was conducted in a detached room, separated from the ordinary classroom. The interview guide was designed by the two main authors of this article (Tuominen, Andersson), and was piloted with two students of the same age as the students who participated in the study, before the interviews were conducted. During this process, the interview guide was slightly revised. One revision was to use a slightly different illustration for the older students, since the illustration in Figure 2 did not seem to challenge or to enable these students to communicate theoretically about relationships. Thus, the final interview guide consisted of different illustrations for the different age groups. Further, directed, although still open, questions were asked, with the same questions for both age groups. White unlined paper and pencils were available for the students to use, in case they wanted to draw and/or write something in order to clarify their reasoning about the relationship between the quantities. The interviews were audio- and video recorded and transcribed.

As mentioned above, the phenomenon of interest was relationships between quantities. In order to enable the students to talk about relationships such that it would be possible to analyse their ways of experiencing (cf. Marton, 2015, pp. 90–91) the phenomenon, an illustration (see Figures 2 & 3, below), inspired by a textbook, was presented initially during the interviews. The textbook is based on Davydov's research and general mathematical structures (Davydov, Gorbov, Mikulina, Saveleva, 2012, p. 22, p. 25). The intention behind using illustrations, and not concrete objects or tasks, was to enable the students to focus on possible mathematical structures and not on manipulating the objects, or merely to focus on calculating tasks as mentioned in the Introduction, in line with what Kilpatrick et al. (2001, pp. 271–272) and Mason et al. (2005, p. 135) stress. Further, the context used should be familiar to the students. Moreover, the assignment should enable the students to discuss the mathematical structures theoretically (Davydov, 2008, pp. 93–94).

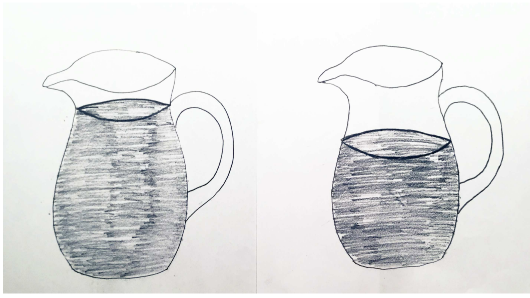

Figure 2: The illustration shown to the students in grade 3.

As an interview started, the students in grade 3 were shown the illustration (Figure 2), displaying two jugs with different quantities. Another illustration (Figure 3), displaying three cylinders with different quantities, was shown to the students in grades 8 and 9.

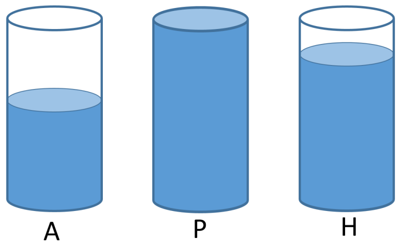

Figure 3: The illustration shown to the students in grades 8 and 9.

The illustration in Figure 3 displays three cylinders, with different quantities. The quantities are identified by the same letters as in the mentioned textbook by Davydov and his colleagues.

Although the illustrations used during the interviews were different for the different grades, the interview guides were identical and the assignments and questions were similar. Initially, during the interviews, the students were told to change the quantities in the jugs or the cylinders. The interviewer said: As much as is here [pointing to one of the quantities], should be there [pointing to the other jug in Figure 2; one other of the quantities in Figure 3]. How can that be solved? After the students in each pair reasoned about how the volume could assume equal quantities, they were asked whether it could be described mathematically and, if so, in what way. Later, the students also were encouraged to write their suggestions on paper. Since the students were interviewed in pairs, they discussed a lot with each other and the interviewer's role was, to a large extent, to listen and to ask supplementary questions when necessary. If the students merely used specific numbers, the interviewer asked: Do we know how much the amounts are? During this discussion, the interviewer asked a question like: Can we label the amount (for example, the amount in the left jug) as A? After that question, the interviewer did not ask further questions regarding general symbols, thus, the students had further discussions on their own. Using quantities and mathematical symbols, Davydov (2008) stresses as following:

> [...] it is only the use of the letter formulas that produces an abstraction of the mathematical relation. But the letter formulas record only the results of real or mental actions with objects, while a graphical representation [...], being a visible quantity (a length), enables the children to perform real transformations whose results can be not merely imagined but also observed (p. 151).

Zuckerman (2003, pp. 184–185, 2004, pp. 10–11) complements Davydov's quote above when claiming that teaching based on general mathematical structures enables students to abstract and to develop sustainable reflective abilities. In the study for this article, we did not actually teach, but similar reasoning was the basis for conducting the interviews through situations where the students were invited to display what Zuckerman would label as abstract and reflective abilities.

2.2 The phenomenographical analysis

According to phenomenography (Eriksson, 1999, pp. 32–36; Marton, 2015, pp. 106–107), and in order to capture the students' ways of experiencing the phenomenon relationships between quantities, the analysis was conducted as follows:

- the audio- and video recordings were studied several times by Tuominen and Andersson,
- the interviews were transcribed and the transcripts were read several times,
- the students' expressions, both oral and written, concerning the phenomenon were highlighted,
- the expressions were analysed and interpreted as ways of experiencing the phenomenon.

Below, the analysis is described in more detail.

In the phenomenographical analysis, it was not what the students said (or did) concerning the phenomenon that was the focus of the analysis, rather it was the possible ways of experiencing underlying their expressions (Eriksson, 1999, p. 33). In the analysis, the students' oral and written signs and actions were taken into consideration. In the students' use of signs, for example mathematical symbols, graphs, written and spoken words, and other different artefacts, such as rulers, helped to identify ways of experiencing the phenomenon (Radford, 2000, pp. 259–262, 2010b, pp. XXXV–XXXVI; Radford, Schubring, Seeger, 2011, p. 150). The process of the analysis focused on what the students expressed concerning the relationships between quantities and on what experiences their expressions maybe based.

During the comparative reading of the transcribed interviews, the students' expressions, relevant for the analysis and the interpretation, were highlighted. An example of an expression which was highlighted is "And then [...] we need to add to C to get the total of nine centimetres, which we call A." The reason for highlighting this expression was that it was an expression reflecting how the students experience the relationship between the quantities called C and A.

In the analysis, when interpreting the students' different expressions we were influenced by what, for example, Mason et al. (2005, p. 135) emphasize concerning that if students are focused on solving tasks by calculating something, it may obscure their possibilities for discerning general mathematical structures. In the analysis we could also draw on previous research, for example, Chevallard (2005, pp. 23–24, pp. 26–27) concerning know how and know why, and Skemp (1978, pp. 13–14, 2006, pp. 90–92) concerning instrumental and relational understanding. Further, the analysis was influenced by Vergnaud (1982, p. 58), concerning additive structures, as well as Davydov (2008, p. 95) concerning relationships between quantities. Thus, previous research supported the analytical work.

In the analysis, students' ways of experiencing the phenomenon were categorised into qualitatively different categories (Marton, Pang, 2006, pp. 203–204). In the Findings section below, students' expressions will be used as illustrations of the categories as excerpts or as descriptions. Thereafter, the analysis and interpretations of the expressions will be presented as students' ways of experiencing the phenomenon.

The findings depicting the qualitatively different ways of experiencing the phenomenon consist of two main categories that together form what in phenomenography is called an *outcome space* (Marton, 1995, p. 164). In the outcome space (Figure 4) identified in this study, the categories could be arranged hierarchically, where one category indicates a more nuanced understanding. Further, the analysis led to two sub categories specifying each of the two main categories.

According to phenomenography, what it means to experience a phenomenon, and what distinguishes between two different ways of experiencing a phenomenon, is called a *critical aspect* of the phenomenon (Pang, 2003, pp. 151–152). In this article, we identified one critical aspect related to the phenomenon relationships between quantities.

2.3 Ethical considerations

All parents or guardians of students involved in the interviews received a letter of formal notice where they were asked to agree (or disagree) with their children participating. It was stated clearly in the letter that audio- and video recordings would take place. The participating students were informed about these intentions. The students were also informed about how the data would be used and handled, for example to include what they would calculate, draw, or write during the interviews. The students were informed several times that they had the right to interrupt the interview without any negative consequences (see Codex; The Swedish Research Council, 2017, pp. 26–27).

3 Findings

In the findings, examples of what the students were saying concerning relationships between quantities are presented, and can be seen as examples of data for the phenomenographical analysis. As mentioned in Methodological Considerations, the students' expressions were analysed and interpreted as their ways of experiencing the phenomenon relationships between quantities. As mentioned above, the interview guides were identical, and the assignments and questions were similar for both age groups. According to phenomenography, the analysis is about capturing qualitatively different ways of experiencing a phenomenon. The findings from this study are presented from all three grades. In those cases where there are a difference between the different grades, it will be noted.

3.1 Students' ways of experiencing relationships between quantities

The students, regardless of grade, talked about the relationships between quantities in more or less general terms. Mostly, the students talked about the specific quantities in the illustration using specific numbers, even though no specific values were given in the assignment. The students' general and specific expressions were different: for example, whether they used specific numbers or general symbols, or whether their reasoning was based on the given illustration or not. One example of when students "went beyond" the shown illustration is when they presented another, similar, example. Furthermore, students sometimes described the specific situation depicted in the illustration, and sometimes not. Some students expressed relationships between quantities, in several ways (see fact family in the Introduction section).

In this section, we present the result of the phenomenographical analysis. Although phenomenography does not focus on how many of the interviewees are represented in a specific category, it may be of interest to know if there were merely one or two, or several students. In order to give that type of information we use words such as some or several in relation to the students. According to the analysis, an outcome space with two main categories was discerned (see Figure 4). In relation to each respective main category, two specifying sub categories were identified and the categories were arranged hierarchically.

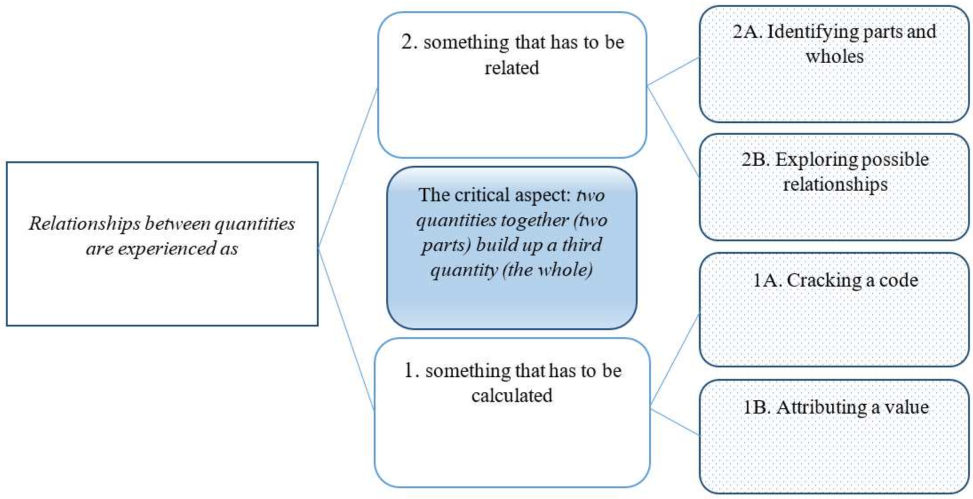

Figure 4: The outcome space concerning the main categories, the sub categories, and the critical aspect are depicted.

As depicted in Figure 4 above, the main category 1: *something that has to be calculated*, with its two sub categories, *cracking a code* and *attributing a value*, represents less qualified ways of experiencing relationships between quantities. Further, this way of experiencing the phenomenon represents most of the students' responses. The main category 2: *something that has to be related*, with the two sub categories, *identifying parts and wholes* and *exploring possible relationships*, involves experiences that are more qualified and shows a deeper understanding of the phenomenon. What distinguishes between the two different ways of experiencing relationships between quantities, is labelled a *critical aspect* (Pang, 2003, pp. 151–152). The critical aspect identified in this study was formulated as *two quantities together (two parts) build up a third quantity (the whole) with the same "value" as the two parts together*. In order to experience relationships between quantities in a more qualified way than "something that has to be calculated," the students need to discern that the quantities are related to each other (see Figure 4 above). The critical aspect will be described in more detail in the last section of Findings.

3.2 Main category 1: Something that has to be calculated

In the main category 1, the students attributed specific numbers to the quantities, sometimes mixed with general symbols, in order to be able to perform a calculation. Some of the students expressed that "if you do not have any values, you cannot solve the task." This finding is in line with what Skemp (1978, p. 14, 2006, p. 90) describes as an instrumental understanding of mathematics. In this category, the students draw on teaching they have previously met, and therefore they express that some kind of numerical answer should be produced.

The students tried to handle the assignment by transforming either the symbols or the quantities into numbers or percentages in the tasks, to which they were then able to calculate an answer. According to the process in the phenomenographical analysis of this study, presented in the Methodological Considerations section, two sub categories were identified. The two sub categories are elaborated below, and then follows a descriptive text, based on the students' expressions. Thereafter, the analysis and interpretation of students' ways of experiencing the phenomenon, is presented.

3.2.1 Sub category 1A: Cracking a code

In sub category 1A, some students in grade 3 converted the letters into numbers, as if the letters were a "code."

Example: A is worth one, since it is the first letter of the alphabet

During the interviews, the interviewer and the students discussed whether quantities could be denoted by general symbols as letters. Some students in grade 3 converted the letters A, B, and C into specific numbers, referring to the letters' respective positions in the alphabet, such as $A = 1$; $B = 2$; $C = 3$, and so on. Converting letter symbols into specific numbers appeared to be problematic when the students and the interviewer discussed how the quantities A, B, and C were related to each other, since the largest quantity was denoted by A. One student disagreed that B plus C could be equal to A since, according to the student's statement that $A = 1$; $B = 2$; $C = 3$ and therefore 2 plus 3 is not equal to 1.

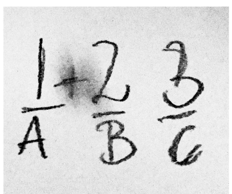

Figure 5: A student converted letters into specific numbers based on the letters' positions in the alphabet.

One student in grade 3 tried to convince the interviewer that only $A+B=C$ or $B+A=C$ describes a possible relationship, not $B+C=A$ or $C+B=A$.

In the analysis, it became clear that the student first focused on the need to calculate something. Because only general symbols were given, it was not possible to calculate anything. Since the student claimed that the given general symbols have a specific value, based on the letters' positions in the alphabet, the student's response was interpreted as a way of experiencing the phenomenon as something that has to be calculated. More specifically, it was interpreted as a code to be cracked (in order to manage to calculate and to solve the assignment). Possibly, the students have previously mostly met mathematics teaching where each calculation requires numbers, and therefore the letters must be converted into numbers.

In sub category 1A, only students from grade 3 were represented.

3.2.2 Sub category 1B: Attributing a value

In sub category 1B, students expressed the specific relationship between the quantities by attributing values to specific numbers, sometimes based on estimated sizes of the quantities visualized in the illustration.

Example: "That's approximately eleven litres"

One example from a student in grade 3 is shown in Excerpt 1, where the student described the sizes of the quantities in the jugs.

> Excerpt 1, grade 3
> Ami: I am thinking about... here it was... maybe this [the jug with larger quantity] is fourteen litres [the student draws two jugs with different quantities] and this one [the jug with a smaller quantity] will be as much [as in the jug with larger quantity]. [...] And then, then that's approximately eleven litres [the jug with a smaller quantity]... and then they need a bit more... then we shall... then we will have this: eleven and fourteen [the student writes "11" above the jug drawn by the student herself with the smaller quantity, and "14" above the jug with the larger quantity.] [...] Then we have to add three [writes "+3" on the side of the jug with a smaller quantity]. And now there is fourteen there [rewrites "11" as "14"].

The student expressed the actual situation as $11 + 3 = 14$, and not one of the other ways to describe the same relationship between the quantities (e.g., $14 - 3 = 11$ or $14 - 11 = 3$).

Another example is a student in grade 9 discussing the quantity H and comparing the quantity with the quantity P (see Figure 3). In Figure 3, there are three quantities, but first the students discussed the relationship between the quantities H and P. This student decided a value by estimating the quantity H, first as two centimetres, and then as one and a half centimetres.

In order to obtain a value, students sometimes measured with their fingers, with their pencils or merely attributed a value, estimated by their "naked eyes." There are also examples of when students in grade 3 constructed their own measuring tools, drawing a ruler next to the jugs on the illustration, thus attributing a value to the quantities inside the jugs.

In the analysis of data such as Excerpt 1, as well as the above example from grade 9, it was identified that the students estimated the quantities, and thereafter attributed specific numbers or values to the quantities on the given illustration. This is shown when the student in grade 3 attributed the quantity of fourteen litres to the jug with larger quantity, and, from that, decided that the value for the other quantity was eleven litres. This is also shown when the student in grade 9 finally estimated the quantity H as one and a half centimetres. According to the students' expressions, it was concluded in the analysis that relationships between quantities are experienced as something that has to be calculated. Accordingly, the students attributed a value, based on an estimate. Possibly, the students have previously met mathematics teaching where tools are required when measurements are to be performed for identifying and attributing correct values of quantities in order to solve an assignment. When there is no tool, you need to find an approximate value.

Example: "Add one to B"

Students also expressed the actual mathematical situation and the specific relationship between the quantities shown in the illustration using specific numbers mixed with general symbols. In an example from grade 3, the interviewer and a student discussed how the quantities in the two jugs could be made equal:

> Excerpt 2, grade 3
> Interviewer: Could this [the larger quantity] be called "A?" Could we call that quantity "A"?
> Ben: And this [the smaller quantity] is "B." [The student points with his pencil to the smaller quantity, and writes "B".] [...]
> Interviewer: If we are talking with "math language" then, what was it we needed to do, what did we say? [To assume an equal quantity in B, as in A]
> Ben: We might have to add one to B. [Ben writes $B + 1 = A$]

When the interviewer encouraged the student to express how the quantities related to each other with "math language", the student expressed the difference between the two quantities (identified by A and B) with a specific number (one).

Another example related to "Add one to B" is when two students in grade 9 discussed and compared the quantity A with the quantity P (see Figure 3), and finally wrote: "$A + 3X = P$". These students in grades 3 and 9 talked about the actual mathematical situation given in the illustration. This is shown when the student in grade 3 said, "We might have to add one to B", and wrote "$B + 1 = A$", and not one of the other ways to describe the same relationship between the quantities (e.g., $A - 1 = B$ or $A - B = 1$).

It appeared that the student in grade 3 accepted the interviewer's proposal to identify the larger quantity as "A", and thereafter, by himself, immediately proposed "B" for the smaller quantity. However, when discussing how to make the quantities equal, the student expressed: "We might have to add one to B." The students in grade 9 also tried to solve their assignment by mixing general symbols with specific numbers. In the analysis, the students' ways of solving the assignments were interpreted as students' ways of experiencing relationships between quantities are as something that has to be calculated. Further, when there is an unknown, it has to be represented by a specific value, and therefore you have to insert a specific number. Possibly, the students have previously met mathematics teaching which suggests that a calculation always requires numbers to find an answer.

Example: "P – 25% = H"

In another example, two students in grade 8 discussed what must be done with the quantity in the full cylinder (quantity P, in Figure 3) to assume a lesser quantity, as illustrated in the right cylinder (quantity H) (see Figure 3). The students explained that the cylinder containing quantity P is absolutely full, "containing 100%" (in the words of the students), and estimated the quantity H to be three quarters of the quantity P, three quarters of the whole, "75%." The students expressed the actual situation as: "one hundred per cent minus twenty-five per cent is seventy-five per cent." The students wrote: 100% – 25% = 75%. The students also expressed that one hundred per cent minus seventy-five per cent is equal to twenty-five per cent. When encouraged to describe what to do when the quantity P has to assume a quantity equal to H, the students wrote: $P - H = 25\%$, and then they corrected the expression to: $P - 25\% = H$. The students attributed the quantity P with the specific value 100%, the quantity H with 75%, and the difference between the two quantities as 25%. This shows that the students expressed the relationship with specific values and not with general symbols. The students did not go beyond the actual situation, since the letter symbols and the chosen percentages correlated to the given illustration. These students converted the quantities to percentages based on their estimates from the illustration. The students' expressions were interpreted in the analysis as students' ways of experiencing the phenomenon are as something that has to be calculated and, in this example, using percentages. Possibly, the students have previously met mathematics teaching where they normally should produce a numerical answer.

In sub category 1B, students from all three grades were represented, although it was only the older students who were talking about percentages.

3.3 Main category 2: Something that has to be related

In main category 2, the students' ways of experiencing relationships between quantities are: as something that has to be related. This is expressed by the students with specific numbers as well as with general symbols. Regardless of how the students expressed the relationship, they focused on mathematical structures, in line with what Mason et al. (2009, pp. 17–18) emphasize as important. The students apparently draw on the mathematics teaching they met earlier, saying that the quantities are related in some way, for example, as parts building up a whole. The students in the study tried to handle the assignment by describing the relationship using the inverses of addition and subtraction, which Greer (2011, pp. 431–433) describes as a basic part of mathematical structures.

3.3.1 Sub category 2A: Identifying parts and wholes

In sub category 2A, students expressed that parts build a whole.

Example: B and C build up A

An example is from grade 3, where a pair of students and the interviewer identified the quantities in the given illustration with A (the larger quantity), B (the smaller quantity), and C (the difference between A and B). During the interview, one of the students drew his own illustration (Figure 6).

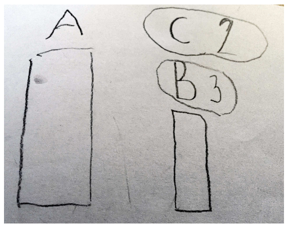

Figure 6: A relationship expressed by a student's own general graphics.

At the same time as the student was drawing, he explained the meaning of his illustration (see Excerpt 3, below).

Excerpt 3, grade 3
Elif: If this maybe, if this... thing [draws a rectangle] maybe is five. Yes, and then you... have this one [draws another rectangle, smaller than the first], maybe is three then. Then that is A [the larger rectangle] then... [the student writes "A" above the larger rectangle]. And then here [referring to the smaller rectangle] you have a B... [writes "B" and

3 above the smaller rectangle]. And then a C there [writes "C" and 2 above the smaller rectangle and the symbol "C" indicating the difference between "A" and "B"].

In Excerpt 3, the student gave an alternative example (using rectangles) of a relationship between quantities, different from the given illustration with the jugs (see Figure 2). The student's example described a situation corresponding to the actual situation based on the jugs. The student chose the same letter symbols for the rectangle quantities as on the given illustration; A, B, and C. Subsequently, the student attributed a specific number to each of the respective rectangles: 5, 3, and 2. In the illustration (Figure 6) the value 5 is not written.

Another example is from grade 9, where a student is discussing the relationships between two of the three quantities; in this example, A and P (see Figure 3).

Excerpt 4, grade 9
Inez: Maybe you have to subtract A from P? [...] then [...] if you have a whole, this is a whole [points to the quantity P]. [...]
And then you should take away this [points to the empty space (in the cylinder) above the quantity].

In Excerpt 4, the student discussed the relationship between the three quantities A, P, and the empty space (the unknown), and mentioned quantity P as "a whole". The student's example was based on the specific illustration. In the analysis, the student's expressions were interpreted as the students' ways of experiencing the phenomenon are as something that has to be related, additionally, as identifying parts and wholes. According to Excerpts 3 and 4, students may have previously met mathematics teaching suggesting that relationships can be described with geometric shapes and implicitly describing the parts that build up a whole. Describing relationships as a "part-whole structure" is mentioned in the Introduction (see Carpenter, Moser, 1982, pp. 17–20; Ng, Lee, 2009, pp. 283–286; Schmittau, 2005, pp. 19–20).

In sub category 2A, students from all three grades were represented.

3.3.2 Sub category 2B: Exploring possible relationships

In sub category 2B, students described relationships between quantities and/or between numbers. They talked about the specific relationship in more than one way: describing both the actual situation, as well as going beyond the actual situation with specific numbers, or a mix of specific numbers and general symbols.

Example: "$4 + 1 = 5$ *as well as* $5 - 1 = 4$"

In one example, a student in grade 3 attributed specific numbers to the quantities in the illustration when stating that the same relationship between the quantities could be described by $4 + 1 = 5$ as well as by $5 - 1 = 4$.

> Excerpt 5, grade 3
> Connie: Eh, that one [pointed at the jug with larger quantity] has five... decilitres and that one [the jug with a smaller quantity] has four and then you can pour another decilitre [the student wrote $4 + 1 = 5$]. [...].
> Interviewer: Can you write that in another way? [The student wrote $5 - 1 = 4$].

Here, the student expressed the same relationship in two different ways. This is shown when the student wrote "$4 + 1 = 5$" which corresponds to the actual situation, when adding a quantity to the smaller quantity, to obtain the larger quantity. The student also wrote "$5 - 1 = 4$," which describes a situation beyond the actual situation, although describing the same relationship. Thus, the student expressed the same relationship with addition and its inverse subtraction. According to the analysis, the students' ways of experiencing the phenomenon were interpreted as something that has to be related, and, further, that the same relationship can be formulated in several ways, as the inverses of addition and subtraction. Possibly, the students have previously met mathematics teaching demonstrating addition and subtraction as related to each other.

Example: "$B + C = A$ *and therefore* $A - B = C$

In another example, the quantities were identified as A, B, and C by some students in grade 3 in discussions during the interviews. When the students in one pair were asked to describe how the quantities (see Figure 2) were related to each other, one of the students wrote: $B + C = A$ and, further, $A - B = C$. The other student in the pair then wrote: $A - C = B$. Another example is from grade 9 when students discussed the relationships between the three quantities A, P, and the empty space, the unknown (see Figure 3). The students denoted the empty space as C and wrote: $A + C = P$ and further, $P - C = A$. Later the students in grade 9 now discussing the relationship between the quantities P, H, and the empty space above H, which they denoted as E and also wrote $P - E = H$ and $P - H = E$.

In these cases, the students described the relationship with general symbols. The students in these examples expressed the same relationship with addition and its inverse subtraction. According to the analysis, the students' ways of experiencing the phenomenon are as something that has to be related and, further, when $B + C = A$ is valid, the same relationship can be formulated by a subtraction. Additionally, it does not matter which part is taken away from a whole, the relationship is still the same. Possibly, the students' responses indicate that they have previously met mathematics teaching that no matter what part is taken away from the whole, the same relationship is described, as well as that addition and subtraction are inverse mathematical relationships.

Example: Four possible ways

In this example, two students in grade 9 talked about a relationship between the symbols within expressions. The students expressed the same relationship in four possible ways and further, one of the four was expressed with the sum to the left of the equals sign.

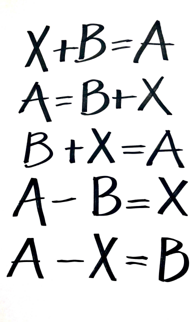

Figure 7: The same relationship, formulated in several ways (the picture is reconstructed due to the poor quality of the original image).

In the analysis, the students' expressions were interpreted as the students' ways of experiencing the phenomenon are as something that has to be related, and that the same relationship can be formulated in four different ways. Possibly, the students have previously met mathematics teaching where the same relationship has been expressed in several ways.

Example: "No matter how much it was from the beginning"

Excerpt 6 illuminates how students expressed that general symbols are always valid, and that relationships between quantities in a general way can be expressed with general symbols.

> Excerpt 6, grade 9
> Gry: No, but, like... it's not, not [one] hundred per cent, but, but... that you show [it] like this, that X, this is X, because it [X] maybe not is [one] hundred per cent, but instead X being like it... I do not know how to explain this, but that... [...]. Yes, but I think that like... yes, for example, if you look at these [the quantities] if you would not know... the size and such things... then you use X and Y... and letters... [...]

> ... so you can know... keep an eye on them... approximately, like that, yes... it is... it. And if you add it together with Y, then you get... yes, K [...].
> Helin: I thought approximately like Gry. But it may be fifty per cent instead of one hundred. But when you remove Y from X, it is always as much as K. No matter how much it was from the beginning.

One of the students in the excerpt above formulated a general expression without any support from the given illustration, although she referred to the illustration when giving a general example, while not knowing the value of the quantities. She also said that the quantities could have any value from the beginning, though the relationship is still the same and the letter symbols are always valid. According to the analysis, the students' ways of experiencing the phenomenon are as something that has to be related, regardless of general symbols or specific numbers. Further, the students related to this example, experiencing the phenomenon as: It does not matter how much a whole assumes from the beginning, it is still the same relationship. Possibly, the students have previously met mathematics teaching where general symbols have been used when describing relationships. This finding may be in line with what Skemp (2006, p. 92) points out when stressing that relational understanding may prepare students to grasp unknown tasks.

In sub category 2B, there were students from all three grades represented. It was, however, only the older students who represented "Four possible ways" and "No matter how much it was from the beginning. "Four possible ways" represents the four different ways of describing the same relationship, and are in line with "the fact family," mentioned in the Introduction (see Davydov, 1975b, p. 131, 2008, p. 128; Schmittau, 2004, pp. 27–28; Schmittau, Morris, pp. 67–70).

Those students who paid attention to the relationship were initially analysing the relationship between the quantities, and thus not merely focusing on finding an answer. When they talked about relationships, they used either specific or general expressions.

3.4 The critical aspect

In this study, two qualitatively different categories were identified, based on the students' expressions concerning relationships between quantities. The expressions were analysed, and interpreted as students' ways of experiencing the phenomenon. According to phenomenography, mentioned above in Methodological Considerations section, what it means to experience a phenomenon, and what *distinguishes* between two different ways of experiencing a phenomenon, is called a *critical aspect* (Pang, 2003, pp. 151–152). The students' ways of experiencing the phenomenon relationships between quantities are, as something that has to be calculated and as something that has to be related. What distinguished these two main categories is the critical aspect identified in this study. In Figure 4, the critical aspect is placed between the two main categories. In order to experience the phenomenon relationships between quantities in a more qualified way than "something that has to be calculated," the students need to discern that the quantities are related to each other. Therefore, teachers need to enable students to discern the critical aspect: *two quantities together (two parts) build up a third quantity (the whole) with the same "value" as the two parts together*. In this way, the critical aspect captures that it is about an additive structure. The same relationship can also be expressed as: *if one of the parts is taken away from the whole, the other part is what remains*. This critical aspect and consequences that this may imply for teaching will be discussed below.

4 Summary of findings and concluding discussion

In the analysis, two qualitatively different categories with two sub categories respectively, depicting students' ways of experiencing the phenomenon relationships between quantities, were identified. Category 2 (something that has to be related) represents a more qualified understanding than category 1 (something that has to be calculated). The main categories and a short description of each, including the two sub categories specifying each category, are summarized in Table 1 below. The critical aspect identified in the study is mentioned below Table 1.

In order to experience relationships between quantities in a more qualified way, teachers need to enable students to discern the critical aspect: *two quantities together (two parts) build up a third quantity (the whole) with the same "value" as the two parts together*. This critical aspect can also be formulated as: *if one of the parts is taken away from the whole, the other part remains*. The critical aspect in this study was identified by comparing the two main categories in the outcome space (Figure 4).

The examples depicted in Table 1 are representative of the categories. Despite the fact that the study involves students from different grades, the analysis shows that the students, regardless of grade, can hold the same or similar ways of experiencing relationships between quantities. Further, from the students' utterances presented above in the Findings section, it was clear that some students hold both ways of experiencing relationships between quantities during an interview, for example, when giving another example. This is consistent with phenomenography, as mentioned previously in the article: the same person can hold different experiences of a phenomenon (Larsson, 1986, p. 36).

Main category and their descriptions The students experience relationships between quantities as...	**Sub categories and their descriptions**	**Examples**
1. something that has to be calculated Students indicate that a numerical answer should be produced.	1A. Cracking a code ...something to be calculated based on converting letters to numbers	• "A is worth one, since it is the first letter of the alphabet"
	1B. Attributing a value ...something to be calculated with attributed values, sometimes estimated from the illustration	• "That's approximately eleven litres" • "Add one to B" • "$P - 25\% = H$"
2. something that has to be related Students describe a relationship between quantities and between numbers.	2A. Identifying parts and wholes ...something that is built up by parts	• B and C build up A
	2B. Exploring possible relationships ...something that can be formulated in several different ways	• "$4 + 1 = 5$ as well as $5 - 1 = 4$" • "$B + C = A$ and therefore $A - B = C$" • Four possible ways • "No matter how much it was from the beginning"

Table 1: The Categories and Their Descriptions.

As mentioned, it was not *how many* students who represent a specific experience which was of interest in the study. Instead, it was about identifying possible qualitatively different *ways of experiencing* (Marton, 1981, pp. 177–178) relationships between quantities. Still, it can be interesting to notice that most of the students' ways of experiencing relationships between quantities are as something that has to be calculated, according to category 1. Students' ways of experiencing the phenomenon as something that has to be related (category 2) were represented by few students.

Considering what we have found so far, there is no previous research specifically exploring students' ways of experiencing relationships between quantities. Thus, the findings in this study can be described as a contribution in relation to previous research. For example, Kilpatrick et al. (2001, pp. 271–272) stress that mathematics teaching tends to focus on answer-oriented tasks. In the assignment that the participating students received in our study, there were no specific values. Despite this, most of the students, regardless of grade, focused on calculating something in order to find an answer. Teaching based on specific numbers may also be mentioned as focusing on instrumental understanding (Skemp, 1978, p. 14, 2006, p. 90). In this study, most of the students were looking for instrumental ways in order to know how (Chevallard, 2005, pp. 23–24) to handle the assignment.

Considering what Mason et al. (2005, p. 23) emphasize concerning mathematics teaching which mainly focuses on specific tasks, a subsequent issue arises as to whether the mathematics teaching the students in this study have previously met may obstruct them from discerning general mathematical structures, in the sense of experiencing relationships between quantities as something that has to be related. This is an assumption, since also the older students in the study mostly focused on calculations, specific values, and the specific situation, based on the specific illustration. This in turn indicates an implication, that mathematics teaching needs to change focus from merely arithmetic teaching (van Oers, 2001, p. 62), mentioned above, to algebraic teaching (Davydov, 2008, p. 121), mentioned in the Introduction section.

The main contribution is, we argue, that it is promising to introduce mathematics (arithmetic) with teaching based on general mathematical structures and, in the case of this article, relationships within additive structures. The argument is based on both previous research, presented in this article, and the findings in this study.

We argue that it is worthwhile to introduce mathematics teaching for young students based on general structures, general concepts and symbols. Some of the students, also in grade 3, talked about relationships between quantities, when they were encouraged. Another argument for introducing mathematics teaching based on general structures, concepts and symbols is that it may enable students to in a larger extent, analyse tasks before solving them. Assuming that it is important to work with general structures, already in early grades, there is need of knowledge concerning what different ways students experience general structures. Thus, we argue that the teaching initially should focus on exploring relationships between quantities and describing the relationships with both general concepts and general symbols (cf. Davydov, 2008, p. 128).

The contribution concerning students' ways of experiencing relationships between quantities can be considered useful for teachers, especially those who are teaching students aged seven to fifteen years old, when planning and conducting lessons in order to enable students to discern relationships between quantities. It is essential for teachers not to take for granted that the students discern relationships in the sense of the critical aspect *two quantities together (two parts) build up a third quantity (the whole) with the same "value" as the two parts together*. This critical aspect may be valuable to take into account when planning lessons in order to enable students to explore relationships between quantities in a more qualified way.

One way of enabling students to experience relationships between quantities according to main category 2, may be to explore quantities with the support of a learning model (see Figure 1, in Introduction), which may function as a mediating tool (Davydov, 2008, pp. 94–95). A learning model may support students to discern the part-whole structure, and thus the additive structure (Vergnaud, 1982, p. 58). If the students discern the additive structure, it will be possible for them to formulate one and the same relationship as the fact family (e.g., Davydov, 2008, p. 128; Schmittau, Morris, 2004, pp. 67–70). This in turn may enable students to reconstruct a seemingly difficult equation (e.g., $-15 = -7 - x$) to an operation, for them, more easy to solve.

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Najpierw związki, potem obliczenia: sposoby doświadczania przez uczniów relacji miedzy ilościami

S t r e s z c z e n i e

Artykuł ten jest przyczynkiem do badań dotyczących sposobów doświadczania przez uczniów ogólnych struktur matematycznych, a w szczególności związków zachodzących w strukturach addytywnych. Jeżeli uczniowie będą prawidłowo dostrzegać związki zachodzące w strukturach addytywnych, może to mieć pozytywne konsekwencje dla ich przyszłych umiejętności wykonywania obliczeń

w zadaniach na dodawanie i odejmowanie. W ramach badań przeprowadzono częściowo strukturyzowane wywiady z uczniami z klas 3, 8 i 9. Punktem wyjścia wywiadów była ilustracja przedstawiająca zestaw różnych ilości, wraz z pytaniem wstępnym: w jaki sposób można doprowadzić do zrównania tych ilości. Uczniowie najpierw dyskutowali możliwe rozwiązania, a następnie byli pytani, czy potrafią te związki zapisać używając symboli matematycznych. Wypowiedzi uczniów dotyczące zjawiska „związków między ilościami" były analizowane z wykorzystaniem fenomenografii jako narzędzia analitycznego. Zgodnie z fenomenografią, istnieje ograniczona liczba sposobów doświadczania określonego zjawiska. Co więcej, nie chodzi o zbadanie, ile osób doświadcza badane zjawisko w określony sposób. W naszych badaniach interesowało nas wychwycenie jakościowo różnych sposobów doświadczania zjawiska zależności między wielkościami. Pomimo tego, że w zadaniu nie podawano żadnych konkretnych liczb, wyrażając relacje miedzy ilościami wielu uczniów przypisywało im określone liczby i wartości. Tylko niekiedy uczniowie wyrażali w sposób ogólny struktury matematyczne, a i to tylko w tych przypadkach, gdy byli do tego zachęcani przez ankietera. Analiza fenomenografczna sposobów doświadczania przez uczniów „relacji między ilościami" wykazała, że taka relacja jest rozumiana jako coś, co należy obliczyć, lub jako coś, co musi być w określonym związku. Pierwsze z tych podejść było najbardziej powszechne we wszystkich klasach. W tym badaniu zidentyfikowano jeden aspekt krytyczny, mianowicie, że ilości są powiązane ze sobą w sposób addytywny. Wynika z tego, że zamiast wprowadzać uczniów w matematykę koncentrując się na zadaniach zorientowanych na odpowiedź, konieczne jest wprowadzenie matematyki w oparciu o ogólne struktury, takie jak struktury addytywne. Nawet jeśli uczniowie nie są zaznajomieni z taką matematyczną „kulturą", warto to robić. Takie podejście znalazło potwierdzenie w naszych badaniach.

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