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Historical Comments on Monge's Ellipsoid and the Configurations of Lines of Curvature on Surfaces

Abstract. This is an essay on the historical landmarks leading to the study of principal confgurations on surfaces in $\mathbb{R}^3$, their structural stability and further generalizations. Here it is pointed out that in the work of Monge, 1796, are found elements of the qualitative theory of differential equations, founded by Poincaré in 1881. Recent development concerning the space $\mathbb{R}^4$ are mentioned. Two open problems are proposed at the end.

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1. Introduction. The book on differential geometry of D. Struik [49] contains key references to the classical works on *principal curvature lines* and their *umbilic* singularities due to L. Euler, G. Monge, C. Dupin, G. Darboux, A. Gullstrand and Carathéodory, among others (see also [48] and, for additional references, also [34]). These papers – notably that of Monge, complemented with Dupin's – can be connected with aspects of the *qualitative theory of differential equations* (*QTDE* for short) initiated by H. Poincaré [44] and consolidated with the study of the *structural stability and genericity* of differential equations in the plane and on surfaces, which was made systematic from 1937 to 1962 due to the seminal works of Andronov, Pontrjagin and Peixoto (see [1, 42, 43]).

The first author established this connection by 1970, after being prompted by the fortunate reading of Struik [49]. The essay [47], recounts how the interpretation of the historical landmarks led to inquire about the *principal configurations* on smooth surfaces, their *structural* *stability* and *bifurcations,* subjects of current interest in Mathematics, Modeling and Applications.

This paper discusses the historical sources for the work on the structural stability of principal curvature lines and umbilic points, developed and further extended with the collaboration of C. Gutierrez [31–33] (see also the papers devoted to other classes of differential equations in classical geometry: the asymptotic lines [13, 18, 26], and the arithmetic, geometric and harmonic mean curvature lines [20–22]). This direction of research led the authors and collaborators to the study of general mean curvature lines in [23] and also to consider other ambient spaces of higher dimension [12, 27, 28, 36, 37].

This paper contains a reformulation of the essentials of [47], updates the historical references, mentioning new developments in three and higher dimensions and proposes two open problems.

Here it is also pointed out that in the work of Monge, [41], are found sugestive elements of the qualitative theory of differential equations, founded by Poincaré in [44].

2. The Landmarks before Poincaré: Euler, Monge and Dupin

Leonhard Euler (1707–1783) [10], founded of the curvature theory of surfaces. He defined the *normal curvature* $k_n(p, L)$ on an oriented surface $\mathbf{S}$ in a tangent direction L at a point p as the curvature, at p, of the planar curve of intersection of the surface with the plane generated by the line L and the positive unit normal N to the surface at p. The *principal curvatures* at p are the extremal values of $k_n(p, L)$ when L ranges over the tangent directions through p. Thus, $k_1(p) = k_n(p, L_1)$ is the *minimal* and $k_2(p) = k_n(p, L_2)$ is the *maximal* of the *normal curvatures*, attained along the *principal directions*: $L_1(p)$, *the minimal*, and $L_2(p)$, *the maximal* (see Fig. 1).

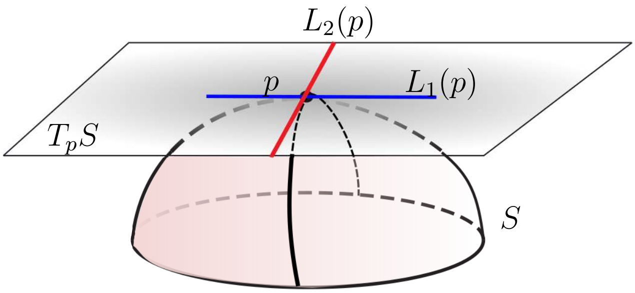

Figure 1: Principal Directions.

Euler's formula expresses the normal curvature $k_n(\theta)$ along a direction making angle θ with the minimal principal direction L_1 as

$$k_n(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

Euler, however, did not consider the integral curves of the principal line fields $L_i : p \to L_i(p)$, $i = 1, 2$, and overlooked the role of the *umbilic points* at which the principal curvatures coincide and the principal line fields are undefined.

Gaspard Monge (1746–1818) coined the mathematical term *umbilic point* in the sense defined above and found the family of integral curves of the *principal line fields* $L_i, i = 1, 2$, for the case of the triaxial ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad a > b > c > 0.$$

In doing this, by direct integration of the differential equations of the principal curvature lines, circa 1779, Monge was led to the first example of a *foliation with singularities* on a surface. The name *principal configuration* of an oriented surface is used to refer to its *umbilic points* together with the pair of orthogonal foliations by principal curvature lines on the complement of such points. The Ellipsoid, endowed with its principal configuration, will be called Monge's Ellipsoid (see Fig. 2).

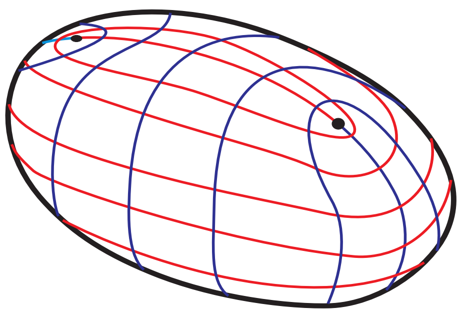

Figure 2: Monge's Ellipsoid.

The motivation found in Monge's paper [41] is a complex interaction of esthetic and practical considerations and of the explicit desire to apply the results of his mathematical research to real world problems. The principal configuration on the triaxial ellipsoid appears in Monge's proposal for the dome of the Legislative Palace for the government of the French Revolution, to be built over an elliptical terrain. The lines of curvature are the guiding curves for the workers to put the stones. The umbilic points, from which were to hang the candle lights, would also be the reference points below which to put the podiums for the speakers. In [46] J. Sakarovitch focuses on the geometric aspects of Monge building proposal and the connection with its practical implementation for construction purposes. He also discusses Monge theory of stone carving.

The ellipsoid depicted in Fig. 2 contains some of the typical features of the qualitative theory of differential equations discussed briefly in a) to d) below:

a) **Singular Points and Separatrices.** The umbilic points play the role of singular points for the principal foliations, each of them has one separatrix for each principal foliation. This separatrix produces a connection with another umbilic point of the same type, for which it is also a separatrix, in fact an umbilic separatrix connection.

b) **Cycles.** The configuration has principal cycles. In fact, all the principal lines, except for the four umbilic connections, are periodic. The cycles fill a cylinder or annulus, for each foliation. This pattern is common to all classical examples, where no surface exhibiting an isolated cycle was known. This fact is derived from the symmetry of the surfaces considered and from the integrability that is present in the classical examples involving the application of Dupin's Theorem for triply orthogonal families of surfaces. See Fig. 3.

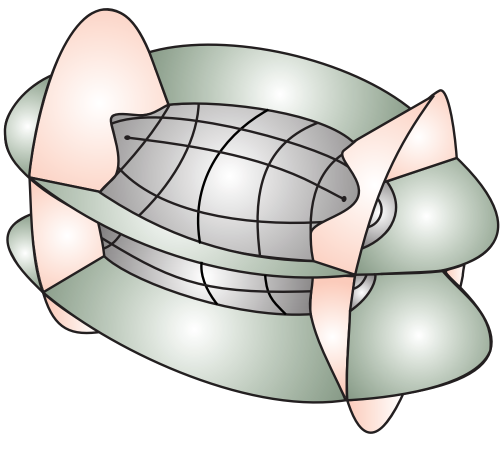

Figure 3: Dupin's Theorem.

As was shown in [31], these configurations are exceptional, i.e. *non-generic*; the generic principal cycle for a smooth surface is a *hyperbolic* limit cycle. In section 3 is recalled the integral characterization of these cycles.

c) **Structural Stability (relative to quadric surfaces).** The principal configuration remains qualitatively unchanged under small perturbations on the coefficients of the quadric polynomial that defines the surface.

d) **Bifurcations.** The drastic changes in the principal configuration exhibited by the transitions from a sphere, to an ellipsoid of revolution and to a triaxial ellipsoid (as in Fig. 2), which after a very small perturbation, is a simple form of a bifurcation phenomenon.

Charles Dupin (1784–1873) considered the surfaces that belong to *triply orthogonal surfaces,* thus extending considerably those whose principal configurations can be found by integration. Monge's Ellipsoid belongs to the family of *homofocal quadrics* (see [49] and Fig. 3).

Remark 2.1 Notice that M. Binet had contributed with the theory of triply orthogonal systems previously to Dupin's publication. See [3, 4]. The authors are grateful to G. Wüstholz for pointing out these references.

The conjunction of Monge's analysis and Dupin extension provides the first global theory of principal configurations, the integrable ones, which for quadric surfaces gives those which are also *principally structurally stable* under small perturbations of the coefficients of their quadratic defining equations. This primary observation is expressed as theorem 2.2.

Theorem 2.2 [47] *In the space of oriented quadrics, identified with the nine-dimensional sphere, those having principal structurally stable configurations are open and dense.*

Historical Remark. *The global structure of umbilic points and lines of principal curvature leading to Monge's Ellipsoid, which is analogous of the phase portrait of a differential equation, contains elements of Poincaré's QTDE, studied 85 years later.*

This remark seems to have been overlooked by Monge's scientific historian René Taton (1915–2004) in his remarkable book [51].

Concerning G. Monge contributions [40,41], see also the more recent works of E. Ghys and R. Langevin [29,30,35].

Further developments on orthogonal systems of coordinates can be found in M. Berger [2], G. Darboux [8] and C. Dupin [9].

3. Differential Equations: Poincaré and Darboux

The exponential role played by **Henri Poincaré** (1854–1912) for the *QTDE* as well as for other branches of mathematics is well known and has been discussed and analyzed in several places (see for instance [5] and [43]).

Here we are concerned with his Memoires [44], where he laid the foundations of the *QTDE*. In this work Poincaré determined the form of the solutions of planar analytic differential equations near their *foci*, *nodes* and *saddles*. He also studied properties of the solutions around cycles and, in the case of polynomial differential equations, also the behavior at infinity.

Gaston Darboux (1842–1917) determined the structure of the lines of principal curvature near a *generic* umbilic point. In his note [7], Darboux uses the theory of singularities of Poincaré. In fact, the Darbouxian umbilics are those whose resolution by blowing up are saddles and nodes (see Figs. 4 and 5).

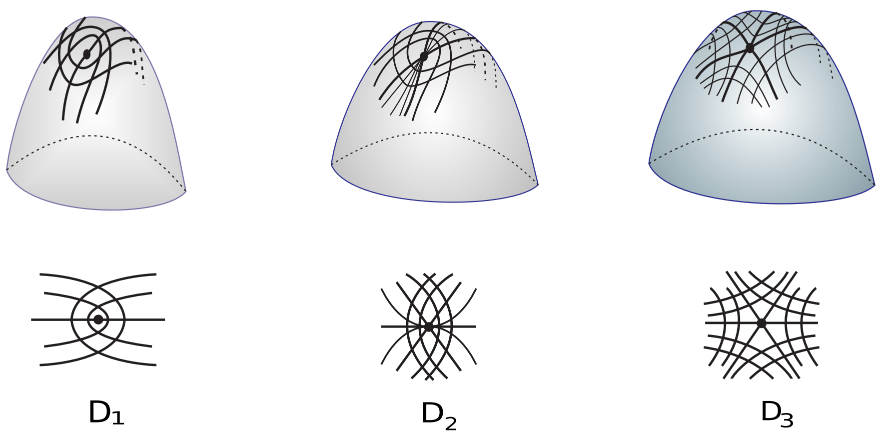

Figure 4: Darbouxian Umbilics.

Let $\mathbf{p}_0 \in \mathbf{S}$ be an umbilic point. Consider a chart $(u, v) : (\mathbf{S}, \mathbf{p}_0) \to (\mathbb{R}^2, \mathbf{0})$ around it, on which the surface has the form of the graph of a function such as

$$\frac{k}{2}(u^2+v^2) + \frac{a}{6}u^3 + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + O[(u^2+v^2)^2].$$

This is achieved by projecting $\mathbf{S}$ onto $T\mathbf{S}(\mathbf{p}_0)$ along $N(\mathbf{p}_0)$ and choosing there an orthonormal chart (u, v) on which the coefficient of the cubic term u^2v vanishes.

An umbilic point is called *Darbouxian* if, in the above expression, the following 2 conditions **T)** and **D)** hold:

T) $b(b-a) \neq 0$

D) either

$D_1 : a/b > (c/2b)^2 + 2$, or

$D_2 : (c/2b)^2 + 2 > a/b > 1,\ a \neq 2b$, or

$D_3 : a/b < 1$.

The corroboration of the pictures in Fig. 4, which illustrate the principal configurations near Darbouxian umbilics, has been given in [31, 33]; see also [6, 26] and Fig. 5 for the Lie-Cartan resolution of a Darbouxian umbilic.

The subscripts refer to the number of *umbilic separatrices,* which are the curves tending to the umbilic point and separating regions whose principal lines have different patterns of approach to the umbilic point.

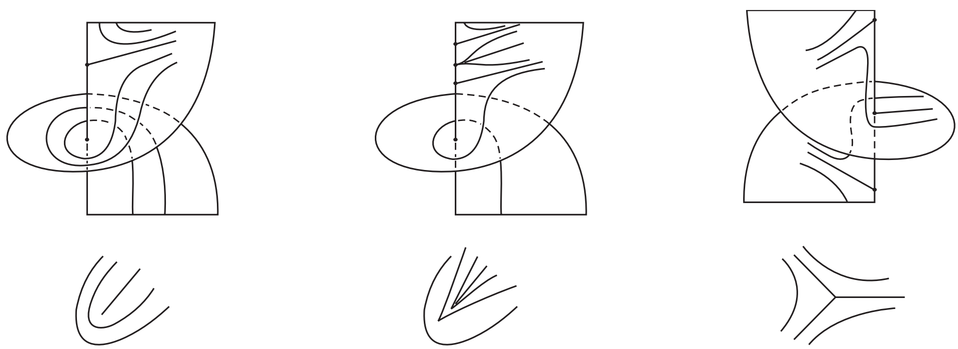

Figure 5: Lie-Cartan Resolution of Darbouxian Umbilics in terms of Poincaré Saddles and Nodes Singularities of Ordinary Differential Equations.

The work of Darboux [7], properly reformulated, amounts to a local generic one theory for principal configurations, the second one after Monge-Dupin global theory for integrable ones.

4. Structurally Stable Principal Configurations on Smooth Surfaces

After the seminal work of Andronov-Pontrjagin [1] on structural stability of differential equations in the plane and its extension to surfaces by Peixoto [42, 43] and in view of the discussion on Monge's Ellipsoid formulated above, an inquiry into the characterization of the oriented surfaces in $\mathbf{S}$ whose principal configuration are structurally stable under small C^r perturbations, for $r \geq 3$, seems unavoidable.

Call $\Sigma(a, b, c, d)$ the set of smooth compact oriented surfaces $\mathbf{S}$ which verify the following conditions.

a) All umbilic points are Darbouxian.

b) All principal cycles are hyperbolic. This means that the corresponding return map is *hyperbolic*; that is, its derivative is different from 1. It has been shown in [31] that hyperbolicity of a principal cycle γ is equivalent to the requirement that

$$\int_\gamma \frac{d\mathcal{H}}{\sqrt{\mathcal{H}^2 - \mathcal{K}}} \neq 0.$$

where $\mathcal{H} = (k_1 + k_2)/2$ is the mean curvature and $\mathcal{K} = k_1 k_2$ is the Gaussian curvature.

c) The limit set of every principal line is contained in the set of umbilic points and principal cycles of $\mathbf{S}$.

The α-(resp. ω) *limit set* of an oriented principal line γ, defined on its maximal interval $\mathcal{I} = (w_-, w_+)$ where it is parametrized by arc length s, is the collection $\alpha(\gamma)$-(resp. $\omega(\gamma)$) of limit point sequences of the form $\gamma(s_n)$, convergent in $\mathbf{S}$, with s_n tending to the left (resp. right) extreme of $\mathcal{I}$. The *limit set* of γ is the set $\Omega = \alpha(\gamma) \cup \omega(\gamma)$.

Examples of surfaces with *non trivial recurrent* principal curves, which violate condition c) are given in [32, 33]. There are no examples of these situations in the classical geometry literature.

d) All umbilic separatrices are separatrices of a single umbilic point.

Separatrices which violate d) are called *umbilic connections*; an example can be seen in the ellipsoid of Fig. 2.

To make precise the formulation of the next theorems, some topological notions must be defined.

A sequence $\mathbf{S}_n$ of surfaces *converges in the C^r sense* to a surface $\mathbf{S}$ provided there is a sequence of real functions $\mathbf{f}_n$ on $\mathbf{S}$, such that $\mathbf{S}_n = (I + f_n N)(\mathbf{S})$ and f_n tends to 0 in the C^r sense; that is, for every chart (u, v) with inverse parametrization X, $f_n \circ X$ converges to

0, together with the partial derivatives of order r, uniformly on compact parts of the domain of X.

A set Σ of surfaces is said to be *open* in the C^r sense if every sequence $\mathbf{S}_n$ converging to $\mathbf{S}$ in Σ in the C^r sense is, for n large enough, contained in Σ.

A set Σ of surfaces is said to be *dense* in the C^r sense if, for every surface $\mathbf{S}$, there is a sequence $\mathbf{S}_n$ in Σ converging to $\mathbf{S}$ the C^r sense.

A surface $\mathbf{S}$ is said to be *C^r-principal structurally stable* if for every sequence $\mathbf{S}_n$ converging to $\mathbf{S}$ in the C^r sense, there is a sequence of homeomorphisms H_n from $\mathbf{S}_n$ onto $\mathbf{S}$, which converges to the identity of $\mathbf{S}$, such that, for n big enough, H_n is a principal equivalence from $\mathbf{S}_n$ onto $\mathbf{S}$. That is, it maps $\mathbf{U}_n$, the umbilic set of $\mathbf{S}_n$, onto $\mathbf{U}$, the umbilic set of $\mathbf{S}$, and maps the lines of the principal foliations $\mathbf{F}_{i,n}$, of $\mathbf{S}_n$, onto those of $\mathbf{F}_i$, $i = 1, 2$, principal foliations for $\mathbf{S}$.

Theorem 4.1 *(Structural Stability of Principal Configurations* [31,33]*)* *The set of surfaces $\Sigma(a, b, c, d)$ is open in the C^3 sense and each of its elements is C^3-principal structurally stable.*

Figure 6 depicts artistically principal configurations of the class $\Sigma(a, b, c, d)$.

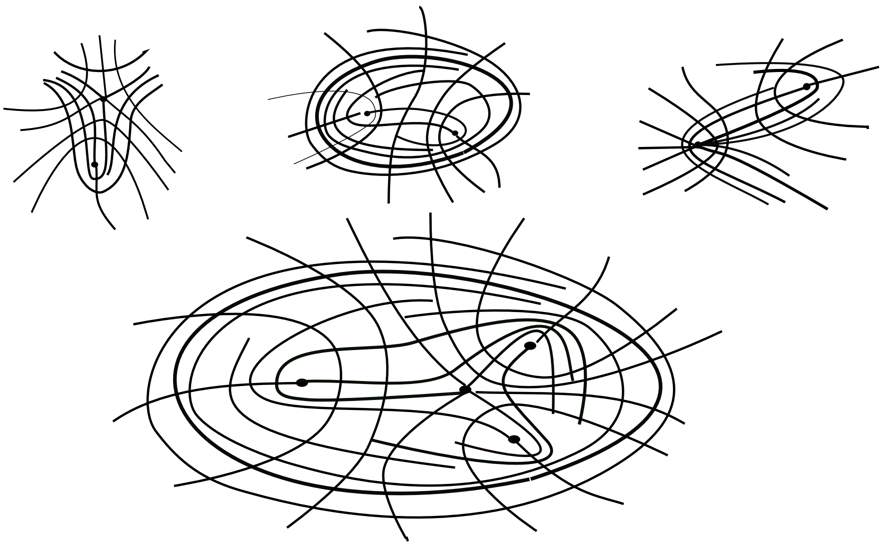

Figure 6: Illustrations of principally structurally stable patterns.

Theorem 4.2 *(Density of Principal Structurally Stable Surfaces,* [32, 33]*) The set* $\Sigma(a, b, c, d)$ *is dense in the* C^2 *sense.*

Extensions of the results for surfaces with generic singularities, algebraic surfaces, surfaces and hypersurfaces in $\mathbb{R}^4$ have been achieved recently (see, for example, [12, 14, 16, 17, 19, 38, 50]).

The bifurcations of umbilic points have been studied in [15], where references to other aspects of the bifurcations of principal configurations are found.

To conclude, two significant problems are stated.

Problem 1 Raise from 2 to 3 the differentiability class in the density Theorem 4.2.

This remains among the most intractable questions in this subject, involving difficulties of Closing Lemma type, [39, 45], which also permeate other differential equations of classical geometry, [23].

See [24] for an example of an embedded torus in $\mathbb{R}^3$ such that all principal lines, in both principal foliations, are dense.

Problem 2 Determine the class of principally structurally stable cubic and higher degree surfaces. Theorem 2.2 deals with the case of degree 2 – quadratic – surfaces.

Partial results, including the behavior of lines of curvature at infinity in non-compact algebraic surfaces, have been established by Garcia and Sotomayor [17, 25].

Non trivial problems concerning the Index of isolated umbilic points derive from the elusive Carathéodory Conjecture. See F. Fontenele and F. Xavier [11].

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Elipsoid Monge'a i konfiguracje linii krzywiznowych na powierzchniach i przestrzeniach euklidesowych – uwagi historyczne

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Streszczenie. W prezentowanej pracy przedstawione są historyczne źródła dotyczące badań nad głównymi konfiguracjami na powierzchniach w $\mathbb{R}^3$, ich strukturalną stabilnością i dalszymi uogólnieniami. Zwrócono uwagę na prace Monge'a z 1796 roku, gdzie pojawiają się elementy jakościowej teorii równań różniczkowych, za twórcę której uważa się Poincarégo. Wspomniano także o najnowszych wynikach dotyczących przestrzeni $\mathbb{R}^4$ oraz zaproponowano dwa otwarte problemy.

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